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INJECTIVE ENVELOPES OF SIMPLE MODULES OVER POLYNOMIAL RINGS

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ABSTRACT. Let A be a polynomial ring over a field and M a simple A-module. We generalize one result of Song about the description of the injective envelope $E_A(M)$ in terms of modules of generalized fractions.

1. Introduction

For any maximal ideal $\mathfrak{m}$ of a polynomial ring $K[X_1, \dots, X_n]$ over a field K, if $\mathfrak{m} = (X_1 - a_1, \dots, X_n - a_n)$, $a_i \in K$, $i = 1, \dots, n$, Song and Kim ([8]) have given a very explicit description of the injective envelope $E(K[X_1, \dots, X_n]/\mathfrak{m})$ of the simple $K[X_1, \dots, X_n]$-module $K[X_1, \dots, X_n]/\mathfrak{m}$ in terms of modules of generalized fractions. For a general maximal ideal $\mathfrak{m}$ of $K[X_1, \dots, X_n]$, it is proved in [9] that there exists a finite normal field extension L of K such that all the maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ of $L[X_1, \dots, X_n]$ which lie over $\mathfrak{m}$ have the form

$$\mathfrak{m}_i = (X_1 - a_{i1}, \dots, X_n - a_{in}),\ a_{ij} \in L,\ i = 1, \dots, t, j = 1, \dots, n.$$

Then one obtains a very explicit description of $E' = \bigoplus_{i=1}^{t} E(L[X_1, \dots, X_n]/\mathfrak{m}_i)$ in terms of modules of generalized fractions and an action of the Galois group $G = \mathrm{Gal}(L/K)$ of L over K on E'. When the order of G is not divisible by the characteristic of K and L is a Galois extension of K, Song ([9]) have shown that the injective envelope $E(K[X_1, \dots, X_n]/\mathfrak{m})$ is isomorphic to the fixed submodule E'^G of E' which can be described very explicitly in terms of modules of generalized fractions. In this note we shall show that the condition on the order of G can be removed. For a general finite normal field extension L of K, we will prove that E'^G is isomorphic to the direct sum of $[L : K]_i$, the inseparability degree of

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L over K, copies of $E(K[X_1,\dots,X_n]/\mathfrak{m})$. As corollaries, if K is perfect then E'^G is isomorphic to $E(K[X_1,\dots,X_n]/\mathfrak{m})$, and E'^G is isomorphic to $E(K[X_1,\dots,X_n]/\mathfrak{m})$ if and only if L is a separable extension of K.

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2. Preliminaries

Let K be a field and $A = K[X_1,\dots,X_n]$ the polynomial ring in indeterminates $X_1,\dots,X_n$ over K. Let $\mathfrak{m}$ be a maximal ideal of A. If there exist $a_i \in K$, $i = 1,\dots,n$, such that $\mathfrak{m} = (X_1 - a_1,\dots,X_n - a_n)$, one can give a very explicit description of the injective envelope $E_A(A/\mathfrak{m})$ in terms of modules of generalized fractions (see [8]). In general, by [9, 2.5], there exists a finite normal field extension L of K such that $\mathfrak{m}$ splits in L, i.e., all the maximal ideals $\mathfrak{m}_1,\dots,\mathfrak{m}_t$ of $L[X_1,\dots,X_n]$ which lie over $\mathfrak{m}$ are such that, for suitable $a_{ij} \in L$, $i = 1,\dots,t$, $j = 1,\dots,n$, $\mathfrak{m}_i = (X_1 - a_{i1},\dots,X_n - a_{in})$, $i = 1,\dots,t$. Let L be a finite normal field extension of K, L is called a splitting field for $\mathfrak{m}$ if L is a minimal element in the set of finite normal field extensions of K in which $\mathfrak{m}$ splits. Then it is easy to see that there exists a splitting field for $\mathfrak{m}$.

PROPOSITION 2.1. *Splitting fields for $\mathfrak{m}$ are unique up to isomorphisms over K.*

Proof. Let $\overline{K}$ be an algebraic closure of K and L a splitting field for $\mathfrak{m}$ in $\overline{K}$. Then there exist $f_1,\dots,f_r \in A$ such that $\mathfrak{m} = (f_1,\dots,f_r)$. Suppose that $\mathfrak{m}_i = (X_1 - a_{i1},\dots,X_n - a_{in})$, $a_{ij} \in L$, $i = 1,\dots,t, j = 1,\dots,n$, are the maximal ideals of $L[X_1,\dots,X_n]$ which lie over $\mathfrak{m}$. Then

$$\begin{aligned} & \{(a_{i1},\dots,a_{in}) : i = 1,\dots,t\} \\ = \ & \{(a_1,\dots,a_n) \in L^n : f_i(a_1,\dots,a_n) = 0, i = 1,\dots,r\}, \end{aligned}$$

and L is the normal closure of the field $K(\{a_{ij} : i = 1,\dots,t, j = 1,\dots,n\})$ in $\overline{K}$. Hence there is only one splitting field for $\mathfrak{m}$ in $\overline{K}$. Since the algebraic closures of K are unique up to isomorphisms over K, it follows that splitting fields for $\mathfrak{m}$ are unique up to isomorphisms over K. □

Let L be a splitting field for $\mathfrak{m}$ and $\mathfrak{m}_i = (X_1 - a_{i1},\dots,X_n - a_{in})$, $i = 1,\dots,t$, the maximal ideals of $B := L[X_1,\dots,X_n]$ which lie over $\mathfrak{m}$. Let

$$U_i = \{(X_1 - a_{i1})^{r_1},\dots,(X_n - a_{in})^{r_n}, 1) : r_i \geq 1, i = 1,\dots,n\}.$$

Then $E'(B/\mathfrak{m}_i) := U_i^{-n-1}B$ is an injective envelope of the B-module $B/\mathfrak{m}_i$. Set $E' = \bigoplus_{i=1}^t E'(B/\mathfrak{m}_i)$. Then the Galois group $G := \mathrm{Gal}(L/K)$ of L over K can act on E' in a natural way (see [9]).

Suppose that L is separable over K. Let $\sigma_i \in G$ such that $\sigma_i(\mathfrak{m}_1) = \mathfrak{m}_i$, $i = 2, \ldots, t$. Set $a_j = a_{1j}$, $j = 1, \ldots, n$, and $K^* = K(a_1, \ldots, a_n)$. Then, by [9, 3.1], the fixed submodule $E'^G := \{e' \in E' : \sigma(e') = e'$ for all $\sigma \in G\}$ can be described very explicitly:

$$E'^G = \{(\delta, \sigma_2^{(1)}(\delta), \ldots, \sigma_t^{(1)}(\delta)) \in E' : \delta \in U_1^{-n-1}K^*[X_1, \ldots, X_n]\}$$

where $\sigma_i^{(1)} : E'(B/\mathfrak{m}_1) \to E'(B/\mathfrak{m}_i)$ is the A-module isomorphism induced from $\sigma_i, i = 2, \ldots, t$.

From the proof of [9, 3.2], we obtain the following

THEOREM 2.2. *If L is a separable extension of K, then E'^G is an essential extension of $A/\mathfrak{m}$ as A-modules.*

For any finite normal field extension L of K, it is easy to prove the following lemma.

LEMMA 2.3. *If L is a finite normal field extension of K, then there exists a subfield K' of L such that K' is a purely inseparable extension of K and L is a separable extension of K'. If $G = \mathrm{Gal}(L/K)$ then $K' = \mathrm{Inv}(G)$, the subfield of L over K of G-invariants, and $G = \mathrm{Gal}(L/K')$.*

3. The results

Let K be a field. We firstly consider Noetherian K-algebras and their tensor products with a finite purely inseparable field extension of K.

LEMMA 3.1. *Let A be a commutative Noetherian K-algebra and K' a finite purely inseparable field extension of K. Let $A' = A \bigotimes_K K'$. Then, for any prime ideal $\mathfrak{p}$ of A, there exists only one prime ideal of A' which lies over $\mathfrak{p}$.*

Proof. Suppose that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two prime ideals of A' which lie over $\mathfrak{p}$. Since, for any $x \in K'$, there exists an integer $e \geq 0$ such that $x^{p^e} \in K$, where $p = \mathrm{char}(K)$, and $\mathfrak{p}_1$ is finitely generated, it follows that there exists an integer $m \geq 0$ such that $\mathfrak{p}_1^m \subseteq \mathfrak{p}A'$. Then $\mathfrak{p}_1^m \subseteq \mathfrak{p}_2$, hence $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$. Similarly $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$, then $\mathfrak{p}_1 = \mathfrak{p}_2$. □

LEMMA 3.2. *Let A, K', A' and $\mathfrak{p}$ be as in 3.1 and $\mathfrak{p}'$ the unique prime ideal of A' which lies over $\mathfrak{p}$. Then the injective envelope $E_{A'}(A'/\mathfrak{p}')$ of*

A'-module $A'/\mathfrak{p}'$ is, as an A-module, isomorphic to the direct sum of $[K' : K]$ copies of the injective envelope $E_A(A/\mathfrak{p})$ of A-module $A/\mathfrak{p}$, i.e.,

$$E_{A'}(A'/\mathfrak{p}') \cong \bigoplus [K' : K] E_A(A/\mathfrak{p}).$$

Proof. By [6, 4.3], $E_{A'}(A'/\mathfrak{p}') \cong \bigoplus e(\mathfrak{p}'/\mathfrak{p})f(\mathfrak{p}'/\mathfrak{p})E_A(A/\mathfrak{p})$ as A-modules, where $e(\mathfrak{p}'/\mathfrak{p})$ is the generalized ramification index of $\mathfrak{p}'$ over $\mathfrak{p}$ and $f(\mathfrak{p}'/\mathfrak{p})$ is the residue class degree of $\mathfrak{p}'$ over $\mathfrak{p}$. Again, by [6, 4.4], $e(\mathfrak{p}'/\mathfrak{p})f(\mathfrak{p}'/\mathfrak{p}) = n(\mathfrak{p})$, the rank of the finitely generated free $A_{\mathfrak{p}}$-module $A'_{\mathfrak{p}}$. Since $A'_{\mathfrak{p}} \cong A' \bigotimes_A A_{\mathfrak{p}} = (A \bigotimes_K K') \bigotimes_A A_{\mathfrak{p}} \cong A_{\mathfrak{p}} \bigotimes_K K'$, it follows that $n(\mathfrak{p}) = [K' : K]$, as required. □

Next, we return to consider polynomial rings. Let $A = K[X_1, \ldots, X_n]$ and $\mathfrak{m}$ a maximal ideal of A. Suppose that L is a splitting field for $\mathfrak{m}$. Let K' be the subfield of L which is such that L is a separable extension of K' and K' is a purely inseparable extension of K and $G = \mathrm{Gal}(L/K)$. Then L is a Galois extension of K' and $G = \mathrm{Gal}(L/K')$. Let $\mathfrak{m}'$ be the unique maximal ideal of $A' = K'[X_1, \ldots, X_n]$ which lies over $\mathfrak{m}$ and $\mathfrak{m}_i = (X_1 - a_{i1}, \ldots, X_n - a_{in})$, $i = 1, \ldots, t$, the maximal ideals of $B = L[X_1, \ldots, X_n]$ which lie over $\mathfrak{m}$. Then L is a splitting field for $\mathfrak{m}'$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ are the maximal ideals of B which lie over $\mathfrak{m}'$.

THEOREM 3.3. *As A'-modules, E'^G is isomorphic to $E_{A'}(A'/\mathfrak{m}')$.*

Proof. By [6, 3.5], E' is an injective A'-module. Since, by 2.2, E'^G is an essential extension of $A'/\mathfrak{m}'$ as A'-modules, it follows that in order to prove $E'^G \cong E_{A'}(A'/\mathfrak{m}')$ it suffices to show that E'^G is a maximal essential extension of $A'/\mathfrak{m}'$ in E'.

Let $x \in E' \setminus E'^G$. We want to show that $E'^G + A'x$ is not an essential extension of E'^G. Suppose that $x = (\delta_1, \delta_2, \ldots, \delta_t)$ (cf., [9] for notations). If $\delta_1 \in U_1^{-n-1}K^*[X_1, \ldots, X_n]$ where $K^* = K'(a_1, \ldots, a_n)$, then

$$x' := x - (\delta_1, \sigma_2^{(1)}(\delta_1), \ldots, \sigma_t^{(1)}(\delta_1)) \neq 0$$

and $x' \in E'^G + A'x$. Since the first component of x' is zero, we see that $A'x' \cap E'^G = 0$. Then $E'^G + A'x$ is not an essential extension of E'^G in this case. Now suppose that $\delta_1 \notin U_1^{-n-1}K^*[X_1, \ldots, X_n]$ and

$$\delta_1 = \sum_{i=1}^{w} \frac{l_i}{((X_1 - a_1)^{\alpha_{i1}}, \ldots, (X_n - a_n)^{\alpha_{in}}, 1)}$$

where $w \geq 1$, $l_1, \ldots, l_w \in L \setminus \{0\}$, $\alpha_{ij} \geq 1$, $i = 1, \ldots, w$, $j = 1, \ldots, n$, and $(\alpha_{i1}, \ldots, \alpha_{in})$, $i = 1, \ldots, w$, are distinct. We may assume that all the $l_i \notin K^*$. Suppose that the set

$$\{(\alpha_{i1}, \ldots, \alpha_{in}) : i = 1, \ldots, w\}$$

has been ordered lexicographically by $<$: $(\alpha_{i1}, \ldots, \alpha_{in}) < (\alpha_{j1}, \ldots, \alpha_{jn})$ if and only if there exists an integer h such that

$$1 \leq h \leq n,\ \alpha_{i1} = \alpha_{j1}, \ldots, \alpha_{i,h-1} = \alpha_{j,h-1},\ \alpha_{ih} < \alpha_{jh}$$

and $(\alpha_{11}, \ldots, \alpha_{1n}) < (\alpha_{21}, \ldots, \alpha_{2n}) < \ldots < (\alpha_{w1}, \ldots, \alpha_{wn})$. For any $f \neq 0 \in A'$,

$$f = \sum_{i1,\ldots,in} f_{i1,\ldots,in}(X_1 - a_1)^{\beta_{i1}} \cdots (X_n - a_n)^{\beta_{in}},\ f_{i1,\ldots,in} \in K^*.$$

Since we are going to consider $f\delta_1$, we may assume that $\beta_{ij} < \alpha_{wj}$, $j = 1, \ldots, n$. Suppose that

$$f = f_{11,\ldots,1n}(X_1 - a_1)^{\beta_{11}} \cdots (X_n - a_n)^{\beta_{1n}} + \cdots$$
$$+ f_{h1,\ldots,hn}(X_1 - a_1)^{\beta_{h1}} \cdots (X_n - a_n)^{\beta_{hn}}$$

where $(\beta_{11}, \ldots, \beta_{1n}) < (\beta_{21}, \ldots, \beta_{2n}) < \ldots < (\beta_{h1}, \ldots, \beta_{hn})$ and $f_{i1,\ldots,in} \neq 0$, $i = 1, \ldots, h$. Note that

$$(\alpha_{w1} - \beta_{11}, \ldots, \alpha_{wn} - \beta_{1n}) > (\alpha_{i1} - \beta_{j1}, \ldots, \alpha_{in} - \beta_{jn})$$

for all (i, j) with $i \neq w$ or $j \neq 1$ and

$$f\delta_1 = \sum_{(\gamma_{i1},\ldots,\gamma_{in})<(\alpha_{w1}-\beta_{11},\ldots,\alpha_{wn}-\beta_{1n})} \frac{l'_i}{((X_1 - a_1)^{\gamma_{i1}}, \ldots, (X_n - a_n)^{\gamma_{in}}, 1)}$$
$$+ \frac{l_w f_{11,\ldots,1n}}{((X_1 - a_1)^{\alpha_{w1}-\beta_{11}}, \ldots, (X_n - a_n)^{\alpha_{wn}-\beta_{1n}}, 1)},\ l'_i \in L.$$

Since $l_w f_{11,\ldots,1n} \notin K^*$, it follows that $f\delta_1 \notin U_1^{-n-1}K^*[X_1, \ldots, X_n]$. Then $fx \notin E'^G$, hence $A'x \cap E'^G = 0$ and $E'^G + A'x$ is not an essential extension of E'^G. Therefore E'^G is a maximal essential extension of $A'/\mathfrak{m}'$ in E', as required. □

THEOREM 3.4. *As A-modules, $E'^G \cong \bigoplus [L : K]_i E_A(A/\mathfrak{m})$, where $[L : K]_i$ is the inseparability degree of L over K.*

Proof. By 3.3, $E'^G \cong E_{A'}(A'/\mathfrak{m}')$ and by 3.2, $E_{A'}(A'/\mathfrak{m}') \cong \bigoplus [K' : K] E_A(A/\mathfrak{m})$. But $[K' : K] = [L : K]_i$, the result follows. □

COROLLARY 3.5. *As A-modules, E'^G is isomorphic to $E_A(A/\mathfrak{m})$ if and only if a splitting field for $\mathfrak{m}$ is separable over K.*

COROLLARY 3.6. *If K is perfect, then E'^G is isomorphic to $E_A(A/\mathfrak{m})$.*

REMARK. It is easy to see that if L is an arbitrary finite normal field extension of K in which $\mathfrak{m}$ splits then above results remain true.

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