Bull. Korean Math. Soc. 42 (2005), No. 1, pp. 189–201

SHAPE OPERATOR A_H FOR SLANT SUBMANIFOLDS IN GENERALIZED COMPLEX SPACE FORMS

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ABSTRACT. In this article, we establish relations between the sectional curvature function K and the shape operator, and also relationship between the k-Ricci curvature and the shape operator for slant submanifolds in generalized complex space forms with arbitrary codimension.

1. Introduction

Nash's imbedding Theorem enables us to regard any Riemannian manifold as a submanifold of Euclidean space and it gives us a natural motivation for the study of submanifolds of Riemannian manifolds. In this case, we have intrinsic invariants as well as extrinsic invariants. Among extrinsic invariants, the shape operator and the squared mean curvature are the most important ones. The sectional, Ricci and scalar curvature are the well-known intrinsic invariants. By setting up some relations between the intrinsic invariants and the extrinsic invariants, some nice results have been achieved like Gauss-Bonnet Theorem, Isoperimetric inequality and Chern-Lashof's Theorem.

In [3] and [4], B. Y. Chen established an inequality relating intrinsic quantities and extrinsic ones for submanifolds in a real space form with arbitrary codimension. In particular, in [3] he investigated a relation between the sectional curvature and the shape operator for submanifolds in Riemannian space forms and, in [4] he established a sharp relation between the k-Ricci curvature and the shape operator. On the other hand, for the above mentioned contents K. Matsumoto, I. Mihai and

Received October 31, 2003.
2000 Mathematics Subject Classification: 53B25, 53C55.
Key words and phrases: shape operator, scalar curvature, squared mean curvature, k-Ricci curvature, generalized complex space form, complex space form, real space form, slant submanifold, invariant and anti-invariant submanifold.
* This work was supported by Kyungpook National University Research Team Fund, 2003.

A. Oiaga ([7]) studied these relations of slant submanifolds in complex space forms.

On the other hand, the notion of constant type for a nearly Kaehler manifold was introduced in [5] and that of RK-manifolds $\tilde{M}(c,\alpha)$ of constant holomorphic sectional curvature c and constant type α and a generalized complex space form $\tilde{M}(f_1,f_2)$ are given in [11] and [13]. For such manifolds, we have the inclusion relation $\tilde{M}(c) \subset \tilde{M}(c,\alpha) \subset \tilde{M}(f_1,f_2)$, where $\tilde{M}(c)$ is the complex space form of constant holomorphic sectional curvature c. Thus it is worthwhile studying relationships between the intrinsic and the extrinsic invariants of submanifold in a generalized complex space form and we study the general case which extends the previous work of [7].

In this paper, we study slant submanifolds of generalized complex space forms with arbitrary codimension and establish the relations between the sectional curvature and the shape operator, and also between the k-Ricci curvature and the shape operator for slant submanifolds in generalized complex space forms.

2. Preliminaries

Let $(\tilde{M},J,g)$ be an almost Hermitian manifold with an almost Hermitian structure J and an almost Hermitian metric g. $\tilde{M}$ is called a *nearly Kaehler manifold* ([5]) if $(\tilde{\nabla}_X J)X = 0$, and a *Kaehler manifold* if $\tilde{\nabla} J = 0$ for all $X \in T\tilde{M}$, where $\tilde{\nabla}$ is Levi-Civita connection of the Riemannian metric g. $\tilde{M}$ is said to be a *para-Kaehler manifold* ([9]) if it satisfies the Kaehler identity, that is,

$$\tilde{R}(X,Y,JZ,JW) = \tilde{R}(X,Y,Z,W), \;\; X,Y,Z,W \in T\tilde{M},$$

where $\tilde{R}$ is the Riemannian curvature tensor. An almost Hermitian manifold with J-invariant Riemannian curvature tensor $\tilde{R}$, that is,

$$\tilde{R}(JX,JY,JZ,JW) = \tilde{R}(X,Y,Z,W), \;\; X,Y,Z,W \in T\tilde{M},$$

is called an *RK-manifold* ([13]). All nearly Kaehler and para-Kaehler manifolds belong to the class of RK-manifolds. There are examples of flat para-Kaehler manifolds (and hence of RK-manifolds) which are not Kaehler ([6], [10], [12]). The notion of constant type was first introduced by A. Gray for nearly Kaehler ([5]). An almost Hermitian manifold $\tilde{M}$ is said to have *(pointwise) constant type* if for each $p \in \tilde{M}$ and for all

$X, Y, Z \in T_p\tilde{M}$ such that
$$g(X,Y) = g(X,Z) = g(X,JY) = g(X,JZ) = 0,\ g(Y,Y) = 1 = g(Z,Z),$$
we have
$$\begin{aligned}&\tilde{R}(X,Y,X,Y) - \tilde{R}(X,Y,JX,JY)\\ &= \tilde{R}(X,Z,X,Z) - \tilde{R}(X,Z,JX,JZ).\end{aligned}$$
It is known that if $\tilde{M}$ is an RK-manifold, then it has (pointwise) constant type if and only if there is a differentiable function α on $\tilde{M}$ satisfying ([13])
$$\begin{aligned}&\tilde{R}(X,Y,X,Y) - \tilde{R}(X,Y,JX,JY)\\ &= \alpha\{g(X,X)g(Y,Y) - g(X,Y)^2 - g(X,JY)^2\}\end{aligned}$$
for all $X, Y \in T\tilde{M}$. Furthermore, $\tilde{M}$ has *global constant type* if α is constant. The function α is called the constant type of $\tilde{M}$. An RK-manifold of constant holomorphic sectional curvature c and constant type α is denoted by $\tilde{M}(c,\alpha)$. For $\tilde{M}(c,\alpha)$ it is known that
$$\begin{aligned}&4\tilde{R}(X,Y)Z\\ &= (c+3\alpha)(g(Y,Z)X - g(X,Z)Y)\\ &\quad + (c-\alpha)(g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ)\end{aligned}$$
for all $X, Y, Z \in T\tilde{M}$ (See [13]). If $c = \alpha$, then $\tilde{M}(c,\alpha)$ is a space of constant curvature. A complex space form $\tilde{M}(c)$ belongs to the class of almost Hermitian manifolds $\tilde{M}(c,\alpha)$ with the constant type zero.

An almost Hermitian manifold $\tilde{M}$ is called a *generalized complex space form* $\tilde{M}(f_1, f_2)$([8, 11]) if its Riemannian curvature tensor $\tilde{R}$ satisfies
$$\tilde{R} = f_1\tilde{R}_1 + f_2\tilde{R}_2 \tag{2.1}$$
where f_1 and f_2 are smooth functions on $\tilde{M}$ and
$$\tilde{R}_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{2.2}$$
$$\tilde{R}_2(X,Y)Z = g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ \tag{2.3}$$
for all $X, Y, Z \in T\tilde{M}$. Therefore, we have the inclusion relation $\tilde{M}(c) \subset \tilde{M}(c,\alpha) \subset \tilde{M}(f_1, f_2)$.

Let M be an n-dimensional submanifold of a Reimannian manifold $\tilde{M}$ with a Riemannian metric g. On M, the induced metric is naturally defined and we will use the same notation as g. Let $\tilde{\nabla}$ be the Riemannian connection on $\tilde{M}$. Then, the Gauss and Weingarten formulae are given respectively by $\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$ and $\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$ for all $X, Y \in TM$ and $N \in T^{\perp}M$, where ∇ and $\nabla^{\perp}$ are respectively the induced Riemannian connection on M and the normal connection on the normal bundle $T^{\perp}M$ of M, and h is the second fundamental form related to A by $g(h(X,Y),N) = g(A_N X, Y)$. Let $\tilde{R}$ and R be the curvature tensors of $\tilde{M}$ and M respectively. Then the equation of Gauss is given by

(2.4)
$$\begin{aligned} &g(\tilde{R}(X,Y)Z,W) \\ &= g(R(X,Y)Z,W) - g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W)) \end{aligned}$$

for all $X, Y, Z, W \in TM$.

Let H be the mean curvature vector, that is,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.5}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space T_pM at $p \in M$. The submanifold M is called totally geodesic in $\tilde{M}$ if $h = 0$, minimal if $H = 0$, and totally umbilical if $h(X,Y) = g(X,Y)H$. An invariant submanifold of a nearly Kaehler or a Kaehler manifold is always minimal.

We denote by

$$h_{ij}^r = g(h(e_i, e_j), e_r) \tag{2.6}$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g^2(h(e_i, e_j), h(e_i, e_j)). \tag{2.7}$$

For $p \in M$ and for any $X \in T_pM$, we put $JX = PX + FX$, where $PX \in T_pM, FX \in T_p^{\perp}M$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM. We put

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j). \tag{2.8}$$

Let L be a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of L such that $e_1 = X$. The $k-$Ricci curvature $Ric_L(X)$ of L at X is defined by

$$Ric_L(X) = K_{12} + K_{13} + \ldots + K_{1k}, \tag{2.9}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i and e_j $(i \neq j)$.

On the other hand, the scalar curvature τ of the $k-$plane section L is given by

$$\tau(L) = \sum_{1\le i<j\le k} K_{ij}. \tag{2.10}$$

For each integer k, $2 \le k \le n$, the Riemannian invariant Θ_k on an n-dimensional Riemannian manifold M is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \ p \in M \tag{2.11}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L. For a submanifold M in a Riemannian manifold, the relative null space of M at a point $p \in M$ is defined by

$$N_p = \{X \in T_pM \mid h(X,Y) = 0 \text{ for all } Y \in T_pM\}. \tag{2.12}$$

3. Sectional curvature and shape operator

B. Y. Chen established a relationship between the sectional curvature function K and the shape operator for submanifolds in real space form ([3]). We prove a similar inequality for a slant submanifold M into a m-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$.

On a submanifold M of almost Hermitian manifold, for a vector $0 \ne X_p \in T_pM$, the angle $\theta(X_p)$ between JX_p and the tangent space T_pM is called the wirtinger angle of X_p. If the wirtinger angle is independent of $p \in M$ and $X_p \in T_pM$, then M is called a *slant submanifold* ([2]). Invariant and anti-invariant submanifolds are slant submanifolds with $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively. Slant submanifolds of almost Hermitian manifolds are characterized by the condition $P^2 + \lambda^2 I = 0$ for a real number $\lambda \in [0, 1]$.

THEOREM 3.1. *Let $x : M \longrightarrow \tilde{M}(f_1, f_2)$ be an isometric immersion of an n-dimensional θ-slant submanifold M into an m-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. If there exists a point $p \in M$ and a number $b > f_1 + 3f_2 \cos^2\theta$ such that $K \ge b$ at p, then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}\{b - f_1 - \frac{3f_2}{n-1}\cos^2\theta\} I_n \quad at \quad p \tag{3.1}$$

where I_n denotes the identity map.

Proof. Suppose that M is a slant submanifold in $\tilde{M}(f_1, f_2)$. Let $p \in M$ and a number $b > f_1 + 3f_2 \cos^2\theta$ such that $K \geq b$ at p. Choose an orthonormal basis $\{e_1, \ldots, e_{n+1}, \ldots, e_{2m}\}$ at p such that e_{n+1} is parallel to the mean curvature vector H and $e_1, \ldots, e_n$ diagonalize the shape operator A_{n+1}. Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & a_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix}, \tag{3.2}$$

$$A_r = (h^r_{ij}), \qquad \sum_{i=1}^{n}(h^r_{ii}) = 0, \quad 1 \leq i, j \leq n, \quad n+2 \leq r \leq 2m.$$

For $i \neq j$, we denote by

$$u_{ij} = a_i a_j. \tag{3.3}$$

From the Gauss equation with $X = Z = e_i$, $Y = W = e_j$, we get

(3.4)

$$u_{ij} \geq b - f_1 - 3f_2 g^2(Pe_i, e_j) - \sum_{r=n+2}^{2m} \{h^r_{ii}h^r_{jj} - (h^r_{ij})^2\}, \quad 1 \leq i, j \leq n.$$

We need the following lemmas in order to complete the proof of the theorem.

LEMMA 3.2. *The following statements hold.*

(1) *For any fixed $i \in \{1, \ldots, n\}$, we have $\sum_{j \neq i} u_{ij} \geq (n-1)\{b - f_1 - \frac{3f_2}{n-1}\cos^2\theta\}$.*

(2) *$u_{ij} \neq 0$ for $i \neq j \in \{1, \ldots, n\}$.*

(3) *For distinct $i, j, k \in \{1, 2, \ldots, n\}$, we have $a_i^2 = u_{ij}u_{ik}u_{jk}^{-1}$.*

Proof. Together with (3.2), (3.3) and (3.4), we get

$$\begin{aligned} \sum_{j \neq i} u_{ij} &\geq (n-1)\{b - f_1 - \frac{3f_2}{n-1}\cos^2\theta\} - \sum_{r=n+2}^{2m} \{h^r_{ii}\sum_{j \neq i} h^r_{jj} - \sum_{i \neq j}(h^r_{ij})^2\} \\ &= (n-1)\{b - f_1 - \frac{3f_2}{n-1}\cos^2\theta\} + \sum_{r=n+2}^{2m}\sum_{j=1}^{n}(h^r_{ij})^2, \end{aligned}$$

which yields statement (1).

For statement (2), if $u_{ij}=0$ for $i\neq j$, then $a_i=0$ or $a_j=0$. $a_i=0$ implies that $u_{it}=0$ for any $i\neq t$. Hence, $\sum_{i\neq t}u_{it}=0$ which contradicts the statement (1).

(3) follows from $u_{ij}u_{ik}=a_i^2a_ja_k=a_i^2u_{jk}$. □

We put $S_k=\{B\subset\{1,\ldots,n\}:|B|=k\}$. For any $B\in S_k$ we denote by $\bar{B}=\{1,\ldots,n\}\backslash B$.

LEMMA 3.3. *For a fixed k, $1\leq k\leq[\frac{n}{2}]$, and each $B\in S_k$, we have*

$$\sum_{j\in B}\sum_{t\in\bar{B}}u_{jt}\geq(n-k)k\{b-f_1-\frac{3f_2}{n-k}\cos^2\theta\}.$$

Proof. Without loss of generality, we may assume $B=\{1,\ldots,k\}$. From (3.3) together with the last equation of (3.2) we find

$$\begin{aligned}\sum_{j\in B}\sum_{t\in\bar{B}}u_{jt}&\geq(n-k)k\{b-f_1-\frac{3f_2}{n-k}\cos^2\theta\}\\&-\sum_{r=n+2}^{2m}\sum_{j=1}^{k}\sum_{t=k+1}^{n}\{h_{jj}^rh_{tt}^r-(h_{jt}^r)^2\}\\&=(n-k)k\{b-f_1-\frac{3f_2}{n-k}\cos^2\theta\}\\&+\sum_{r=n+2}^{2m}\{\sum_{j=1}^{k}\sum_{t=k+1}^{n}(h_{jt}^r)^2+\sum_{j=1}^{k}(h_{jj}^r)^2\},\end{aligned}$$

which implies the lemma. □

LEMMA 3.4. *For any $1\leq i\neq j\leq n$, we have $u_{ij}>0$.*

Proof. Assume $u_{1n}<0$. Then, by statement (3) of Lemma 3.2, we get $u_{1i}u_{in}<0$ for $1<i<n$. Without loss of generality, we may assume

$$\begin{cases}u_{12},\ldots,u_{1l},u_{(l+1)n},\ldots,u_{(n-1)n}>0,\\u_{1(l+1)},\ldots,u_{1n},u_{2n},\ldots,u_{ln}<0\end{cases}\tag{3.5}$$

for some $[\frac{n+1}{2}]\leq l\leq n-1$.

If $l=n-1$, then $u_{1(n)}+u_{2(n)}+\cdots+u_{(n-1)n}<0$ which contradicts the statement (1) of Lemma 3.2. Thus, $l<n-1$. From statement (3) of Lemma 3.2, we get

$$a_n^2=\frac{u_{in}u_{tn}}{u_{it}}>0,\tag{3.6}$$

where $2 \leq i \leq l$ and $l+1 \leq t \leq n-1$. By (3.4) and (3.5), we have $u_{it} < 0$ which implies

$$\sum_{i=1}^{l}\sum_{t=l+1}^{n} u_{it} = \sum_{i=2}^{l}\sum_{t=l+1}^{n-1} u_{it} + \sum_{i=2}^{l} u_{in} + \sum_{t=l+1}^{n} u_{1t} < 0.$$

This contradicts Lemma 3.3. □

Now, we return to the proof of Theorem 3.1. From Lemma 3.4, it follows that $a_1, \ldots, a_n$ are of the same sign. Assume $a_j > 0$ for all $j \in \{1, \ldots, n\}$. Then from the statement (1) of Lemma 3.2, we get

$$\begin{aligned} a_i n\|H\| - a_i^2 &= a_i(a_1 + \cdots + a_n) - a_i^2 \\ &= a_i \sum_{i \neq j} a_j = \sum_{i \neq j} a_i a_j = \sum_{i \neq j} u_{ij} \\ &\geq (n-1)\{b - f_1 - \frac{3f_2}{n-1}\cos^2\theta\}. \end{aligned}$$

This inequality implies that

$$a_i\|H\| > \frac{n-1}{n}\{b - f_1 - \frac{3f_2}{n-1}\cos^2\theta\},$$

and consequently (3.1) is established. This completes the proof of the theorem. □

COROLLARY 3.5. *We have the following table:*

Manifold	**Submanifold**	A_H **(Shape operator)**
$\tilde{M}(f_1, f_2)$	totally real	$A_H > \frac{n-1}{n}\{b - f_1\}I_n$
$\tilde{M}(f_1, f_2)$	invariant	$A_H > \frac{n-1}{n}\{b - f_1 - \frac{3f_2}{n-1}\}I_n$
$\tilde{M}(c, \alpha)$	θ-slant	$A_H > \frac{n-1}{n}\{b - \frac{(c+3\alpha)}{4} - \frac{3(c-\alpha)}{4(n-1)}\cos^2\theta\}I_n$
$\tilde{M}(c, \alpha)$	totally real	$A_H > \frac{n-1}{n}\{b - \frac{(c+3\alpha)}{4}\}I_n$
$\tilde{M}(c, \alpha)$	invariant	$A_H > \frac{n-1}{n}\{b - \frac{(c+3\alpha)}{4} - \frac{3(c-\alpha)}{4(n-1)}\}I_n$
$\tilde{M}(c)$	θ-slant	$A_H > \frac{n-1}{n}\{b - \frac{c}{4} - \frac{3c}{4(n-1)}\cos^2\theta\}I_n$
$\tilde{M}(c)$	totally real	$A_H > \frac{n-1}{n}\{b - \frac{c}{4}\}I_n$
$\tilde{M}(c)$	invariant	$A_H > \frac{n-1}{n}\{b - \frac{c}{4} - \frac{3c}{4(n-1)}\}I_n$
$R(c)$		$A_H > \frac{n-1}{n}\{b - c\}I_n$

4. k-Ricci curvature and shape operator

In this section, we establish a relation between the k-Ricci curvature and the shape operator for an n-dimensional slant submanifold M into an m-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$.

THEOREM 4.1. *Let $x : M \longrightarrow \tilde{M}(f_1, f_2)$ be an isometric immersion of an n-dimensional θ-slant submanifold M into an m-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then, for any integer k, $2 \leq k \leq n$, and any point $p \in M$, we have*

(i) *If $\Theta_k(p) \neq (f_1 + \frac{3f_2}{n-1}\cos^2\theta)$, then shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}\{\Theta_k(p) - f_1 - \frac{3f_2}{n-1}\cos^2\theta\}I_n \text{ at } p, \tag{4.1}$$

where I_n denotes the identity map of T_pM.

(ii) *If $\Theta_k(p) = f_1 + \frac{3f_2}{n-1}\cos^2\theta$, then $A_H \geq 0$ at p.*

(iii) *A unit vector $X \in T_pM$ satisfies*

$$A_H X = \frac{n-1}{n}\{\Theta_k(p) - f_1 - \frac{3f_2}{n-1}\cos^2\theta\}X \tag{4.2}$$

if and only if $\Theta_k(p) = f_1 + \frac{3f_2}{n-1}\cos^2\theta$ and $X \in N_p$.

(iv)

$$A_H = \frac{n-1}{n}\{\Theta_k(p) - f_1 - \frac{3f_2}{n-1}\cos^2\theta\}I_n \text{ at } p$$

if and only if p is a totally geodesic point.

Proof. *Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM. Denote by $L_{i_1\ldots i_k}$ the k-plane section spanned by $e_{i_1}, \ldots, e_{i_k}$. It follows from (2.9) and (2.10) that*

$$\tau(L_{i_1\ldots i_k}) = \frac{1}{2}\sum_{i\in\{i_1,\ldots,i_k\}} Ric_{L_{i_1\ldots i_k}}(e_i), \tag{4.3}$$

$$\tau(p) = \frac{1}{\binom{n-2}{k-2}}\sum_{1\leq i_1<\cdots<i_k\leq n} \tau(L_{i_1\ldots i_k}). \tag{4.4}$$

Combining (2.11), (4.3) and (4.4), we find

$$\tau(p) \geq \frac{n(n-1)}{2}\Theta_k(p). \tag{4.5}$$

From the equation of Gauss (2.4) with $X = Z = e_i, Y = W = e_j$, by summing over $\{1, 2, \cdots, n\}$ with respect to i and j $(i \neq j)$, we obtain

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - n(n-1)f_1 - 3f_2 n\cos^2\theta. \tag{4.6}$$

We choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\}$ at p in such a way that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, \ldots, e_n$ diagonalize the shape operator A_{n+1}. Then we have the relations (3.2) and (3.3). From (4.6) we get

$$n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 - n(n-1)f_1 - 3f_2 n\cos^2\theta. \tag{4.7}$$

On the other hand, since

$$0 \leq \sum_{i<j}(a_i - a_j)^2 = (n-1)\sum_{i=1}^{n} a_i^2 - 2\sum_{i<j} a_i a_j,$$

which implies

$$n^2\|H\|^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \leq n\sum_{i=1}^{n} a_i^2. \tag{4.8}$$

We have from (4.7) and (4.8)

$$n^2\|H\|^2 \geq 2\tau + n\|H\|^2 - n(n-1)f_1 - 3f_2 n\cos^2\theta, \tag{4.9}$$

or, equivalently

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - f_1 - \frac{3f_2}{n-1}\cos^2\theta. \tag{4.10}$$

From (4.5) and (4.10), we have

$$\begin{aligned} \|H\|^2(p) &\geq \Theta_k(p) - f_1 - \frac{3f_2}{n-1}\cos^2\theta \\ &= \Theta_k(p) - f_1 - \frac{3f_2}{n-1}\cos^2\theta. \end{aligned} \tag{4.11}$$

This show that $H(p) = 0$ may occur only when $\Theta_k(p) \leq f_1 + \frac{3f_2}{n-1}\cos^2\theta$. Consequently, if $H(p) = 0$, statements (i) *and* (ii) *hold automatically. Therefore, without loss of generality, we may assume $H(p) \neq 0$. From the Gauss' equation we get*

$$a_i a_j = K_{ij} - f_1 - 3f_2 g^2(Pe_i, e_j) - \sum_{r=n+2}^{2m}\{h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\}. \tag{4.12}$$

By (4.12) we have

(4.13)

$$a_1(a_{i_2}+\cdots+a_{i_k}) = Ric_{L_{1i_2\dots i_k}}(e_1)-(k-1)f_1-3f_2\sum_{j=2}^{k}g^2(e_1,Pe_j)$$
$$+\sum_{r=n+2}^{2m}\sum_{j=2}^{k}\{(h^r_{1i_j})^2-h^r_{11}h^r_{i_ji_j}\},$$

which yields

(4.14)

$$a_1(a_2+\cdots+a_n)=\frac{1}{\binom{n-2}{k-2}}\sum_{2\le i_2<\cdots<i_k\le n}Ric_{L_{1i_2\dots i_k}}(e_1)$$
$$-(n-1)f_1-3f_2\sum_{j=2}^{n}g^2(e_1,Pe_j)+\sum_{r=n+2}^{2m}\sum_{j=1}^{n}(h^r_{1j})^2.$$

From (8), (11) and (4.14) we have

$$a_1(a_2+\cdots+a_n)\ge(n-1)\{\Theta_k(p)-f_1-\frac{3f_2}{n-1}\cos^2\theta\}. \tag{4.15}$$

Then

$$\begin{aligned}a_1(a_1+\cdots+a_n)&=a_1^2+a_1(a_2+\cdots+a_n)\\&\ge a_1^2+(n-1)\{\Theta_k(p)-f_1-\frac{3f_2}{n-1}\cos^2\theta\}\\&\ge(n-1)\{\Theta_k(p)-f_1-\frac{3f_2}{n-1}\cos^2\theta\}.\end{aligned} \tag{4.16}$$

Similar inequalities hold when the index 1 *is replaced by* $j\in\{2,\dots,n\}$. *Hence, we have*

$$a_j(a_1+\cdots+a_n)\ge(n-1)\{\Theta_k(p)-f_1-\frac{3f_2}{n-1}\cos^2\theta\},\quad j\in\{1,\dots,n\},$$

which yields

$$A_H\ge\frac{n-1}{n}\{\Theta_k(p)-f_1-\frac{3f_2}{n-1}\cos^2\theta\}I_n.$$

The equality does not hold because $H(p)\neq 0$. *Thus, (4.1) is valid. The statement* (ii) *is obvious.*

(iii) *Let* X *be a unit vector in* T_pM *satisfying (4.2). By (4.16) and (4.14), one has* $a_1=0$ *and* $h^r_{1j}=0$, *for all* $j\in\{1,\dots,n\}, r\in\{n+2,\dots,2m\}$, *respectively. It follows that* $\Theta_k(p)=f_1+\frac{3f_2}{n-1}\cos^2\theta$ *and*

$X \in N_p$. *The converse is clear.*

(iv) *The equality (4.2) holds for any $X \in T_pM$ if and only if $N_p = T_pM$, i.e., p is a totally geodesic point. This completes the proof of the theorem.* □

COROLLARY 4.2. *Let $x : M \longrightarrow \tilde{M}(f_1, f_2)$ be an isometric immersion of an n-dimensional invariant submanifold M into an m-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then, for any integer k, $2 \le k \le n$, and any point $p \in M$, we have*

(i) If $\Theta_k(p) \neq (f_1 + \frac{3f_2}{n-1})$, then shape operator at the mean curvature vector satisfies

$$A_H > \frac{n-1}{n}\{\Theta_k(p) - f_1 - \frac{3f_2}{n-1}\}I_n \quad \text{at} \quad p, \tag{4.1}$$

where I_n denotes the identity map of T_pM.

(ii) If $\Theta_k(p) = f_1 + \frac{3f_2}{n-1}$, then $A_H \ge 0$ at p.

(iii) A unit vector $X \in T_pM$ satisfies

$$A_H X = \frac{n-1}{n}\{\Theta_k(p) - f_1 - \frac{3f_2}{n-1}\}X \tag{4.2}$$

if and only if $\Theta_k(p) = f_1 + \frac{3f_2}{n-1}$ and $X \in N_p$.

(iv)

$$A_H = \frac{n-1}{n}\{\Theta_k(p) - f_1 - \frac{3f_2}{n-1}\}I_n \quad \text{at} \quad p$$

if and only if p is a totally geodesic point.

COROLLARY 4.3. *Let $x : M \longrightarrow \tilde{M}(f_1, f_2)$ be an isometric immersion of an n-dimensional anti-invariant submanifold M into an m-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then, for any integer k, $2 \le k \le n$, and any point $p \in M$, we have*

(i) If $\Theta_k(p) \neq f_1$, then shape operator at the mean curvature vector satisfies

$$A_H > \frac{n-1}{n}\{\Theta_k(p) - f_1\}I_n \quad \text{at} \quad p, \tag{4.1}$$

where I_n denotes the identity map of T_pM.

(ii) If $\Theta_k(p) = f_1$, then $A_H \ge 0$ at p.

(iii) A unit vector $X \in T_pM$ satisfies

$$A_H X = \frac{n-1}{n}\{\Theta_k(p) - f_1\}X \tag{4.2}$$

if and only if $\Theta_k(p) = f_1$ and $X \in N_p$.

(iv)

$$A_H = \frac{n-1}{n}\{\Theta_k(p) - f_1\}I_n \text{ at } p$$

if and only if p is a totally geodesic point.

REMARK 4.4. B. -Y. Chen established a sharp relationship between the Ricci curvature and the shape operator for submanifolds in a real space form ([4]). A similar inequality for an n-dimensional slant submanifold M of a $2m$-dimensional complex space form $\tilde{M}(c)$ is proved in [7]. Theorem 4.1 is a natural generalization to the above two kinds of results.

References

[1] B.-Y. Chen, *Geometry of Slant submanifolds*, Katholieke Univ. Leuven, Belgium.
[2] ———, *Geometry of submanifolds*, Marcel Dekker, New York, 1973.
[3] ———, *Mean curvature and shape operator of isometric immersions in real space form*, Glasg. Math. J. **38** (1996), 87–97.
[4] ———, *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*, Glasg. Math. J. **41** (1999), 33–41.
[5] A. Gray, *Nearly Kaehler manifolds*, J. Differential Geom. **4** (1970), 283–309.
[6] J. J. Konderak, *An example of an almost Hermitian flat manifold which is not Hermitian*, Riv. Mat. Univ. Parma (4), **17** (1991), 3159–318.
[7] K. Matsumoto, I. Mihai, and A. Oiaga, *Shape operator for slant submanifold in complex space forms*, Bull. Yamagata Univ. Natur. Sci. **14** (2000), 169–177.
[8] Z. Olszak, *On the existance of generalized complex space forms*, Israel J. Math. **65** (1989), 214–218.
[9] G.-B. Rizza, *Varieta parakahleriane*, Ann. Mat. Pura Appl. **98** (1974), 47–61.
[10] F. Triccrri, L. Vanhecke, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc. **267** (1981), 365–398.
[11] ———, *Flat almost Hermitian manifolds which are not Kahler manifolds*, Tensor (N.S.) **13** (1977), 149–154.
[12] M. M. Tripathi, *Some remarks on almost Hermitian Manifolds*, Riv. Mat. Univ. Parma **3** (1994), no. 5, 229–230.
[13] L. Vanhecke, *Almost Hermitian manifolds with J-invariant Riemann curvature tensor*, Rend. Mat. **34** (1975/76), 487–498.

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