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STRONGLY IRREDUCIBLE SUBMODULES

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Abstract. This paper is motivated by the results in [6]. We study some properties of strongly irreducible submodules of a module. In fact, our objective is to investigate strongly irreducible modules and to examine in particular when submodules of a module are strongly irreducible. For example, we show that prime submodules of a multiplication module are strongly irreducible, and a characterization is given of a multiplication module over a Noetherian ring which contain a non-prime strongly irreducible submodule.

1. Introduction

Throughout this paper all rings will be commutative with identity and all modules will be unitary. If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M, $\mathrm{Ann}(M)$. An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N : M)$. An R-submodule N of M is said to be irreducible if N is not the intersection of two submodules of M that properly contain it. An ideal of R which is a strongly irreducible (irreducible) module is called a strongly irreducible (irreducible) ideal.

A proper submodule N of a module M over a ring R is said to be prime submodule (primary submodule) if for each $r \in R$ the R-endomorphism of M/N produced by multiplication by r is either injective or zero (either injective or nilpotent), so $(0 : M/N) = P$ ($\mathrm{nilrad}(M/N) = P'$) is a prime ideal of R, and N is said to be P-prime submodule (P'-primary submodule). So N is prime (N is primary) in M if and only if whenever $rm \in N$, for some $r \in R$, $m \in M$, then either $m \in N$ or $rM \subseteq N$ (either $m \in N$ or $r^sM \subseteq N$ for some s), so every prime submodule of M is primary.

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Let M be an R-module. We say that $r \in R$ is a zero-divisor for M if there is a non-zero $m \in M$ such that $rm = 0$, and otherwise that r is M-regular. The set of zero-divisors of M is written $Z_R(M)$. Elements of R that are not zero-divisors are called regular. A regular ideal of R is one that contains a regular element. A submodule N of M is said to be regular if it possesses a N-regular element. A ring R is said to be arithmetical if for all ideals, I, J, and K of R, we have $(I + J) \cap K = (I \cap K) + (J \cap K)$. This property is equivalent to the condition that for all ideals I, J, and K of R, we have $(I \cap J) + K = (I + K) \cap (J + K)$. We use "$\subset$" for strict inclusion.

2. Strongly irreducible modules

DEFINITION 2.1. *A submodule N of an R-module M is said to be strongly irreducible if for submodules N_1 and N_2 of M, the inclusion $N_1 \cap N_2 \subseteq N$ implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$.*

In this section we list some basic properties concerning strongly irreducible modules.

LEMMA 2.2. *Let R be a ring, M an R-module, and N an R-submodule of M. Set $(N : M) = I$. Then:*

(1) $Z_R(M/N)$ is a prime ideal of R if and only if $Z_{R/I}(M/N)$ is a prime ideal of R/I.

(2) $Z_R(R/I)$ is a prime ideal of R if and only if $Z_{R/I}(R/I)$ is a prime ideal of R/I.

Proof. (1) Assume that $Z_R(M/N)$ is a prime ideal of R and let $r + I, s + I \in Z_{R/I}(M/N) = J$. Then there are elements $m, n \in M - N$ such that $(r+I)(m+N) = rm+N = N$ and $(s+I)(n+N) = sn+N = N$, so $s, r \in Z_R(M/N)$, and hence there exists $k \in M - N$ such that $(r-s)k \in N$. It follows that $(r+I)-(s+I) \in J$. Clearly, if $(t+I) \in R/I$ and $(r + I) \in J$, then $(r + I)(t + I) = (t + I)(r + I) \in J$. Therefore, J is an ideal of R/I. Assume that $(r_1 + I)(r_2 + I) \in J$ for some $r_1, r_2 \in R$. Then there exists $a \in M - N$ such that $r_1 r_2 a \in N$, so $r_1 r_2 \in Z_R(M/N)$, and hence either $r_1 \in Z_R(M/N)$ or $r_2 \in Z_R(M/N)$ since $Z_R(M/N)$ is prime. Therefore it follows that either $(r_1 + I) \in J$ or $(r_2 + I) \in J$, so J is a prime ideal of R/I. The other direction is clear.

(2) This proof is similar to that of case (1) and we omit it. $\square$

LEMMA 2.3. *Let M be a module over a commutative ring R, and let $m, n \in M$. Then $Rm \cap Rn = (Rm : Rn)n = (Rn : Rm)m$. Moreover, if N is a submodule of M such that $N \subseteq Rm$, then $N = (N : Rm)m$.*

Proof. Clearly, $(Rm : Rn)n \subseteq Rm \cap Rn$. For the other direction, if $X \in Rm \cap Rn$, then $X = rm = sn$ for some $r, s \in R$. It is clear that $r \in (Rn : Rm)$, and hence $X \in (Rn : Rm)m$. Similarly, $Rm \cap Rn = (Rm : Rn)n$.

For the last statement, assume that N is a submodule of M such that $N \subseteq Rm$. Then it is clear that $(N : Rm)m \subseteq N$, and if $a \in N \subseteq Rm$, then $a = tm$ for some $t \in R$, so $t \in (N : Rm)$, and hence $a = tm \in (N : Rm)m$, as required. □

LEMMA 2.4. *Let R be a ring, M an R-module, and N an R-submodule of M. Then:*

(1) *If N is strongly irreducible, then N is irreducible. In particular, if M is Noetherian, then N is a primary submodule of M.*

(2) *To show that N is strongly irreducible, it suffices to show that if Rn and Rm are cyclic submodules of M such that $Rm \cap Rn \subseteq N$, then either $m \in N$ or $n \in N$.*

(3) *If N is strongly irreducible and if K is a submodule of M contained in N, then N/K is strongly irreducible in M/K.*

Proof. (1) Assume that N is strongly irreducible and let N_1 and N_2 be submodules of M such that $N_1 \cap N_2 = N$. Then $N_1 \cap N_2 \subseteq N$, so either $N_1 \subseteq N$ or $N_2 \subseteq N$, and it then follows that either $N = N_1$ or $N = N_2$, so N is irreducible. Finally, if M is Noetherian, then [13, Proposition 4.13] show that irreducible is primary.

(2) Let N_1 and N_2 be submodules of M such that $N_1 \cap N_2 \subseteq N$. Assume that $N_1 \not\subseteq N$, so there exists $n_1 \in N_1$ such that $n_1 \notin N$. Then for all $a \in N_2$ it holds $Rn_1 \cap Rn_2 \subseteq N_1 \cap N_2 \subseteq N$, so $n_2 \in N$, as required.

(3) Let N_1 and N_2 be submodules of M such that $(N_1/K) \cap (N_2/K) \subseteq N/K$. Then $(N_1 + K) \cap (N_2 + K) \subseteq N + K = N$, so either $N_1 \subseteq N$ or $N_2 \subseteq N$ since N is strongly irreducible, and hence either $N_1/K \subseteq N/K$ or $N_2/K \subseteq N/K$, as required. □

REMARK 1. Let R be a commutative ring, M an R-module, and S a multiplicatively closed set in R. If B is a submodule of M_S, define $B \cap M = \varphi^{-1}(B)$ where $\varphi : M \to M_S$ is the natural homomorphism. Clearly, $B \cap M$ is a submodule of M.

LEMMA 2.5. *Let R be a commutative ring, M an R-module, and N an R-submodule of M. If S is a multiplicatively closed set in R and if*

N is primary submodule of M such that $\mathrm{Rad}((N:M)) \cap S = \emptyset$, *then* $N_S \cap M = N$.

Proof. Clearly, $N \subseteq N_S \cap M$. Let $m \in N_S \cap M$. Then there are elements $n \in N$ and $s \in S$ such that $m/1 = n/s$. There exists $t \in S$ such that $stm = tn \in N$. It follows that $m \in N$ since $st \notin \mathrm{Rad}((N:M))$, as required. □

LEMMA 2.6. *Let R be a commutative ring, M an R-module, and N an R-submodule of M. If S is a multiplicatively closed set in R and if N_S is strongly irreducible, then $N_S \cap M$ is strongly irreducible.*

Proof. Assume that N_S is strongly irreducible and let H and G be submodules of M such that $H \cap G \subseteq N_S \cap M$. Then $G_S \cap H_S \subseteq N_S$. For if $a_1/s_1 = a_2/s_2 \in G_S \cap H_S$ (where $a_1 \in G, a_2 \in H$ and $s_1, s_2 \in S$), then $a_1 s_2 t_1 = a_2 s_1 t_1 \in H \cap G \subseteq N_S \cap M$ for some $t_1 \in S$. Therefore, there are elements $n \in N$ and $s \in S$ such that $(a_1 s_2 t_1)/1 = (a_2 s_1 t_1)/1 = n/s$, so there exists $t_2 \in S$ such that $a_1 s_2 t_1 t_2 s = t_2 n$, and hence $t_2(a_1 s_2 t_1 s s_1 - s_1 n) = 0$. Thus $a_1/s_1 = n/(s_2 t_1 s_1 s) \in N_S$. It follows that either $G_S \subseteq N_S$ or $H_S \subseteq N_S$, so either $H \subseteq N_S \cap M$ or $G \subseteq N_S \cap M$, as required. □

PROPOSITION 2.7. *Let R be a commutative ring, M an R-module, and N an R-submodule of M. If S is a multiplicatively closed set in R and if N is strongly irreducible primary submodule of M such that* $\mathrm{Rad}((N:M)) \cap S = \emptyset$, *then N_S is strongly irreducible.*

Proof. Assume that N is strongly irreducible primary submodule of M and let H and G be submodule of N_S such that $H \cap G \subseteq N_S$. Then $(H \cap M) \cap (G \cap M) \subseteq N_S \cap M = N$ by lemma 2.4. So either $H \cap M \subseteq N$ or $G \cap M \subseteq N$ since N is strongly irreducible. Therefore it follows that either $G = (G \cap M)_S \subseteq N_S$ or $H = (H \cap M)_S \subseteq N_S$, and hence N_S is strongly irreducible. □

PROPOSITION 2.8. *Let R be a commutative ring, M an R-module, and N an R-submodule of M. If N is P-primary and N_P is strongly irreducible, then N is strongly irreducible.*

Proof. By lemma 2.5, $N_P \cap M$ is strongly irreducible. Now the assertion follows from lemma 2.4. □

3. Multiplication modules

Let R be a commutative ring with non-zero identity. Then R is a cyclic multiplication R-module. Thus strongly irreducible ideals are strongly irreducible submodules of the cyclic multiplication R-module R.

THEOREM 3.1. *Let R be a ring, and M a multiplication R-module. If N is a prime submodule of M, then N is strongly irreducible.*

Proof. Assume that N is a prime and let N_1 and N_2 be submodules of M such that $N_1 \cap N_2 \subseteq N$ but $N_1 \nsubseteq N$ and $N_2 \nsubseteq N$. We can write $N_1 = I_1M$ and $N_2 = I_2M$ for some ideals I_1 and I_2 of R, so there are $r_1 \in I_1$, $r_2 \in I_2$ and $m_1, m_2 \in M$ such that $r_1m_1 \notin N$ and $r_2m_2 \notin N$. It follows that $r_1r_2m_1 \in N_1 \cap N_2 \subseteq N$, so $r_2M \subseteq N$ since N is prime. Thus $r_2m_2 \in N$, a contradiction, as required. □

PROPOSITION 3.2. *Let R be a ring, and M a finitely generated multiplication R-module. Then:*

(1) *A submodule N of M is strongly irreducible if and only if there exists a strongly irreducible ideal I of R such that $N = IM$.*

(2) *A submodule N of M is irreducible if and only if there exists a irreducible ideal I of R such that $N = IM$.*

Proof. (1) Suppose first that N is a strongly irreducible submodule of M. There exists an ideal I of R such that $N = IM$. Let I_1 and I_2 be ideals of R such that $I_1 \cap I_2 \subseteq I$. It follows from [5, Corollary 1.6] that

$$(I_1 + \mathrm{Ann}M)M \cap (I_2 + \mathrm{Ann}M)M = (I_1 \cap I_2)M \subseteq N,$$

and hence either $(I_1 + \mathrm{Ann}M)M \subseteq N$ or $(I_2 + \mathrm{Ann}M)M \subseteq N$. As $\mathrm{Ann}M \subseteq (N : M) = I$ we get (by [12, p. 231 Corollry]) either $I_1 \subseteq I_1 + \mathrm{Ann}M \subseteq I$ or $I_2 \subseteq I_2 + \mathrm{Ann}M \subseteq I$, so it follows that I is strongly irreducible. Conversely, assume that I is strongly irreducible and let N_1 and N_2 be submodules of M such that $N_1 \cap N_2 \subseteq N$. There are ideals J_1 and J_2 of R such that $N_1 = J_1M$, $N_2 = J_2M$, so $((J_1 + \mathrm{Ann}M) \cap (J_2 + \mathrm{Ann}M))M = N_1 \cap N_2 \subseteq IM = N$, and hence either $J_1 = J_1 + \mathrm{Ann}M \subseteq I$ or $J_2 = J_2 + \mathrm{Ann}M \subseteq I$. It follows that N is strongly irreducible.

(2) This proof is similar to that of case (1) and we omit it. □

PROPOSITION 3.3. *Let R be a ring, and M a finitely generated multiplication R-module. Then a primary submodule of M over a* **UFD** *is strongly irreducible.*

Proof. Assume that N is a P-primary submodule of M and let $N = IM$ for some ideal I of R. There exists a principal primary ideal I_P of R_P such that $N_P = I_PM_P$ since R_P is **DVR**. It then follows from

[6, Lemma 2.2 (10)] and proposition 3.2 that N_P is strongly irreducible, and hence N is strongly irreducible by Proposition 2.8. □

REMARK 2. Why is the hypothesis "M **is a multiplication module**" needed?

(1) Let R be a local Dedekind domain with maximal ideal $P = Rp$. The module $E = E(R/P)$, the injective hull of R/P, is pure-injective and secondary (see [3], Theorem 1.1). Set $A_n = (0 :_E P^n)$ $(n \geq 1)$. Then every nonzero proper submodule L of E is of the form $L = A_m$ for some m and E is Artinian module with a strictly increasing sequence of submodules $A_1 \subset A_2 ...$, where they are not prime in E (see [4], p. 324), but they are strongly irreducible (so primary).

The mapping $f : E \to E$ defined by $x \mapsto p^n x$ $(n \geq 1)$ is a module surjective homomorphism with $\mathrm{Ker}(f) = A_n$, so $E/A_n \cong E$. Similarly, the mapping $g : E \to P^n E$ $(n \geq 1)$ by $x \mapsto p^n x$ is a surjective homomorphism with $\mathrm{Ker}(g) = A_n$, and hence $E \cong E/A_n \cong P^n E$. Thus E is not multiplication (compare with theorem 3.1).

(2) Suppose that R is a field. Then any R-module M is torsion-free (vector space) and every proper submodule of M is prime (so primary). But M is not multiplication. Let $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a R-basis of an R-module M (so it is Noetherian and Artinian). Set $N_1 = Rx_1 + Rx_2 + Rx_3$, $N_2 = Rx_1 + Rx_5$ and $N = Rx_1 + Rx_6$. Then $N_1 \cap N_2 \subseteq N$ but $N_1 \not\subseteq N$ and $N_2 \not\subseteq N$, so N is not strongly irreducible (compare with Proposition 3.3 and theorem 3.1).

(3) A submodule N of M is said to be a maximal submodule of M if (i) $M \neq N$ and (ii) there is no proper submodule of M strictly containing N. It is well known that every non-zero finitely generated R-module possesses a maxmal submodule. If N is a maximal submodule of M, then N is prime in M (since M/N is a simple R-module and $(N : M)$ is a maximal ideal of R). Therefore it follows that every non-zero finitely generated multiplication R-module possesses a strongly irreducible submodule by Proposition 3.1 (4). In particular, every non-zero cyclic R-module possesses a strongly irreducible submodule.

(4) Let M be a module over a ring R. If the zero submodule of M is irreducible, then the zero submodule of M is strongly irreducible.

PROPOSITION 3.4. *Let R be an arithmetical ring, M a finitely generated multiplication R-module, and N an R-submodule of M. Then:*

(1) *N is strongly irreducible if and only if N is irreducible.*

(2) *N is strongly irreducible if and only if the set of zero-divisors of M/N is a prime ideal of R.*

(3) *If N is a primary submodule of M, then N is irreducible.*

Proof. (1) By Lemma 2.4, it is enough to show that if N is irreducible, then N is strongly irreducible. Let N_1 and N_2 be submodules of M such that $N_1 \cap N_2 \subseteq N$, so $N = (N_1 \cap N_2) + N = (N_1 + N) \cap (N_2 + N)$ since M is distributive by [10, Theorem 5]. It then follows that either $N_1 \subseteq N$ or $N_2 \subseteq N$, as required.

(2) Assume that N is strongly irreducible and let P be the set of zero-divisors of M/N. P is not empty since $0 \in P$. To prove P is an ideal of R, assume that $r_1, r_2 \in P$. Then there are elements m_1 and m_2 of M such that $r_1m_1, r_2m_2 \in N$ and $m_1, m_2 \notin N$. If $Rm_1 \cap Rm_2 \neq 0$, then $t_1m_1 = t_2m_2 \neq 0$ for some $t_1, t_2 \in R$, so $(r_1 - r_2)(t_1m_1) \in N$, and hence $r_1 - r_2 \in P$. If $Rm_1 \cap Rm_2 = 0$, then we have $N = (Rm_1 \cap Rm_2) + N = (N + Rm_1) \cap (N + Rm_2)$ since M is distributive. It then follows that either $Rm_1 \subseteq N$ or $Rm_2 \subseteq N$ since N is irreducible, so either $(r_1 - r_2)m_1 \in N$ or $(r_1 - r_2)m_2 \in N$. Thus $r_1 - r_2 \in P$. Clearly, if $r \in R$ and $r_1 \in P$, then $rr_1 = r_1r \in P$. Therefore, P is an ideal of R. It remains only to show that P is prime. Assume that $rs \in P$ for some $r, s \in R$. There exists $m \in M - N$ such that $rsm \in N$. If $sm \in N$, then $s(m + N) = 0$, so $s \in P$. If $sm \notin N$, then $sm \neq 0$ and $r(sm + N) = 0$, and hence $r \in P$, so P is prime. For the other direction, assume the set of zero-divisors of M/N is a prime ideal of R. There exists an ideal I of R such that $N = IM$ where $I = \mathrm{Ann}(M/N)$. It is easy to see that M/N is a faithful multiplication R/I-module. Now the assertion follows from Lemma 2.2, Proposition 3.2, [5, Lemma 4.3] and [6, Lemma 2.2(3)].

(3) Assume that N is a primary submodule of M and let $N = IM$ for some ideal I of R. Then $I = \mathrm{Ann}(M/N)$ is a primary ideal of R by [9, sec 2.8 Proposition 18]. Let N_1 and N_2 be submodules of M such that $N = N_1 \cap N_2$. There are ideals I_1 and I_2 of R such that $N_1 = I_1M$ and $N_2 = I_2M$, so $(I_1 \cap I_2)M = IM$ by [5, Corollary 1.6] (since $\mathrm{Ann}(M) \subseteq I_1, I_2$ and I). It then follows from [12, p. 231 Corollary] that $I = I_1 \cap I_2$, so either $I = I_1$ or $I = I_2$ since I is irreducible by [7, Theorem 6], and hence either $N = N_1$ or $N = N_2$, as required. □

LEMMA 3.5. *Let (R, P) be a quasi-local ring, M a cyclic R-module, and N a strongly irreducible P-primary submodule of M. Assume that $N \subset (N : PM)M$. Then:*

(1) *$(N : PM)M$ is a cyclic module.*

(2) *$N = (N : PM)PM$.*

(3) *For each submodule K of M either $K \subseteq N$ or $(N : PM)M \subseteq K$.*

Proof. (1) Since $N \subset (N : PM)M$, there exists $x \in (N : PM)M - N$. We cliam that $(N : PM)M = Rx$. If $(N : PM)M \neq Rx$, then let $y \in (N : PM)M - Rx$. Then $Rx \cap Ry = (Rx : Ry)y$ by Lemma 2.3.

$(Rx : Ry)y \subseteq N$. For if $ry \in (Rx : Ry)y$ with $r \in (Rx : Ry)$, then there are elements $m \in M$ and $s \in (N : PM)$ such that $y = sm$ (since M is cyclic), and hence $ry = srm \in N$ since $r \in (Rx : Ry) \subseteq P$. However, N is srongly irreducible, so $Rx \cap Ry \subseteq N$ implies that either $Rx \subseteq N$ or $Ry \subseteq N$, hence $y \in N$. It follows that $(N : PM)M = Rx \cup N$, so either $Rx \subseteq N$ or $N \subseteq Rx$, and hence $(N : PM)M = Rx$, a contradiction, as required.

(2) There exists an ideal $I = (N : M)$ of R such that $N = IM$ since M is multiplication, so $N \subset (N : PM)M = Rx$, and hence $N = (N : Rx)x$ by Lemma 2.3. $P = (N : Rx)$. Otherwise, there are elements $r \in P$ and $s \in R$ such that $rsx \notin N$ since R is quasi-local with maximal ideal P. Since M is cyclic and $x \in (N : PM)M$, $x = tm$ for some $m \in M$ and $t \in (N : PM)$, so $rsx = t(rsm) \in N$, and this is a contradiction. Thus $N = Px = (N : PM)PM$.

(3) Let K be a submodule of M. It may clearly be assumed that $K \nsubseteq N$, so it remains to show that $(N : PM)M \subseteq K$; that is, $x \in K$. If $x \notin K$, then let $a \in K$, so $x \notin Ra$. Therefore, $Rx \cap Ra = (Ra : Rx)x$ (by Lemma 2.3) $\subseteq Px = N$. It follows that either $Rx \subseteq N$ or $Ra \subseteq N$ since N is strongly irreducible, so $a \in N$, and hence $K \subseteq N$, a contradiction, as required. □

PROPOSITION 3.6. *Let R be a Noetherian ring, M a multiplication R-module, and N a strongly irreducible R-submodule of M. Let* $\mathrm{Rad}((N : M)) = P$, *and assume that $I = (N : M) \neq P$. Then:*

(1) *$(N_P : P_P M_P)M_P$ is a cyclic R_P-submodule of M_P.*

(2) *$N_P = (N_P : P_P M_P)P_P M_P$.*

(3) *For each submodule K of M either $K \subseteq N$ or $(N_P : P_P M_P)M_P \subseteq K_P$.*

Proof. By Lemma 2.4, N is a strongly irreducible P-primary submodule of M (since every multiplication module over a Noetherian ring is Noetherian). Also, N_P is strongly irreducible by Proposition 2.7, so (1)–(3) follow from Lemma 3.5 (note that any multiplication module over a quasi-local ring is cyclic by [2, Proposition 4]). □

PROPOSITION 3.7. *Let (R, P) be a local ring, M a multiplication R-module, and N a strongly irreducible P-primary submodule of M with $(N : M) \neq P$. Then $N = \bigcup\{K : K$ is a submodule of M and $K \subset (N : PM)M\}$ and $(N : PM)M = \bigcap\{K : K$ is a submodule of M and $N \subset K\}$.*

Proof. Set $H = \bigcap\{K : K$ is a submodule of M and $N \subset K\}$. Clearly, $H \subseteq (N : PM)M$. If K is a submodule of M such that $N \subset K$, then

$(N : PM)M \subseteq K$ by Lemma 3.5 (3), so $(N : PM)M \subseteq H$, and hence $H = (N : PM)M$.

Set $L = \bigcup\{K : K$ is a submodule of M and $K \subset (N : PM)M\}$. Clearly, $N \subseteq L$. If K is a submodule of M such that $K \subset (N : PM)M$, then $(N : PM)M \nsubseteq K$, so $K \subseteq N$, and hence $L \subseteq N$. Thus $L = N$, as required. □

THEOREM 3.8. *Let M be a multiplication module over a Noetherian ring R. A submodule N of M is a non-prime strongly irreducible module if and only if there exist submodules H and G of M such that $N \subset H \subseteq G$ and:*

(1) *G is prime;*

(2) *N is $(G : M)$-primary (set $P = (G : M)$);*

(3) *for all submodules K of M either $K \subseteq N$ or $H_P \subseteq K_P$. Also if this holds, then $H_P = (N_P :_{R_P} G_P)M_P$. In particular, a finitely generated multiplication module over a Noetherian ring R contains a non-prime strongly irreducible submodule if and only if there exists a submodule N of M satisfying these conditions.*

Proof. Since every multiplication module over a Northerian ring is Noetherian, so N is primary by Lemma 2.4, hence N is P-primary (where $P = \mathrm{Rad}(N : M)$). Moreover, $G = PM$ is a prime submodule of M by [5, Corollary 2.11], so $N \neq G$, and hence $N \subset H = (N : PM)M$. Now the assertion follows from Propositions 3.6 and 3.7.

For the converse, assume that N is P-primary. By Proposition 2.7, it suffices to show that N_P is strongly irreducible, so it may be assumed that R is local with maximal ideal P. Let K and L be submodules of M such that $K \cap L \subseteq N$. If $K \nsubseteq N$ and $L \nsubseteq N$, then $N \subset H = (N : PM)M \subseteq K \cap L$, and this is a contradiction, as required.

Finally, since $G = PM \nsubseteq N$, so $G \subseteq H = (N : PM)M$ by Proposition 3.6, and hence $PM = (N : PM)M$. □

LEMMA 3.9. *Let R be a Noetherian ring, M a finitely generated multiplication R-module, and N a strongly irreducible R-submodule of M. Let $\mathrm{Rad}((N : M)) = P$, and assume that $I = (N : M) \neq P$ and $\mathrm{ht}(P) > 0$. Then:*

(1) *N is a strongly irreducible $R/\mathrm{Ann}(M)$-submodule of M,*

$$\mathrm{Rad}((N :_{R/\mathrm{Ann}(M)} M)) = P/\mathrm{Ann}(M), I/\mathrm{Ann}(M) \neq P/\mathrm{Ann}(M)$$

and $\mathrm{ht}(P/\mathrm{Ann}(M)) > 0$.

(2) *If I is a regular ideal of R, then $I/\mathrm{Ann}(M)$ is a regular ideal of $R/\mathrm{Ann}(M)$.*

Proof. (1) Clearly, M is multiplication as an $R/\mathrm{Ann}(M)$-module. Also, N is a strongly irreducible $R/\mathrm{Ann}(M)$-submodule of M by [6, Lemma 2.2 (8)] and Proposition 3.2. It is clear that N satisfies the stated conditions.

(2) If r is a regular element of I and $sI = 0$, then $s = 0$, so $\mathrm{Ann}(I) = 0$. By the [11, Lemma 2.6], we get

$$\mathrm{Ann}_M(I) = \{m \in M : Im = 0\} = \mathrm{Ann}_R(I).M = 0.$$

If $(t + \mathrm{Ann}(M))(I/\mathrm{Ann}(M)) = 0$, then $tI \subseteq \mathrm{Ann}(M)$, so $I(rM) = 0$. By the above consideration, we have $rM = 0$, and hence $\mathrm{Ann}_{R/\mathrm{Ann}(M)}$ $(I/\mathrm{Ann}(M)) = 0$, as required. □

PROPOSITION 3.10. *Let R be a Noetherian ring, M a finitely generated multiplication R-module, and N a strongly irreducible R-submodule of M. Let* $\mathrm{Rad}((N : M)) = P$, *and assume that* $I = (N : M) \neq P$ *and* $\mathrm{ht}(P) > 0$. *Then N_P is a regular module.*

Proof. By Lemma 3.9, it may be assumed that M is a faithful finitely generated multiplication R-module. Also, by Proposition 3.2, I is strongly irreducible, so by hypothesis, I_P is a regular ideal of R_P. We claim that there is an element $x \in I_P$ such that $xs = 0$ for all $0 \neq s \in M_P$. Otherewise, for each $x \in I_P$, there exists $0 \neq s \in M_P$ such that $xs = 0$, so $I_P \subseteq Z(M_P) = Z(R_P)$ (by [5], Lemma 4.3), and this is a contradiction since I_P contains a regular element. Thus there is an element $x \in R_P$ such that $xt = 0$ for all $0 \neq t \in N_P$, and hence N_P is regular module. □

THEOREM 3.11. *Let R be a Noetherian ring, M a finitely generated multiplication R-module, and N a non-prime R-submodule of M with* $\mathrm{ht}((N : M) = I) > 0$. *Then N is strongly irreducible if and ony if N is primary, R_P is a* **DVR**, *where* $P = \mathrm{Rad}(I)$, *and* $I = P^n$ *for some integer* $n > 1$.

Proof. ($\Leftarrow$) As N is primary, we conclude that I is a primary ideal of R. Since R_P is a **DVR**, I_P is strongly irreducible (because the ideals of R_P are linearly ordered), and since I is P-primary, this implies that I is strongly irreducible by [6, Lemma 2.2(6)]. It follows from proposition 3.2 that N is strongly irreducible.

($\Rightarrow$) Since over a Noetherian ring, every multiplication module is Noetherian, we conclude that N is primary by Lemma 2.4 (1). As N is strongly irreducible, it follows from Proposition 3.2 that I is strongly irreducible. Now the ideal I satisfies the stated conditions of [6, Theorem 3.4], as required. □

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