Bull. Korean Math. Soc. **42** (2005), No. 1, pp. 75–85

MOTION OF VORTEX FILAMENTS IN 3-MANIFOLDS

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ABSTRACT. In this paper, the visco-Da-Rios equation;

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial s} \wedge \frac{D}{ds}\frac{\partial \gamma}{\partial s} + \nu \frac{\partial \gamma}{\partial s} \tag{0.1}$$

is investigated on 3-dimensional complete orientable Riemannian manifolds. The *global existence* of solution is discussed by transforming (0.1) into a cubic nonlinear Schrödinger equation for complete orientable Riemannian 3-manifolds of constant curvature.

1. Introduction

In the early years of the 20th century, Da-Rios derived the Localized Induction Equation(LIE)

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial s} \times \frac{\partial^2 \gamma}{\partial s^2} \tag{1.1}$$

in order to represent the motion of isolated thin vortex filaments as an object embedded in an infinite domain entirely filled by a homogeneous, incompressible, inviscid fluid. Here the operation $\times$ denotes the cross product in 3-dimensional Euclidean space. This law being commonly referred to as the *localized induction approximation*(LIA) says that the vortex filament moves with a velocity proportional to its local curvature toward the binormal direction. Even though this equation has been rediscovered with the growth of interest in the fluid mechanics and also in different fields of physics such as ferromagnetic media, it had been little known about the solutions of the Da-Rios equation (1.1) before Hasimoto transformed the equation (1.1) into a non-linear Schrödinger equation(NLS-equation)([6]).

Received July 3, 2003.

2000 Mathematics Subject Classification: 58J72, 22E30, 76B03, 35Q55.

Key words and phrases: nonlinear Schrödinger equation, Da-Rios equation, incompressible fluid, Hasimoto transform, Schrödinger maps.

The present research was conducted by the research fund of Dankook university in 2004.

Lately, it has been questioned an analog of the LIE in 3 dimensional manifolds such as S^3, H^3, $\cdots$ by K. Uhlenbeck, C.L. Terng and many other geometors. For a few recent years, C.L. Terng and K. Uhlenbeck have formulated this problem by developing Schrödinger maps on Kähler manifolds (hence it has a complex structure)([11]). In [4], the Cauchy problem for Schrödinger maps in a compact Riemannian surface (with a complex structure) and its global solution have been considered. In [8], one can find a complete description of the LIE-NLS connection on complete 3-dimensional orientable Riemannian manifolds. In [9], a natural generalization of the LIE in higher dimensional spaces is introduced.

In this paper, we consider the case when viscosity effects are present on the dynamics of the fluids in a complete 3-dimensional orientable Riemannian manifold M. That is, the *viscosity* (denoted by a non-negative constant ν) in the tangential direction of the vortex filament is counted as it moves. Hence the visco-Da-Rios equation on M is expressed by

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial s} \wedge \frac{D}{ds}\frac{\partial \gamma}{\partial s} + \nu \frac{\partial \gamma}{\partial s}. \tag{1.2}$$

When the viscosity ν is zero, this equation is reduced to the Da-Rios equation which is described in [8], and when $M = \mathbb{R}^3$, it is just the same as the classical Da-Rios equation (1.1).

The aims of this paper are: (1) to investigate the possibility of the movement of a given curve obeying the LIA with viscosity effects inside a 3-dimensional complete orientable Riemannian manifold M, (2) to find a scalar-valued differential equation which is equivalent to the visco-Da-Rios equation on M, and (3) to investigate the role of Riemannian curvature in transforming the visco-Da-Rios equation on M.

In Section 2, a comprehensive development for the Hasimoto Transform on Riemannian 3-manifolds is introduced, and in Section 3, the Hasimoto Transform is applied to the visco-Da-Rios equation (1.2) on M to show that it is equivalent to a non-linear Scrödinger equation of the type;

$$\psi_t = i\psi_{ss} + \nu\psi_s + F(\psi)\psi,$$

for some complex-valued function F (which will be described in (3.9)). Also, when M is a 3-dimensional complete orientable Riemannian manifold with constant sectional curvature, the *global existence* of solution is proved by transforming (1.2) into a cubic nonlinear Schrödinger equation;

$$\psi_t = i\psi_{ss} + \nu\psi_s + \frac{i}{2}\left(|\psi|^2 + \widetilde{D}(t)\right)\psi,$$

for some real-valued function $\widetilde{D}(t)$.

2. Hasimoto transform on Riemannian manifolds

In the following discussion, $(M, \langle\cdot,\cdot\rangle)$ represents a 3-dimensional complete orientable Riemannian manifold, and ∇ the Levi-Civita connection associated with $\langle\cdot,\cdot\rangle$.

Let $\gamma : \mathbb{R} \to M$ be a C^2-curve parametrized by arc-length;

$$s(\tilde{x}) = \int_0^{\tilde{x}} |\gamma_x(x)| dx,$$

where $|\gamma_x| \equiv \sqrt{\langle \gamma_x, \gamma_x \rangle}$. First, we will find an orthonormal frame field along γ. We denote the *unit tangent vector* by $\mathbf{t} = \frac{d\gamma}{ds}$, the *unit normal vector* by $\mathbf{n} \equiv \frac{D\mathbf{t}}{ds} / \left|\frac{D\mathbf{t}}{ds}\right|$, and define the *geodesic curvature* κ_g by

$$\frac{D\mathbf{t}}{ds} = \kappa_g \mathbf{n}. \tag{2.1}$$

A vector field X can be identified with its dual 1-form $X^* \equiv \langle X, \cdot \rangle$. Since M is oriented, we may select an orientation on all tangent spaces T_pM, hence also on all cotangent spaces T_p^*M in a consistent manner. A 1-form Y^* can be regarded as the 2-form $*(Y^*)$ by the $*$-operation, hence the *vector field* $X \wedge Y$ of vector fields X and Y is defined such a way that its dual $(X \wedge Y)^*$ is determined by the following identification;

$$(X \wedge Y)^* \equiv *(X^* \wedge Y^*) \cong X^* \wedge Y^*. \tag{2.2}$$

Note that there is a notational abuse using the wedge product. Then we get

$$X \wedge Y = -Y \wedge X,$$
$$\langle X \wedge Y, Z \rangle = \langle X, Y \wedge Z \rangle,$$

for vector fields X, Y and Z. For an orthonormal positive basis $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$ for a cotangent space T_p^*M, we note that $\mathbf{e}_3^* = *(\mathbf{e}_1^* \wedge \mathbf{e}_2^*)$. Hence, defining the *unit binormal vector* $\mathbf{b}$ by

$$\mathbf{b}(s) \equiv \mathbf{t}(s) \wedge \mathbf{n}(s), \quad s \in \mathbb{R},$$

it is clear that $\mathbf{b}(s)$ is *the* unit vector and that $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ forms an orthonormal positive basis for each tangent space $T_{\gamma(s)}G$. Hence $\frac{D}{ds}\mathbf{b}(s)$ is parallel to $\mathbf{n}(s)$ on $T_{\gamma(s)}G$ for all $s \in \mathbb{R}$, that is,

$$\frac{D}{ds}\mathbf{b}(s) = -\tau(s)\, \mathbf{n}(s) \tag{2.3}$$

for some $\tau(s)$, called the *(geodesic) torsion* of γ. In all, it can be summarized as the Frenet-Serret formula:

LEMMA 2.1 (Frenet-Serret formula on M). *We have*

$$\frac{D}{ds}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & 0 \\ -\kappa_g & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \tag{2.4}$$

Proof. This lemma can be induced from the idea that the Lie algebra $\mathcal{O}(3)$ of the orthogonal group $O(3)$ is the set of 3×3 skew-symmetric matrices. Let $X \equiv (\mathbf{t}, \mathbf{n}, \mathbf{b})^T$, and A be a 3×3-matrix satisfying the equation $\frac{D}{ds}X = AX$. We recognize the first and the third rows of A directly from (2.1) and (2.3). Therefore it suffices to show that A is a skew-symmetric matrix. We solve the ordinary differential equation $\frac{D}{ds}X = AX$ to get

$$X = e^{\int_0^s A(x)dx} X_0, \tag{2.5}$$

where $\int_0^s A(x)dx$ means the integration on each component of A with respect to s and $X_0 \equiv (\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0)^T$. Here $\mathbf{t}_0$ ($\mathbf{n}_0$, $\mathbf{b}_0$) denotes the parallel vector field along γ with the value $\mathbf{t}(0)$($\mathbf{n}(0)$, $\mathbf{b}(0)$, respectively) at the point $\gamma(0)$. Then from the facts that $X^* \equiv (\mathbf{t}^*, \mathbf{n}^*, \mathbf{b}^*)^T$ and

$$I_{3\times 3} = X^* X^T = (e^{\int_0^s A(x)dx} X_0^*)(X_0)^T e^{\int_0^s A^T(x)dx} = e^{\int_0^s (A(x)+A^T(x))dx},$$

we can see that A is skew-symmetric. □

The Frenet-Serret formula illustrates the Leibniz' rule;

$$\frac{D}{ds}(X \wedge Y) = \frac{D}{ds}X \wedge Y + X \wedge \frac{D}{ds}Y, \tag{2.6}$$

for any vector fields X and Y along γ.

We may rewrite the second and third rows of (2.4) above as follows:

$$\frac{D}{ds}\begin{pmatrix} \mathbf{n} \\ \mathbf{b} \end{pmatrix} = -\kappa_g \begin{pmatrix} \mathbf{t} \\ 0 \end{pmatrix} - \tau \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \tag{2.7}$$

We observe that the second term in (2.7), $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, is a complex structure, i.e., $J^2 = -I$, and also there is a natural field-monomorphism $\mathbb{C} \to GL(2, \mathbb{R})$ defined by

$$a + bi \mapsto a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence by writing $\begin{pmatrix} \mathbf{n} \\ \mathbf{b} \end{pmatrix} \cong \mathbf{n} + i\mathbf{b}$, we may rewrite (2.7) as

$$\frac{D}{ds}\{\mathbf{n}+i\mathbf{b}\} = -\kappa_g \mathbf{t} - \tau i \{\mathbf{n}+i\mathbf{b}\}. \tag{2.8}$$

This is a linear first-order ordinary differential equation with respect to $\{\mathbf{n}+i\mathbf{b}\}$, and so, multiplying the integrating factor $e^{i\int\tau}$ on both sides of (2.8), it can be rewritten as

$$\frac{D}{ds}\mathbf{N} = -\psi(s)\mathbf{t}, \tag{2.9}$$

where we set $\mathbf{N}(s) \equiv (\mathbf{n}+i\mathbf{b})\, e^{i\int^s \tau(x)dx}$ and $\psi(s) \equiv -\kappa_g e^{i\int^s \tau(x)dx}$. The following lemma shows a way of changing the old moving frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ into the new complex valued frame $\{\mathbf{t}, \mathbf{N}, \overline{\mathbf{N}}\}$, where $\overline{\mathbf{N}}$ represents the complex conjugate of $\mathbf{N}$.

LEMMA 2.2. *We have*

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}\bar{z} & \frac{1}{2}z \\ 0 & -\frac{i}{2}\bar{z} & \frac{i}{2}z \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{N} \\ \overline{\mathbf{N}} \end{pmatrix},$$

where $z \equiv e^{i\int\tau}$.

The Hasimoto Transform $\mathcal{H}$ is defined as follows:

$$\mathcal{H}(\kappa_g, \tau) = \kappa_g e^{i\int\tau} \equiv \psi,$$

where κ_g and τ denote the geodesic curvature and the torsion of a curve, respectively. For example, a circle maps to a constant function, $\mathcal{H}(1,0) = 1$, and a helix to the exponential function $\mathcal{H}(1,1) = e^{ix}$. The inverse Transform is

$$\mathcal{H}^{-1}\psi = \left(|\psi|, \ \frac{1}{i}\left(\frac{d}{ds}\frac{\psi}{|\psi|}\right)\left(\frac{|\psi|}{\psi}\right)\right),$$

if $\psi \neq 0$, and $\mathcal{H}^{-1}0 = (0,0)$. Some basic properties of Hasimoto Transform are presented;

$$\begin{aligned} \mathcal{H}(a\kappa_g, \tau) &= a\mathcal{H}(\kappa_g, \tau), \\ \mathcal{H}(\kappa_g^1 + \kappa_g^2, \tau) &= \mathcal{H}(\kappa_g^1, \tau) + \mathcal{H}(\kappa_g^2, \tau), \\ \mathcal{H}(\kappa_g^1\kappa_g^2, \tau_1 + \tau_2) &= \mathcal{H}(\kappa_g^1, \tau_1) + \mathcal{H}(\kappa_g^2, \tau_2), \\ \frac{d}{ds}\mathcal{H}(\kappa_g, \tau) &= \mathcal{H}(\frac{d}{ds}\kappa_g, \tau) + i\tau\mathcal{H}(\kappa_g, \tau). \end{aligned}$$

3. The visco-Da-Rios equation and NLS-equation

In this section, we show as an application of the Hasimoto Transform that the visco-Da-Rios equation is equivalent to a non-linear Scrödinger equation, and using this, in the case of a Riemannian manifold with constant curvature, we show that there exists a global solution for the visco-Da-Rios equation under the category of Sobolev spaces.

PROPOSITION 3.1. *The visco-Da-Rios equation on a 3-dimensional complete orientable Riemannian manifold;*

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial s} \wedge \frac{D}{ds}\frac{\partial \gamma}{\partial s} + \nu \frac{\partial \gamma}{\partial s} \tag{3.1}$$

is equivalent to a non-linear Scrödinger equation;

$$\psi_t = i\psi_{ss} + \nu\psi_s + F(\psi)\psi,$$

for some complex-valued function F which will be defined in page 82. Here ψ is the Hasimoto Transform of γ; that is, $\psi = \mathcal{H}(\kappa_g, \tau)$, κ_g is the geodesic curvature of γ and τ is the torsion of γ.

The proof of the Proposition above will use the two auxiliary Lemmas that follow.

LEMMA 3.2. *Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the orthonormal moving frame defined in Section 2 for the curve γ satisfying the visco-Da-Rios equation (3.1). Then we have*

$$\begin{aligned}
\frac{D}{dt}\mathbf{t} &= (\nu\kappa_g - \kappa_g\tau)\,\mathbf{n} + \frac{d\kappa_g}{ds}\mathbf{b} \\
\frac{D}{dt}\mathbf{n} &= (\kappa_g\tau - \nu\kappa_g)\,\mathbf{t} - A(s,t)\mathbf{b} \\
\frac{D}{dt}\mathbf{b} &= -\frac{d\kappa_g}{ds}\mathbf{t} + A(s,t)\mathbf{n},
\end{aligned}$$

for some real-valued function $A(s,t)$.

Proof. By the symmetry of Levi-Civita connection, we have

$$\begin{aligned}
\frac{D}{dt}\mathbf{t} = \frac{D}{dt}\left(\frac{\partial}{\partial s}\gamma\right) &= \frac{D}{ds}\left(\frac{\partial}{\partial t}\gamma\right)\\
&= \frac{D}{ds}\left(\frac{\partial\gamma}{\partial s}\wedge\frac{D}{ds}\frac{\partial\gamma}{\partial s}+\nu\frac{\partial\gamma}{\partial s}\right)\\
&= \frac{D}{ds}\left(\mathbf{t}\wedge\kappa_g\mathbf{n}+\nu\mathbf{t}\right)\\
&= \frac{D}{ds}(\kappa_g\mathbf{b})+\nu\kappa_g\mathbf{n}\\
&= (\nu\kappa_g-\kappa_g\tau)\,\mathbf{n}+\frac{\partial\kappa_g}{\partial s}\mathbf{b}.
\end{aligned}$$

Let $\frac{D}{dt}\mathbf{n} \equiv C_1\mathbf{t}+C_2\mathbf{n}+C_3\mathbf{b}$. Then differentiate both sides of the identity $\langle\mathbf{n},\mathbf{n}\rangle = 1$ to get $C_2 = 0$. Since $0 = \langle\mathbf{n},\mathbf{t}\rangle$, we have

$$0 = \left\langle\frac{D\mathbf{n}}{dt},\mathbf{t}\right\rangle + \left\langle\mathbf{n},\frac{D\mathbf{t}}{dt}\right\rangle = C_1+\nu\kappa_g-\kappa_g\tau.$$

Also, since $0 = \langle\mathbf{n},\mathbf{b}\rangle$, we have $C_3 + \langle\mathbf{n},\frac{D\mathbf{b}}{dt}\rangle = 0$. Letting $A \equiv \langle\mathbf{n},\frac{D\mathbf{b}}{dt}\rangle$, we have the second identity. Similarly, we can get the third one. □

LEMMA 3.3. *Let R be the Riemannian curvature tensor. Let $\{\mathbf{t},\mathbf{N},\overline{\mathbf{N}}\}$ be the complex-valued frame defined in page 79 for the curve γ satisfying the visco-Da-Rios equation (3.1). Then we obtain*

$$R(\gamma_t,\gamma_s)\mathbf{N} = \psi(R_{1213}+iR_{1313})\mathbf{t}+i|\psi|R_{1323}\mathbf{N}, \tag{3.2}$$

where $R_{1213} = R(\mathbf{t},\mathbf{n},\mathbf{t},\mathbf{b})$, $R_{1313} = R(\mathbf{t},\mathbf{b},\mathbf{t},\mathbf{b})$ *and* $R_{1323} = R(\mathbf{t},\mathbf{b},\mathbf{n},\mathbf{b})$.

Proof. It follows directly from the fact that $R(\gamma_t,\gamma_s)\mathbf{N} = \psi R(\mathbf{b},\mathbf{t})\mathbf{n}+i\psi R(\mathbf{b},\mathbf{t})\mathbf{b}$. □

Proof of Proposition 3.1. From Lemma 3.2, we notice that

$$\begin{aligned}
\frac{D}{dt}\mathbf{N} &= \frac{D}{dt}\left(\mathbf{n}+i\mathbf{b}\right)e^{i\int^s\tau(x,\cdot)dx}\\
&= -(i\psi_s+\nu\psi)\mathbf{t}+iA(s,t)\left(\mathbf{n}+i\mathbf{b}\right)e^{i\int^s\tau dx}\\
&\qquad + \left(i\int^s\tau_t(x,\cdot)dx\right)\left(\mathbf{n}+i\mathbf{b}\right)e^{i\int^s\tau dx}\\
&= -(i\psi_s+\nu\psi)\mathbf{t}+iB(s,t)\mathbf{N},
\end{aligned}$$

where we set $B(s,t) \equiv A(s,t) + \int^s \tau_t(x,t)dx$. Hence by Lemma 2.1, 2.2 and (2.9), we get

$$\text{(3.3)} \quad \frac{D}{ds}\frac{D}{dt}\mathbf{N} = -\left(i\frac{\partial^2\psi}{\partial s^2} + \nu\frac{\partial\psi}{\partial s} + iB\psi\right)\mathbf{t} \\ -\frac{1}{2}\left(i\frac{\partial\psi}{\partial s}\overline{\psi} + \nu\psi\overline{\psi} + i\frac{\partial B}{\partial s}\right)\mathbf{N} - \frac{1}{2}\left(i\frac{\partial\psi}{\partial s} + \nu\psi\right)\psi\overline{\mathbf{N}}.$$

Also, the equation (2.9), Lemma 2.2 and 3.2 yield

$$\frac{D}{dt}\frac{D}{ds}\mathbf{N} = -\psi_t\mathbf{t} - \left((\nu\kappa_g - \kappa_g\tau)\,\mathbf{n} + \frac{\partial\kappa_g}{\partial s}\mathbf{b}\right)\psi$$

$$\text{(3.4)} \quad = -\psi_t\mathbf{t} + \frac{1}{2}\left(i\frac{\partial\overline{\psi}}{\partial s} - \nu\overline{\psi}\right)\psi\mathbf{N} - \frac{1}{2}\left(i\frac{\partial\psi}{\partial s} + \nu\psi\right)\psi\overline{\mathbf{N}}.$$

From the well-known curvature identity, we have

$$\text{(3.5)} \quad R(\gamma_t,\gamma_s)\mathbf{N} = \frac{D}{ds}\frac{D}{dt}\mathbf{N} - \frac{D}{dt}\frac{D}{ds}\mathbf{N}$$

(e.g. [10], p. 261). Substituting (3.2), (3.3), (3.4) into (3.5), we get

$$\text{(3.6)} \quad \psi_t = i\psi_{ss} + \nu\psi_s + iB\psi + (R_{1213} + iR_{1313})\psi$$

$$\text{(3.7)} \quad i|\psi|R_{1323} = -\frac{i}{2}\left\{\psi_s\overline{\psi} + \psi\overline{\psi}_s\right\} + i\frac{\partial B}{\partial s}.$$

From (3.7), we get

$$B(s,t) = \frac{1}{2}|\psi(s,t)|^2 + D(t) + \int_0^t |\psi(s,\tau)|R_{1323}(s,\tau)d\tau,$$

for some real valued function $D(t)$. Plugging this into (3.6), we obtain a non-linear Scrödinger equation;

$$\text{(3.8)} \quad \psi_t = i\psi_{ss} + \nu\psi_s + F(\psi)\psi,$$

where we set

$$\text{(3.9)} \quad F(\psi) \equiv \frac{i}{2}|\psi|^2 + iD + iR_{1313} + R_{1213} + i\int_0^t |\psi(\tau)|R_{1323}(\cdot,\tau)d\tau.$$

□

In the case of Riemannian manifolds with constant sectional curvature, Proposition 3.1 assures the global existence of the solutions for the visco-Da-Rios equation under the category of Sobolev spaces. For this, we make use of the semi-group theory on the Cauchy problem for the NSL-equation generated by the Hasimoto Transform.

LEMMA 3.4. *Let $(M, \langle , \rangle)$ be a 3-dimensional orientable Riemannian manifold with constant sectional curvature. Then for any orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, on M the Ricci curvature $Ric(\mathbf{e}_i, \mathbf{e}_j)$, $i \neq j$ is identically zero;*

$$Ric(\mathbf{e}_i, \mathbf{e}_j) = 0 \quad \text{for all} \quad i \neq j.$$

Proof. Note that M has constant sectional curvature equal to K_0 if and only if $R = K_0 R'$, where

$$\langle R'(X,Y)W, Z\rangle = \langle X, W\rangle\langle Y, Z\rangle - \langle Y, W\rangle\langle X, Z\rangle,$$

X, Y, W, $Z \in T_pM$. Hence $Ric(\mathbf{e}_i, \mathbf{e}_j) = \langle R(\mathbf{e}_i, \mathbf{e}_k)\mathbf{e}_j, \mathbf{e}_k\rangle = 0$, $i \neq k \neq j$. □

THEOREM 3.5. *The visco-Da-Rios equation (3.1) on a 3-dimensional complete orientable Riemannian manifold with constant sectional curvature is equivalent to a cubic non-linear Scrödinger equation;*

$$\psi_t = i\psi_{ss} + \nu\psi_s + \frac{i}{2}\left(|\psi|^2 + \widetilde{D}(t)\right)\psi,$$

for some real-valued function $\widetilde{D}(t)$. Therefore for any given curve γ_0 with its geodesic curvature $\kappa_g = |\frac{D}{ds}\frac{\partial}{\partial s}\gamma_0| \in W^{1,2}(\mathbb{R})$, there exists a global solution $\gamma(s,t)$ (in time) for the visco-Da-Rios equation (3.1) such that its geodesic curvature $|\frac{D}{ds}\frac{\partial}{\partial s}\gamma|$ is staying in $C([0,\infty), W^{1,2}(\mathbb{R}))$.

Proof. From Lemma 3.4, we know that $R_{1323} = R_{1213} = 0$. Hence the NSL-equation (3.8) is reduced to

$$\psi_t = i\psi_{ss} + \nu\psi_s + \frac{i}{2}\left(|\psi|^2 + D(t) + K_0\right)\psi, \tag{3.10}$$

where K_0 is the given sectional curvature. Letting $\widetilde{D}(t) = D(t) + K_0$, we have the first result.

Looking at this evolution equation (3.10) as a Cauchy problem, we may apply the semi-group theory to get a global solution for a semi-linear partial differential equation (3.10) inside appropriate function spaces. Set

$$\varphi \equiv \psi \exp\left\{-\frac{i}{2}\left(\int_0^t (D(\tau) + K_0 + \nu^2)d\tau + \nu s\right)\right\}.$$

Then the NLS-equation (3.10) is reduced to

$$\varphi_t = i\varphi_{ss} + \frac{i}{2}|\varphi|^2\varphi - \frac{\nu^2}{4}\varphi. \tag{3.11}$$

Also, multiplying $e^{\frac{\nu^2}{4}t}$ on both sides of (3.11) and letting $\phi \equiv e^{\frac{\nu^2}{4}t}\varphi$, we get a *defocusing cubic NLS-equation*

$$\frac{1}{i}\phi_t = \phi_{ss} + \frac{1}{2}e^{-\frac{\nu^2}{2}t}|\phi|^2\phi. \tag{3.12}$$

Applying the (semi-)group $e^{-it\Delta}\left(\equiv \frac{e^{-is^2/4t}}{\sqrt{2\pi t}} *\right)$ associated with the generator $-i\Delta = -i\frac{\partial^2}{\partial s^2}$ on both sides of (3.12), we get the (semi-)group representation for (3.12);

$$\phi(s,t) = e^{-it\Delta}\phi(0) + \frac{i}{2}\int_0^t e^{-i(t-\tau)\Delta}e^{-\frac{\nu^2}{2}\tau}|\phi|^2\phi\, d\tau. \tag{3.13}$$

Then it is well-known that the Banach fixed point Theorem yields a local existence of the solution of (3.13) in $C([0,T], W^{1,2}(\mathbb{R}))$, and furthermore, in the case of qubic non-linear Scrödinger equations of the type (3.12), the solution never blows up inside of the Sobolev space $W^{1,2}(\mathbb{R})$ even though it is a defocusing NLS-equation, i.e., the defocusing qubic NLS-equation such as (3.12) has the global solution in $C([0,\infty), W^{1,2}(\mathbb{R}))$ for any initial data $\phi(0) \in W^{1,2}(\mathbb{R})$ (see [3], p. 112). This completes the proof. □

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