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SPACES OF CONJUGATION-EQUIVARIANT FULL HOLOMORPHIC MAPS

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Abstract. Let $\mathrm{RRat}_k(\mathbb{C}P^n)$ denote the space of basepoint-preserving conjugation-equivariant holomorphic maps of degree k from S^2 to $\mathbb{C}P^n$. A map $f : S^2 \to \mathbb{C}P^n$ is said to be full if its image does not lie in any proper projective subspace of $\mathbb{C}P^n$. Let $\mathrm{RF}_k(\mathbb{C}P^n)$ denote the subspace of $\mathrm{RRat}_k(\mathbb{C}P^n)$ consisting of full maps. In this paper we determine $H_*(\mathrm{RF}_k(\mathbb{C}P^2); \mathbb{Z}/p)$ for all primes p.

1. Introduction

Let $\mathrm{Rat}_k(\mathbb{C}P^n)$ denote the space of based holomorphic maps of degree k from the Riemannian sphere $S^2 = \mathbb{C} \cup \infty$ to the complex projective space $\mathbb{C}P^n$. Since $PSL(n+1, \mathbb{C})$ acts on $\mathbb{C}P^n$ transitively, we can choose the basepoint condition as $f(\infty) = [1, 0, \ldots, 0]$. Such holomorphic maps are given by rational functions:

(1.1)
$$\mathrm{Rat}_k(\mathbb{C}P^n) = \{(p_0(z), \ldots, p_n(z)) : \text{the following (i) and (ii) hold}\}.$$

(i) Each $p_i(z)$ $(0 \le i \le n)$ has the form

$$p_0(z) = z^k + a_{0,1}z^{k-1} + \cdots + a_{0,k} \quad \text{and}$$
$$p_i(z) = a_{i,1}z^{k-1} + \cdots + a_{i,k} \ (1 \le i \le n), \text{ where } a_{i,j} \in \mathbb{C}.$$

(ii) There are no roots common to all $p_i(z)$ for $0 \le i \le n$.

A map $f : S^2 \to \mathbb{C}P^n$ is said to be full if its image does not lie in any proper projective subspace of $\mathbb{C}P^n$. If f is given by a rational function in (1.1), then f is full if and only if the polynomials $p_i(z)$ $(0 \le i \le n)$ are linearly independent in $\mathbb{C}[z]$. Let $\mathrm{F}_k(\mathbb{C}P^n)$ be the subspace of $\mathrm{Rat}_k(\mathbb{C}P^n)$ consisting of full maps. (The motivation for studying $\mathrm{F}_k(\mathbb{C}P^n)$ is explained in [3].) We have the following sequence of inclusions:

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$$\mathrm{F}_k(\mathbb{C}P^n) \hookrightarrow \mathrm{Rat}_k(\mathbb{C}P^n) \hookrightarrow \Omega^2_k\mathbb{C}P^n \simeq \Omega^2 S^{2n+1}.$$

We already know the following results about these inclusions:

(1) About $\mathrm{Rat}_k(\mathbb{C}P^n)$ we have

(i) It is proved in [7] that the inclusion $\mathrm{Rat}_k(\mathbb{C}P^n) \hookrightarrow \Omega^2_k\mathbb{C}P^n \simeq \Omega^2 S^{2n+1}$ is a homotopy equivalence up to dimension $k(2n-1)$. (Throughout this paper, to say that a map $f : X \to Y$ is a homotopy equivalence up to dimension d is intended to mean that f induces isomorphisms in homotopy groups in dimensions less than d, and an epimorphism in dimension d.)

(ii) The stable homotopy type of $\mathrm{Rat}_k(\mathbb{C}P^n)$ was described in [2] as follows. Let $\Omega^2 S^{2n+1} \underset{s}{\simeq} \bigvee_{1\leq q} D_q(S^{2n-1})$ be Snaith's stable splitting of $\Omega^2 S^{2n+1}$. Then

$$\mathrm{Rat}_k(\mathbb{C}P^n) \underset{s}{\simeq} \bigvee_{q=1}^{k} D_q(S^{2n-1}). \tag{1.2}$$

(2) About $\mathrm{F}_k(\mathbb{C}P^n)$ we have from [3] that

(i) Particular examples are: $\mathrm{F}_k(\mathbb{C}P^1) = \mathrm{Rat}_k(\mathbb{C}P^1)$ for $1 \leq k$; $\mathrm{F}_k(\mathbb{C}P^n) = \emptyset$ for $k < n$; and $\mathrm{F}_n(\mathbb{C}P^n) \cong \mathbb{C}^n \times GL(n,\mathbb{C})$.

(ii) The inclusion $\mathrm{F}_k(\mathbb{C}P^n) \hookrightarrow \mathrm{Rat}_k(\mathbb{C}P^n)$ is a homotopy equivalence up to dimension $2(k-n)+1$.

(iii) $H_*(\mathrm{F}_k(\mathbb{C}P^2);\mathbb{Z}/p)$ was determined for all primes p.

In [4] and [5] a conjugation-equivariant version of these results was studied. Let $\mathrm{Map}^T_k(\mathbb{C}P^1,\mathbb{C}P^n)$ denote the space of continuous basepoint-preserving conjugation-equivariant maps of degree k from $\mathbb{C}P^1$ to $\mathbb{C}P^n$. We set $\mathrm{RRat}_k(\mathbb{C}P^n) = \mathrm{Map}^T_k(\mathbb{C}P^1,\mathbb{C}P^n) \cap \mathrm{Rat}_k(\mathbb{C}P^n)$. An element $(p_0(z),\ldots,p_n(z)) \in \mathrm{Rat}_k(\mathbb{C}P^n)$ belongs to $\mathrm{RRat}_k(\mathbb{C}P^n)$ if and only if each $p_i(z)$ has real coefficients. Finally we set $\mathrm{RF}_k(\mathbb{C}P^n) = \mathrm{RRat}_k(\mathbb{C}P^n) \cap \mathrm{F}_k(\mathbb{C}P^n)$. Then we have the following sequence of inclusions:

$$\mathrm{RF}_k(\mathbb{C}P^n) \hookrightarrow \mathrm{RRat}_k(\mathbb{C}P^n) \hookrightarrow \mathrm{Map}^T_k(\mathbb{C}P^1,\mathbb{C}P^n) \simeq \Omega S^n \times \Omega^2 S^{2n+1}.$$

Similarly to the above (1) and (2), we know the following results:

(3) About $\mathrm{RRat}_k(\mathbb{C}P^n)$ we have

(i) The inclusion $\mathrm{RRat}_k(\mathbb{C}P^n) \hookrightarrow \mathrm{Map}^T_k(\mathbb{C}P^1,\mathbb{C}P^n) \simeq \Omega S^n \times \Omega^2 S^{2n+1}$ is a homotopy equivalence up to dimension $(k+1)(n-1)-1$.

(ii) When $n = 1$, Brockett ([1]) and Segal ([7]) showed that

$$\mathrm{RRat}_k(\mathbb{C}P^1) \simeq \coprod_{i=0}^{k} \mathrm{Rat}_{\min(i,k-i)}(\mathbb{C}P^1). \tag{1.3}$$

(iii) For $n \geq 2$, the following structure of $H_*(\mathrm{RRat}_k(\mathbb{C}P^n);\mathbb{Z}/p)$ was proved in [4]: We define the weight of an element of $H_*(\Omega S^n;\mathbb{Z}/p)$ as usual, but that of $H_*(\Omega^2 S^{2n+1};\mathbb{Z}/p)$ we define as being twice the usual one. Then, as a vector space, $H_*(\mathrm{RRat}_k(\mathbb{C}P^n);\mathbb{Z}/p)$ is isomorphic to the subspace of $H_*(\Omega S^n \times \Omega^2 S^{2n+1};\mathbb{Z}/p)$ spanned by monomials of weight $\leq k$.

(4) About $\mathrm{RF}_k(\mathbb{C}P^n)$ we have from [5] that

(i) Particular examples are: $\mathrm{RF}_k(\mathbb{C}P^1) = \mathrm{RRat}_k(\mathbb{C}P^1)$ for $1 \leq k$; $\mathrm{RF}_k(\mathbb{C}P^n) = \emptyset$ for $k < n$; and $\mathrm{RF}_n(\mathbb{C}P^n) \cong \mathbb{R}^n \times GL(n,\mathbb{R})$.

(ii) The inclusion $\mathrm{RF}_k(\mathbb{C}P^n) \hookrightarrow \mathrm{RRat}_k(\mathbb{C}P^n)$ is a homotopy equivalence up to dimension $k-n$.

(2)(ii) shows that the most interesting part of the homology of $\mathrm{F}_k(\mathbb{C}P^n)$ is the classes in dimensions $\geq 2(k-n)+1$. Then, in (2)(iii), the homology $H_*(\mathrm{F}_k(\mathbb{C}P^2);\mathbb{Z}/p)$ was determined completely. The result shows that the inclusion $\mathrm{F}_k(\mathbb{C}P^2) \hookrightarrow \mathrm{Rat}_k(\mathbb{C}P^2)$ has a nontrivial kernel in homology in dimensions $\geq 2(k-n)+1$. Similarly, it is interesting to study the homology of $\mathrm{RF}_k(\mathbb{C}P^n)$ in dimensions $\geq k-n$. In connection with this, the following result was proved in [5]:

(5) Let $SO(k)/SO(k-n)$ be the Stiefel manifold of orthonormal n-frames in $\mathbb{R}^k$. (When $k=n$, we understand this as $O(n)$.) Then there is a map $\alpha_{k,n} : \mathrm{RF}_k(\mathbb{C}P^n) \to SO(k)/SO(k-n)$ so that $\alpha_{k,n}$ is a homotopy equivalence up to dimension $n-1$.

But the result corresponding to (2)(iii) is left unknown. Hence, the purpose of this paper is to determine $H_*(\mathrm{RF}_k(\mathbb{C}P^2);\mathbb{Z}/p)$ completely.

In order to state our main result, we define the notation

$$\frac{S^1 \times \mathrm{Rat}_l(\mathbb{C}P^1)}{S^1 \wedge D_l(S^1)} \tag{1.4}$$

as follows: From (1.2), $D_l(S^1)$ is a stable summand in $\mathrm{Rat}_l(\mathbb{C}P^1)$. Since $\Sigma(K \times L) \simeq \Sigma(K \wedge L) \vee \Sigma K \vee \Sigma L$, $S^1 \wedge D_l(S^1)$ is a stable summand in $S^1 \times \mathrm{Rat}_l(\mathbb{C}P^1)$. Then (1.4) is defined to be the identification space obtained from $S^1 \times \mathrm{Rat}_l(\mathbb{C}P^1)$ by collapsing $S^1 \wedge D_l(S^1)$ to a point.

Then our main result is as follows.

THEOREM A . *Let p be a prime. Then, we have the following isomorphisms of vector spaces:*

(i) *When $k = 2l$.*

$$H_*(\mathrm{RF}_{2l}(\mathbb{C}P^2);\mathbb{Z}/p) \cong H_*(\mathrm{RRat}_{2l-2}(\mathbb{C}P^2);\mathbb{Z}/p) \oplus \bigoplus_{i=0}^{l-2} H_{*-(2l-2)}(S^1 \times \mathrm{Rat}_i(\mathbb{C}P^1);\mathbb{Z}/p) \oplus H_{*-(2l-2)}\left(\frac{S^1 \times \mathrm{Rat}_l(\mathbb{C}P^1)}{S^1 \wedge D_l(S^1)};\mathbb{Z}/p\right).$$

(ii) *When $k = 2l + 1$.*

$$H_*(\mathrm{RF}_{2l+1}(\mathbb{C}P^2);\mathbb{Z}/p) \cong H_*(\mathrm{RRat}_{2l-1}(\mathbb{C}P^2);\mathbb{Z}/p) \oplus \bigoplus_{i=0}^{l-1} H_{*-(2l-1)}(S^1 \times \mathrm{Rat}_i(\mathbb{C}P^1);\mathbb{Z}/p).$$

2. Proof of theorem A

In what follows, we consider $\mathrm{RF}_k(\mathbb{C}P^n)$ only for $n = 2$ and set

$$X_k = \mathrm{RRat}_k(\mathbb{C}P^2) - \mathrm{RF}_k(\mathbb{C}P^2). \tag{2.1}$$

LEMMA 2.1. *There is a homeomorphism*

$$X_k \cong S^1 \times_{\mathbb{Z}/2} \mathrm{RRat}_k(\mathbb{C}P^1).$$

Here $\mathbb{Z}/2$ acts on S^1 by antipodal and acts on $\mathrm{RRat}_k(\mathbb{C}P^1)$ by

$$(-1) \cdot (v_0(z), v_1(z)) = (v_0(z), -v_1(z)),$$

where $(v_0(z), v_1(z)) \in \mathrm{RRat}_k(\mathbb{C}P^1)$ is given in the form (1.1) with $a_{i,j} \in \mathbb{R}$.

Proof. We write an element $(u_0(z), u_1(z), u_2(z)) \in X_k$ in the form (1.1) with $a_{i,j} \in \mathbb{R}$. Since $u_1(z)$ and $u_2(z)$ are linearly dependent but not both are zero, there exist $\xi \in (\mathbb{R}^2)^*$ and a real polynomial $\phi(z)$ so that $(u_1(z), u_2(z)) = \xi\phi(z)$. Let $\mathbb{R}^*$ act on $\mathrm{RRat}_k(\mathbb{C}P^1)$ by

$$r \cdot (v_0(z), v_1(z)) = (v_0(z), \frac{1}{r}v_1(z)),$$

where $r \in \mathbb{R}^*$. Then a homeomorphism $X_k \xrightarrow{\cong} (\mathbb{R}^2)^* \times_{\mathbb{R}^*} \mathrm{RRat}_k(\mathbb{C}P^1)$ is given by $(u_0(z), u_1(z), u_2(z)) \mapsto (\xi, (u_0(z), \phi(z)))$. Since $(\mathbb{R}^2)^* \times_{\mathbb{R}^*} \mathrm{RRat}_k(\mathbb{C}P^1) \cong S^1 \times_{\mathbb{Z}/2} \mathrm{RRat}_k(\mathbb{C}P^1)$, the result follows. □

PROPOSITION 2.2. *We have the following long exact sequence:*

$$\begin{aligned}\cdots \to H_*(\mathrm{RF}_k(\mathbb{C}P^2);\mathbb{Z}/p) &\to H_*(\mathrm{RRat}_k(\mathbb{C}P^2);\mathbb{Z}/p)\\ &\xrightarrow{J} \bigoplus_{i=0}^{[\frac{k}{2}]} H_{*-(k-1)}(S^1 \times \mathrm{Rat}_i(\mathbb{C}P^1);\mathbb{Z}/p)\\ &\to H_{*-1}(\mathrm{RF}_k(\mathbb{C}P^2);\mathbb{Z}/p) \to \cdots,\end{aligned}$$

where the homomorphism J will be specified later.

Proof. Consider the homology sequence of the pair $(\mathrm{RRat}_k(\mathbb{C}P^2), \mathrm{RF}_k(\mathbb{C}P^2))$. We first prove the following:

LEMMA 2.3. *There is an isomorphism*

$$\begin{aligned}&H_*(\mathrm{RRat}_k(\mathbb{C}P^2), \mathrm{RF}_k(\mathbb{C}P^2);\mathbb{Z}/p)\\ \cong &H_{*-(k-1)}(S^1 \times_{\mathbb{Z}/2} \mathrm{RRat}_k(\mathbb{C}P^1);\mathbb{Z}/p).\end{aligned}$$

Proof. The lemma is a real version of [3, (5.2)] and a proof is given as follows. Let $D^{k-1} \to \nu \to X_k$ be the closed normal disk bundle of X_k in $\mathrm{RRat}_k(\mathbb{C}P^2)$, where X_k is defined in (2.1). By excision (see [6], Corollary 11.2), we have an isomorphism

$$H_*(\mathrm{RRat}_k(\mathbb{C}P^2), \mathrm{RF}_k(\mathbb{C}P^2);\mathbb{Z}/p) \cong H_*(\nu, \partial\nu;\mathbb{Z}/p).$$

By the Thom isomorphism (see [6], Theorem 10.4), we have an isomorphism

$$H_*(\nu, \partial\nu;\mathbb{Z}/p) \cong H_{*-(k-1)}(X_k;\mathbb{Z}/p).$$

Now the lemma follows from Lemma 2.1. □

Next we study $S^1 \times_{\mathbb{Z}/2} \mathrm{RRat}_k(\mathbb{C}P^1)$. It is easy to see that the $\mathbb{Z}/2$-action on the right-hand side of (1.3) is given as follows: When $k = 2m$, each $\mathrm{Rat}_i(\mathbb{C}P^1)$ $(0 \le i \le m-1)$ appears twice and $\mathrm{Rat}_m(\mathbb{C}P^1)$ once in $\mathrm{RRat}_{2m}(\mathbb{C}P^1)$. Then $\mathbb{Z}/2$ exchanges two copies of $\mathrm{Rat}_i(\mathbb{C}P^1)$ $(0 \le i \le m-1)$ and acts on $\mathrm{Rat}_m(\mathbb{C}P^1)$ by the involution $T : \mathrm{Rat}_m(\mathbb{C}P^1) \to \mathrm{Rat}_m(\mathbb{C}P^1)$ defined by $T(p_0(z), p_1(z)) = (p_0(z), -p_1(z))$. When $k = 2m+1$, the $\mathbb{Z}/2$-action is given similarly.

Now when $k = 2m+1$, Proposition 2.2 follows from Lemma 2.3. On the other hand, when $k = 2m$, we need the following:

LEMMA 2.4. *We have an isomorphism*

$$H_*(S^1 \times_T \mathrm{Rat}_m(\mathbb{C}P^1);\mathbb{Z}/p) \cong H_*(S^1 \times \mathrm{Rat}_m(\mathbb{C}P^1);\mathbb{Z}/p).$$

Proof. Since $T_* : H_*(\mathrm{Rat}_m(\mathbb{C}P^1);\mathbb{Z}/p) \to H_*(\mathrm{Rat}_m(\mathbb{C}P^1);\mathbb{Z}/p)$ is the identity mapping, the local system for the fibration $\mathrm{Rat}_m(\mathbb{C}P^1) \to S^1 \times_T \mathrm{Rat}_m(\mathbb{C}P^1) \to S^1$ is simple. Hence Lemma 2.4 holds. This completes the proof of Proposition 2.2. □

We study the homomorphism J in Proposition 2.2. Let us introduce the following homomorphisms φ and ψ:

$$H_*(\Omega^2 S^5;\mathbb{Z}/p) \xrightarrow{\varphi} H_*(\Omega^2 S^3;\mathbb{Z}/p) \xrightarrow{\psi} \bigoplus_{0\le i} H_*(\mathrm{Rat}_i(\mathbb{C}P^1);\mathbb{Z}/p).$$

For that purpose, we recall the structure of $H_*(\Omega^2 S^{2n+1};\mathbb{Z}/p)$. There is a (torsion free) generator $\iota_{2n-1} \in H_{2n-1}(\Omega^2 S^{2n+1};\mathbb{Z}/p) \cong \mathbb{Z}/p$, and the following hold.

(i) For $p=2$,

$$H_*(\Omega^2 S^{2n+1};\mathbb{Z}/2) \cong \mathbb{Z}/2[\iota_{2n-1}, Q_1(\iota_{2n-1}), \dots, Q_1\cdots Q_1(\iota_{2n-1}), \dots].$$

(ii) For an odd prime p,

$$H_*(\Omega^2 S^{2n+1};\mathbb{Z}/p) \cong \bigwedge(\iota_{2n-1}, Q_1(\iota_{2n-1}), \dots, Q_1\cdots Q_1(\iota_{2n-1}), \dots) \otimes \mathbb{Z}/p[\beta Q_1(\iota_{2n-1}), \dots, \beta Q_1\cdots Q_1(\iota_{2n-1}), \dots].$$

In (i) and (ii), Q_1 is the first Dyer-Lashof operation (it takes a class of dimension d to a class of dimension $dp+p-1$) and β is the mod p Bockstein operation.

Now we define the homomorphisms φ and ψ as follows:

(1) Let $x \in H_*(\Omega^2 S^3;\mathbb{Z}/p)$ be a monomial. We define $\varphi(x)$ to be the element obtained from x by changing all ι_1 to ι_3. Note that if $\overline{\mathrm{w}}$ is the usual weight on $H_*(\Omega^2 S^3;\mathbb{Z}/p)$, then

$$\deg \varphi(x) = \deg x + 2\overline{\mathrm{w}}(x).$$

(2) Let $x \in H_*(\Omega^2 S^3;\mathbb{Z}/p)$ be a monomial. Then, from (1.2), we can regard that $x \in H_*(\mathrm{Rat}_{\overline{\mathrm{w}}(x)}(\mathbb{C}P^1);\mathbb{Z}/p)$. We set

$$\psi(x) = (0,\dots,0,x,0,\dots) \in \bigoplus_{0\le i} H_*(\mathrm{Rat}_i(\mathbb{C}P^1);\mathbb{Z}/p),$$

where the element x in the right-hand side belongs to the direct summand indexed by $i = \overline{\mathrm{w}}(x)$.

(3) In the situation of (2), using the generators $1 \in H_0(S^1;\mathbb{Z}/p)$ and $e_1 \in H_1(S^1;\mathbb{Z}/p)$, we construct elements $1 \otimes \psi(x)$ and $e_1 \otimes \psi(x)$ of

$\bigoplus_{0\le i} H_*(S^1 \times \mathrm{Rat}_i(\mathbb{C}P^1); \mathbb{Z}/p)$ belonging to the direct summand indexed by $i = \overline{\mathrm{w}}(x)$.

The following lemma describes the homomorphism

$$J : H_*(\mathrm{RRat}_k(\mathbb{C}P^2); \mathbb{Z}/p) \to \bigoplus_{i=0}^{[\frac{k}{2}]} H_{*-(k-1)}(S^1 \times \mathrm{Rat}_i(\mathbb{C}P^1); \mathbb{Z}/p).$$

LEMMA 2.5. *Let us denote by* w *the weight on* $H_*(\mathrm{RRat}_k(\mathbb{C}P^2); \mathbb{Z}/p)$ *defined in Section* 1 (3)(iii). *Let* $\alpha \in H_*(\mathrm{RRat}_k(\mathbb{C}P^2); \mathbb{Z}/p)$ *be a monomial. We write* $\alpha = u_1^i \otimes \varphi(x)$, *where* $u_1 \in H_1(\Omega S^2; \mathbb{Z}/p)$ *is the generator. (Recall that* $H_*(\Omega S^2; \mathbb{Z}/p) \cong \mathbb{Z}/p\,[u_1]$.*) Then*

(i) *When* $\mathrm{w}(\alpha) \le k-2$, *we have* $J(\alpha) = 0$.

(ii) *When* $\mathrm{w}(\alpha) = k-1$, *we have* $J(\alpha) = 1 \otimes \psi(x)$. *In particular,* $J(u_1^{k-1}) = 1 \otimes 1$.

(iii) *When* $\mathrm{w}(\alpha) = k$, *we have* $J(\alpha) = e_1 \otimes \psi(x)$. *In particular,* $J(u_1^k) = e_1 \otimes 1$.

Proof. (i), (ii) and (iii) are modifications of Theorem 4.11, Lemmas 5.5 and 5.4 in [3], respectively, and are proved similarly. We check the degree shift. From the above (1), we have

$$\deg J(\alpha) = \mathrm{w}(\alpha) + \deg x - k + 1.$$

Hence, if $\mathrm{w}(\alpha) = k - 1 + \epsilon$ $(\epsilon = 0, 1)$, then $\deg J(\alpha) = \deg x + \epsilon$. This completes the proof of Lemma 2.5. □

Now it is easy to prove Theorem A from Proposition 2.2 and Lemma 2.5. □

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