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STUDY ON THE JOINT SPECTRUM

Dong Hark Lee

Abstract. We introduce the Joint spectrum on the complex Banach space and on the complex Hilbert space and the tensor product spectrums on the tensor product spaces. And we will show $\sigma[P(T_1, T_2, \ldots, T_n)] = \sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n)$ on $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ for a polynomial P.

1. Introduction

Let $BL(X)$ denote the algebra of bounded linear operator acting on the complex Banach space X. If $T \in BL(X)$, then write $N(T)$ and $R(T)$ for the null space and the range of T ; $\alpha(T) = \dim(N(T))$; $\beta(T) = \operatorname{codim}(R(T))$; $\sigma(T)$ for the spectrum of T .

An operator $T \in BL(X)$ is called Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension.

The index of a Fredholm operator $T \in BL(X)$ is givin by $i(T) = \alpha(T) - \beta(T)$.

An operator $T \in BL(X)$ is called Weyl if it is Fredholm of index zero.

An operator $T \in BL(X)$ is called Browder if it is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$.

The essential spectrum $\sigma_e(T)$, the Weyl's spectrum $w(T)$, and the Browder's spectrum $\sigma_b(T)$ are defined by

$$\begin{aligned}
\sigma_e(T) &= \{\lambda \in \mathbb{C} | T - \lambda I \text{ is not Fredholm}\}; \\
w(T) &= \{\lambda \in \mathbb{C} | T - \lambda I \text{ is not Weyl}\}; \\
\sigma_b(T) &= \{\lambda \in \mathbb{C} | T - \lambda I \text{ is not Browder}\}; \\
\sigma(T) &= \{\lambda \in \mathbb{C} | T - \lambda I \text{ is not invertible}\}.
\end{aligned}$$

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Then $\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) \subseteq \sigma_e(T) \cup acc(\sigma(T)) \subseteq \sigma(T)$ ([15]) , where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$.

Let $T = (T_1, T_2, \ldots, T_n)$ be an n-tuple of commuting operators(bounded linear operators) on a complex Banach space X.

Then the joint spectrum of $T = (T_1, T_2, \ldots, T_n)$ with respect to $BL(X)$ is to be the set of n-tuples $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of complex numbers for which the system $T - \lambda = (T_1 - \lambda_1, T_2 - \lambda_2, \ldots, T_n - \lambda_n)$ generates a proper left or right ideal in $BL(X)$. The joint spectrum of T is denoted by $\sigma(T) = \sigma^l(T) \cup \sigma^r(T)$, where $\sigma^l(T) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | I \notin \sum_i BL(X)(T_i - \lambda_i)\}$ and $\sigma^r(T) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | I \notin \sum_i (T_i - \lambda_i) BL(X), 1 \le i \le n\ \}$ ([9]).

Let $T = (T_1, T_2, \ldots, T_n)$ be n-tuples of commuting bounded linear operators defined on a complex Banach space X and $T_i \in BL(X), i = 1, 2, \ldots, n$. Define $M_k(T) = R(T_1^k) + R(T_2^k) + \cdots + R(T_n^k)$ for $k \in N$, where $T(x) = (T_1, T_2, \ldots, T_n)(x) = T_1(x) + T_2(x) + \cdots + T_n(x)$ for each $x \in X$.Clearly, $X = M_0(T) \supseteq M_1(T) \supseteq \ldots$ holds. Set $R^\infty(T) = \cap_{k=1}^\infty M_k(T)$. We say that $T = (T_1, T_2, \ldots, T_n)$ is lower semi-Fredholm, that is; $T \in \Phi -^{(n)} (X)$, if codim$(M_1(T)) =$ codim$(R(T_1) + R(T_2) + \cdots + R(T_n)) < \infty$.$T = (T_1, T_2, \ldots, T_n)$ is lower semi-Weyl, that is; $T \in W -^{(n)} (X)$, if codim$(R^n(T)) =$ codim$(\cap_{k=1}^n M_k(T)) < \infty$ for all $n \ge 2$. And $T = (T_1, T_2, \ldots, T_n)$ is lower semi-Browder, that is; $T \in B -^{(n)} (X)$, if codim$(R^\infty(T)) =$ codim$(\cap_{k=1}^\infty M_k(T)) < \infty$. Since codim$(M_1(T)) < \infty$ implies codim$(M_k(T)) < \infty$ for every $k \in N$ ([13]).

We have the inclusion $\Phi -^{(n)} (X) \subseteq W -^{(n)} (X) \subseteq B -^{(n)} (X)$.

The lower semi-Fredholm spectrum $\sigma_\Phi - (T)$; the lower simi-Weyl spectrum $\sigma_W - (T)$; the lower semi-Browder spectrum $\sigma_B - (T)$ and the defect spectrum $\sigma_{\delta(T)}$ of $T = (T_1, T_2, \ldots, T_n)$ are defined by

$$\begin{aligned}
\sigma_\Phi - (T) &= \{\lambda \in \mathbb{C}^n | T - \lambda I \notin \Phi -^{(n)} (X)\}; \\
\sigma_W - (T) &= \{\lambda \in \mathbb{C}^n | T - \lambda I \notin W -^{(n)} (X)\}; \\
\sigma_B - (T) &= \{\lambda \in \mathbb{C}^n | T - \lambda I \notin B -^{(n)} (X)\}; \\
\sigma_\delta - (T) &= \{\lambda \in \mathbb{C}^n | \text{codim}(M_1(T - \lambda I)) \ne 0\} \qquad ([1], [3]).
\end{aligned}$$

We say that $T = (T_1, T_2, \ldots, T_n)$ is upper semi-Fredholm, that is; $T \in \Phi +^{(n)} (X)$, if the map $T : X \to X^n$ defined by $T(x) = (T_1(x), \ldots, T_n(x))$ is upper semi-Fredholm; equivalently, if T has finite dimensional null space and closed range;, $T = (T_1, T_2, \ldots, T_n)$ is upper semi-Weyl, that is, $T \in W +^{(n)} (X)$, if $T \in \Phi +^{(n)} (X)$ and dim$(N^n(T)) =$ dim$(\cup_{k=1}^n [N(T_1^k) \cap$

$N(T_2^k) \cap \cdots \cap N(T_n^k)]) < \infty$ for all $n \geq 2$, and $T = (T_1, T_2, \ldots, T_n)$ is upper semi-Browder, that is, $T \in B +^{(n)} (X)$, if $T \in \Phi +^{(n)} (X)$ and $\dim(N^\infty(T)) = \dim(\cup_{k=1}^\infty [N(T_1^k) \cap N(T_2^k) \cap \cdots \cap N(T_n^k)]) < \infty$.

The upper semi-Fredholm spectrum $\sigma_\Phi + (T)$, the uppper semi-Weyl spectrum $\sigma_w + (T)$, the upper semi-Browder spectrum $\sigma_B + (T)$, and the approximate point spectrum $\sigma_\pi(T)$ of $T = (T_1, T_2, \ldots, T_n)$ are defined by([1], [13]);

$$\begin{aligned} \sigma_\Phi + (T) &= \{\lambda \in \mathbb{C}^n | T - \lambda I \notin \Phi +^{(n)} (X)\}; \\ \sigma_W + (T) &= \{\lambda \in \mathbb{C}^n | T - \lambda I \notin W +^{(n)} (X)\}; \\ \sigma_B + (T) &= \{\lambda \in \mathbb{C}^n | T - \lambda I \notin B +^{(n)} (X)\}; \\ \sigma\pi + (T) &= \{\lambda \in \mathbb{C}^n | N(T - \lambda I) \neq 0 \text{ or } R(T - \lambda I) \text{is not closed}\}. \end{aligned}$$

Let $T = (T_1, T_2, \ldots, T_n)$ be a commuting n-tuple of bounded linear operators defined on a complex Banach space X and let $C - (T) = \{\lambda \in \mathbb{C}^n | N(T - \lambda I)$ has not a direct complement in X^n, where a map $T : X^n \to X$ defined by $T(x_1, x_2, \ldots, x_n) = T_1(x_1) + T_2(x_2) + \cdots + T_n(x_n)\}$.

We say that $SP_e - (T) = \sigma_\Phi - (T) \cup C - (T)$ is the lower split semi-Fredholm spectrum of T, $SR_W - (T) = \sigma_w - (T) \cup C - (T)$ is the lower split semi-Weyl spectrum of T, $SP_B - (T) = \sigma_B - (T) \cup C - (T)$ is the lower split semi-browder spectrum of T, and $SP_\delta(T) = \sigma_\delta(T) \cup C - (T)$ is the split defect spectrum of T.

Let $T = (T_1, T_2, \ldots, T_n)$ be an n-tuple of commuting bounded linear operators defined on a complex Banach space X.

Define the map $T : X \to X^n$ by $T(x) = (T_1(x), T_2(x), \cdots, T_n(x))$. Let $C + (T) = \{\lambda \in \mathbb{C}^n | R(T - \lambda I)$ has not a direct complement in $X^n\}$.

We say that $SP_e + (T) = \sigma_\Phi + (T) \cup C + (T)$ is the upper split semi-Fredholm spectrum of T, $SP_W + (T) = \sigma_W + (T) \cup C + (T)$ is the upper split semi-Weyl spectrum of T, $SP_B + (T) = \sigma_B + (T) \cup C + (T)$ is the upper split semi-Browder spectrum of T, and $SP_\pi(T) = \sigma_\pi(T) \cup C + (T)$ is the split approximate point spectrum of T.

Let $X_1, \ldots, X_n$ be the complex Banach spaces and $X = X_1 \overline{\otimes} \cdots \overline{\otimes} X_n$ be the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some uniform, reasonable cross-norm([5], [11]). Let I_k be the identity operator on X_k and A_k an arbitrary bounded linear operator on $X_k, 1 \leq$

$k \leq n$.

$$\begin{aligned} Set \quad T_1 &= A_1 \otimes I_2 \otimes \cdots \otimes I_n \\ &\vdots \\ T_n &= I_1 \otimes I_2 \otimes \cdots \otimes A_n. \end{aligned}$$

By ([4], [6]), $\sigma(T_k) = \sigma(A_k), 1 \leq k \leq n$.

We call $\sigma(T_1, T_2, \ldots, T_n) = \prod_{k=1}^{n} \sigma(T_k) = \prod_{k=1}^{n} \sigma(A_k) = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | \lambda_k \in \sigma(T_k), 1 \leq k \leq n\}$ the Joint spectrum of $T = (T_1, T_2, \ldots, T_n)$ on $X_1 \otimes X_2 \otimes \cdots \otimes X_n$. If $\sigma(T, X)$ is the Joint spectrum of $T = (T_1, T_2, \ldots, T_n)$ on $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$, then by ([6, theorem 1]), we have
$\sigma(T, X) = \prod_{k=1}^{n} \sigma(A_k), 1 \leq k \leq n$.

2. Main result

Theorem 2.1. *Let $T = (T_1, T_2, \ldots, T_2)$ be a commuting n-tuple of bounded linear operators defined on a complex Banach space X. Then the following statements hold:*
(1) $\sigma_\Phi - (T) \subseteq \sigma_W - (T) \subseteq \sigma_B - (T) \subseteq \sigma_\delta(T)$;
(2) $\sigma_\Phi + (T) \subseteq \sigma_W + (T) \subseteq \sigma_B + (T) \subseteq \sigma_\pi(T)$;
(3) $\sigma_\Phi - (T) \subseteq SP_e - (T) \subseteq SP_W - (T) \subseteq SP_B - (T) \subseteq SP_\pi(T)$;
(4) $\sigma_\Phi + (T) \subseteq SP_e + (T) \subseteq SP_W + (T) \subseteq SP_B + (T) \subseteq SP_\pi(T)$.

Proof. Since $\text{codim}(M_1(T)) < \infty$ implies $\text{codim}(M_k(T)) < \infty$ for every $k \in N$ and $\text{codim}(R^n(T)) = \text{codim}(\cap_{k=1}^{n} M_k(T)) < \infty$ for all $n \geq 2$,

$$\text{codim}(R^\infty(T)) = \text{codim}(\cap_{k=1}^{\infty}(M_k(T)) < \infty \qquad ([13], [1])$$

Easy calculations show that (1) and (2) hold. Since $SP_e - (T) = \sigma_\Phi - (T) \cup C - (T), SP_e + (T) = \sigma_\Phi + (T) \cap C + (T), \sigma_B - (T) \subseteq \sigma_\delta(T)$ and $\sigma_B + (T) \subseteq \sigma_\pi(T)$ ([1]), and by [12], it is easy to see that (3) and (4) hold. □

We now see that the joint spectrum we have introduced satisfy the main spectral properties.

Theorem 2.2. *Let $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ be a tensor product of the complex Hilbert space $X_i, 1 \leq i \leq n$. Let $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ be the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to* *some cross norm and let A_i be a bounded linear operator on $X_i, 1 \leq i \leq n$. Suppose that T_i is the operator on $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ defined by $T_i = I_1 \otimes I_2 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes I_{i+1} \otimes \cdots \otimes I_n$ and $T_n = I_1 \otimes I_2 \otimes \cdots \otimes I_{n-1} \otimes A_n$, where I_i is the identity operator on $X_i, 1 \leq i \leq n$. Then, $\sigma_\Phi(T_1, T_2, \cdots, T_n) \subseteq \sigma_W(T_1, T_2, \ldots, T_n) \subseteq \sigma_B(T_1, T_2, \ldots, T_n) \subseteq \sigma(T_1, T_2, \ldots, T_n)$.*

Proof. Since the operators T_i obviously commute, we have $\sigma(T_i) = \sigma(A_i), 1 \leq i \leq n$ ([4], [6]). A complex vector $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma(T_1, T_2, \ldots, T_n)$ *if and only if* $\lambda_i \in \sigma(A_i), 1 \leq i \leq n$, that is, $\sigma(T_1, T_2, \ldots, T_n) = \prod_{i=1}^{n} \sigma(A_i)$ [7,2.1 theorem], [4, Theorem 1]). And so we can verify the following results:
$\sigma_\Phi(T_1, T_2, \ldots, T_n) = \prod_{i=1}^{n} \sigma_e(A_i) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | \lambda_i \in \sigma_e(A_i), 1 \leq i \leq n\}$
$\sigma_W(T_1, T_2, \ldots, T_n) = \prod_{i=1}^{n} \sigma_W(A_i) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | \lambda_i \in \sigma_W(A_i), 1 \leq i \leq n\}$ and $\sigma_B(T_1, T_2, \ldots, T_n) = \prod_{i=1}^{n} \sigma_b(A_i) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n | \lambda_i \in \sigma_b(A_i), 1 \leq i \leq n\}$.
Since $\sigma_e(T) \subseteq W(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$, we have $\sigma_\Phi(T_1, T_2, \ldots, T_n) \subseteq \sigma_W(T_1, T_2, \ldots, T_n) \subseteq \sigma_B(T_1, T_2, \ldots, T_n) \subseteq \sigma(T_1, T_2, \ldots, T_n)$. □

Let $T_1, T_2, \ldots, T_n$ be a bounded linear operators on a Hilbert space X and let $T_1 \otimes T_2 \otimes \cdots \otimes T_n$ be a tensor product on space $X_1 \otimes X_2 \otimes \cdots \otimes X_n$. Then $\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \sigma(T_1)\sigma(T_2) \cdots \sigma(T_n) = \{\lambda \in \mathbb{C} | \lambda = \lambda_1 \lambda_2 \cdots \lambda_n, \lambda_i \in \sigma(T_i), 1 \leq i \leq n\}$ ([3]).

We see that the tensor product operator we have introduced satisfy the following spectral properties.

Theorem 2.3. *Let $T_1, T_2, \ldots, T_n$ be a bounded linear operators on a complex Hilbert space. Then, $\sigma_\Phi(T_1 \otimes T_2 \otimes \cdots \otimes T_n) \subseteq \sigma_W(T_1 \otimes T_2 \otimes \cdots \otimes T_n) \subseteq \sigma_B(T_1 \otimes T_2 \otimes \cdots \otimes T_n) \subseteq \sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n)$.*

Proof. By [3, 95-96], We have that $\sigma_\Phi(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \sigma_e(T_1)\sigma_e(T_2)$ $\sigma_e(T_3) \cdots \sigma_e(T_n) = \{\lambda = \lambda_1 \lambda_2 \cdots \lambda_n \in \mathbb{C} | \lambda_i \in \sigma_e(T_i),\ 1 \leq i \leq n\}, \sigma_W(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = w(T_1)w(T_2) \cdots w(T_n) = \{\lambda = \lambda_1 \lambda_2 \cdots \lambda_n \in \mathbb{C} | \lambda_i \in w(T_i), 1 \leq i \leq n\}$, and $\sigma_B(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \sigma_b(T_1)\sigma_b(T_2) \cdots \sigma_b(T_n) = \{\lambda = \lambda_1 \lambda_2 \cdots \lambda_n \in \mathbb{C} | \lambda_i \in \sigma_b(T_i), 1 \leq i \leq\ n\}$.
Since $\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$, we obtain the desired result. □
Let $X_1, X_2, \ldots, X_n$ be complex Banach space and let $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ be the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some cross norm.

Let T_i be the operator on $X_1\overline{\otimes}X_2\overline{\otimes}\cdots\overline{\otimes}X_n$ defined by $T_i = I_1 \otimes I_2 \otimes \cdots \otimes A_i \otimes I_{i+1} \otimes \cdots \otimes I_n$ for $A_i \in BL(X_i), 1 \leq i \leq n$. Since the operators T_i obviously commute, $\sigma(T_i) = \sigma(A_i), 1 \leq i \leq n([4],[6])$. Let $P(z_1, z_2, \ldots, z_n)$ be a polynomial in n variables. Then we can form the operator $P(T_1, T_2, \ldots, T_n)([2])$.

THEOREM 2.4. *Let $X_1, X_2, \ldots, X_n$ be complex Banach spaces and let $X_1\overline{\otimes}X_2\overline{\otimes}\cdots\overline{\otimes}X_n$ be the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some cross norm. If T_i is the operator on $X_1\overline{\otimes}X_2\overline{\otimes}\cdots\overline{\otimes}X_n$ defined by $T_i = I_1 \otimes I_2 \otimes \cdots \otimes A_i \otimes I_{i+1} \otimes \cdots \otimes I_n$ for $A_i \in BL(X_i), 1 \leq i \leq n$, then*

$$\sigma_\Phi\,[P(T_1, T_2, \ldots, T_n)] \subseteq \sigma_W[P(T_1, T_2, \ldots, T_n)] \subseteq$$
$$\sigma_B\,[P(T_1, T_2, \ldots, T_n)] \subseteq \sigma[P(T_1, T_2, \ldots, T_n)].$$

Proof. Martin Schechter show that $\sigma(T_i) = \sigma(A_i), 1 \leq i \leq n$ and $\sigma[P(T_1, T_2, \ldots, T_n)] = P[\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_n)] = \{P(\lambda_1, \lambda_2, \ldots, \lambda_n)|\lambda_i \in \sigma(T_i) = \sigma(A_i), 1 \leq i \leq n\}$ ([2, Theorem 2.1], [9, Proposition 1.2]). Then we have
$\sigma_\Phi[p(T_1, T_2, \ldots, T_n)] = P(\sigma_e(T_1), \sigma_e(T_2), \ldots, \sigma(T_n))\{P(\lambda_1, \lambda_2, \ldots, \lambda_n)$ $|\lambda_i \in \sigma_e(T_i) = \sigma_e(A_i), 1 \leq i \leq n\}$,
$\sigma_w[P(T_1, T_2, \ldots, T_n)] = P(w(T_1), w(T_2), \ldots, w(T_n)) = \{P(\lambda_1, \lambda_2, \ldots, \lambda_n)|\lambda_i \in w(T_i) = w(A_i), 1 \leq i \leq n\}$, *and*
$\sigma_B[P(T_1, T_2, \ldots, T_n)] = P(\sigma_b(T_1), \sigma_b(T_2), \ldots, \sigma_b(T_n)) = \{P(\lambda_1, \lambda_2, \ldots, \lambda_n)$ $|\lambda_i \in \sigma_b(T_i) = \sigma_b(A_i), 1 \leq i \leq n\}$.

Since $\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$ we obtain the desired result. □

Let $P(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a polynomial on n variables such that $P(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_1\lambda_2\cdots\lambda_n \in \mathbb{C}$ for $\lambda_i \in \mathbb{C}, 1 \leq i \leq n$.
Then we can obtain the following result.

THEOREM 2.5. *Let X_i be a complex Banach space and $X_1\overline{\otimes}X_2\overline{\otimes}\cdots\overline{\otimes}X_n$ the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some cross norm. Let A_i be a bounded linear operators on $X_i, 1 \leq i \leq n$ and let T_i be the operators on $X = X_1\overline{\otimes}X_2\overline{\otimes}\cdots\overline{\otimes}X_n$ defined by $T_1 = A_1 \otimes I_2 \otimes I_3 \otimes \cdots \otimes I_n$, and, in general, $T_i = I_1 \otimes I_2 \otimes \cdots \otimes I_{i-1} \otimes A_i \otimes I_{i+1} \otimes \cdots \otimes I_n, 1 \leq i \leq n$ where I_i is the identity operator on $X_i, 1 \leq i \leq n$.*
Suppose that $P(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a polynomial in n variables such that

$P(\lambda_1, \lambda_2, \dots, \lambda_n) = \lambda_1\lambda_2 \cdots \lambda_n$ for $\lambda_i \in \mathbb{C}, 1 \leq i \leq n$.
Then $\sigma[P(T_1, T_2, \dots, T_n)] = \sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n)$.

Proof. I_n[3], Brown and Pearcy show that $\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \sigma(T_1)\sigma(T_2) \cdots \sigma(T_n)$, where $T_1 \otimes T_2 \otimes \cdots \otimes T_n$ is the tensor product of $T_1, T_2, \dots, T_n$ acting on a Hilbert space $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$. But in [2], Martin Schechter show that for a complex Banach space X.

$$\begin{aligned} \sigma[P(T_1, T_2, \dots, T_n)] &= P[\sigma(T_1), \sigma(T_2), \dots, \sigma(T_n)] \\ &= \sigma(T_1)\sigma(T_2) \cdots \sigma(T_n). \end{aligned}$$

In[6, Corollary 3], B.P. Rynne shows that the spectrum of the operator $T_1 \otimes T_2 \otimes \cdots \otimes T_n$ on $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ is given by $\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \{\lambda_1\lambda_2 \cdots \lambda_n \in \mathbb{C} | \lambda_i \in \sigma(T_i), 1 \leq i \leq n\}$. That is, $\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \sigma(T_1)\sigma(T_2) \cdots \sigma(T_n)$.
In[2], Martin Schechter show that $\sigma(T_i) = \sigma(A_i), 1 \leq i \leq n$. We obtain the desired result. □

Let $B_1, B_2, \dots$, and B_n be subsets of $\mathbb{C}$ and let $P(B_1, B_2, \dots, B_n)$ be a polynomial in n variables such that $P(B_1, B_2, \dots, B_n) = B_1 \times B_2 \times \cdots \times B_n = \{z = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n | a_i \in B_i, 1 \leq i \leq n\}$.
Let us state the following result.

THEOREM 2.6. *Let $X_1, X_2, \dots$, and X_n be complex Banach spaces and let A_k be bounded linear operators on $X_k, 1 \leq k \leq n$. Let $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ be the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some uniform, reasonable crossnorm, and let $T_k = I_1 \otimes I_2 \otimes \cdots \otimes A_k \otimes I_{k+1} \otimes \cdots \otimes I_n$ on $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n, 1 \leq k \leq n$, where I_k is the identity operator on X_k. Then $\sigma[P(T_1, T_2, \dots, T_n)] = \sigma[(T_1, T_2, \dots, T_n)] = \sigma[(A_1, A_2, \dots, A_n)]$.*

Proof. In[2] Martin Schechter show that $\sigma[P(T_1, T_2, \dots, T_n)] = P(\sigma(T_1), \sigma(T_2), \dots, \sigma(T_n))$.
And by Definition of $P(B_1, B_2, \dots, B_n)$,

$$\begin{aligned} P[\sigma(T_1), \sigma(T_2), \dots, \sigma(T_n)] &= \sigma(T_1) \times \sigma(T_2) \times \ldots \times \sigma(T_n) \\ &= \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n | \lambda_k \in \sigma(T_k), 1 \leq k \leq n\}. \end{aligned}$$

On the other hand, in[6] B.P. Rynne show that $\sigma[(T_1, T_2, \dots, T_n)] = \sigma(T_1) \times \sigma(T_2) \times \cdots \times \sigma(T_n)$. In[2], Martin Schechter show that $\sigma(T_k) = \sigma(A_k), 1 \leq k \leq n$. We obtain the desired result. □

References

[1] Enrico Boasso, Tensor Products and the semi-Browder Joint Spectra, J. Operator Theory 47(2002), 79-89.

[2] Martin Schechter, On the Spectra of Operators on Tensor Products. Journal of Functional Analysis 4(1969), 95-99.

[3] Arlen Brown and Carl Pearcy, Spectra of Tensor Products of Operators. Proc. Amer. Math. Soc. 17(1966), 162-165.

[4] A.T. Dash and M. Schechter, Tensor products and Joint spectra. Israel J. Math. Vol. 8(1970), 191-193.

[5] Volker Wrobel, Tensor Products of Linear Operators in Banach spaces and Taylors' Joint Spectrum. Operator Theory16(1986), 273-275.

[6] B. P. Rynne, Tensor Products and Taylors' Joint Spectrum in Banach Spaces. R. Ir. Acad. Vol.87A, No.1,(1987), 83-88.

[7] Zoia Ceausescu and F. H. Vasilescu, Tensor Products and Taylors' Joint Spectrum. Studia Mathematica, T. LXII.(1978), 305-310.

[8] Takashi Ichinose, Spectra Properties of Tensor Products of Linear Operator. I. Transactions of the America Mathematical Society Volume 235, (1978), 75-81.

[9] Robin Harte, The Spectral Mapping Theorem in Several Variables. Bulletin of the American Mathematical Society Volume 78, 5(1972), 871-872.

[10] Jörg Eschmeier at Münster, Tensor Products and Elementary Operators. J. Reine Angew. Math. 390(1988), 49-58.

[11] A.T. Dash, Joint Browder Spectra and Tensor Products. Bull. Austral. Math. Soc. Vol. 32(1985), 119-126.

[12] Vladimir Kordula, Vladimir Müller and Vladimir Rakocevic, On the semi-Browder Spectrum. Studia Mathematica 123(I), (1997), I-8.

[13] Josep L.Taylor, A Joint Spectrum for several Commuting Operators, Journal of Functional Analysis, 6 (1970), 172-173.

[14] R.E. Harte, Invertibility and Singularity for Bounded Linear Operators, Dekker, New York, 1988, 493-525.

[15] L.A. Coburn and Martin Schechter, Joint Spectra and Interpolation of Operators, Journal of Functional Analysis 2, (1968), 226-237.

Department of Mathematics Education
Kangwon National University
Chuncheon 200-701, Korea
E-mail: dslee@cc.kangwon.ac.kr