

STABILITY OF A GENERALIZED QUADRATIC FUNCTIONAL EQUATION WITH JENSEN TYPE

YOUNG WHAN LEE

ABSTRACT. In this paper we solve a generalized quadratic Jensen type functional equation

$$\begin{aligned} & m^2 f\left(\frac{x+y+z}{m}\right) + f(x) + f(y) + f(z) \\ &= n^2 \left[f\left(\frac{x+y}{n}\right) + f\left(\frac{y+z}{n}\right) + f\left(\frac{z+x}{n}\right) \right] \end{aligned}$$

and prove the stability of this equation in the spirit of Hyers, Ulam, Rassias, and Găvruta.

1. Introduction

S. M. Ulam (see [26]) proposed the stability problem : “When is it true that by slightly changing the hypothesis of a theorem one can still assert that the thesis of the theorem remains true or approximately true?” The case of approximately additive mappings was solved by D. H. Hyers [3]. Th. M. Rassias [17] proved a substantial generalization of the result of Hyers and also P. Găvruta [2] obtained a further generalization of the Hyers-Ulam-Rassias theorem (see also [1-4]). Later, many Rassias and Găvruta type theorem concerning the stability of different functional equations were obtained by numerous authors (see, for instance, [5-24]).

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In this paper we deal with a generalized quadratic Jensen type functional equation

$$(1) \quad \begin{aligned} & m^2 f\left(\frac{x+y+z}{m}\right) + f(x) + f(y) + f(z) \\ &= n^2 \left[f\left(\frac{x+y}{n}\right) + f\left(\frac{y+z}{n}\right) + f\left(\frac{z+x}{n}\right) \right]. \end{aligned}$$

where m and n are nonzero integers with $m+1=2n$. The author [15] solved the quadratic Jensen type functional equation

$$(2) \quad \begin{aligned} & 9f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) \\ &= 4 \left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \right] \end{aligned}$$

and investigated the Hyers-Ulam-Rassias stability of this equation. T. Trif ([25]) generalized the above result for n variables with a quadratic equation deriving from an inequality of Popoviciu for convex functions. But the equation (1) is an another generalized form of the equation (2) for two variables.

In section 2, we solve the equation (1). In section 3, we prove the stability of the equation (2) in the spirit of Hyers, Ulam, Rassias and Găvruta.

2. Solution of the equation (1)

If $f : R \rightarrow R$ satisfies (1) for all $x, y, z \in R$ then $f(x) = ax^2 + bx + c$ is a solution of (1). In particular, if $m^2 + 3 \neq 3n^2$ then $c = 0$.

THEOREM 2.1. *Let X and Y be real linear spaces. A function $f : X \rightarrow Y$ satisfies (1) for all $x, y, z \in X$ if and only if there exist a quadratic function $Q : X \rightarrow Y$, an additive function $A : X \rightarrow Y$, and an element $B \in Y$ such that*

$$f(x) = Q(x) + A(x) + B$$

for all $x \in X$. In particular, if $m^2 + 3 \neq 3n^2$ then $B = 0$.

Proof. (Necessity) Let $Q(x) := \frac{1}{2}[f(x) + f(-x)] - f(0)$, $A(x) := \frac{1}{2}[f(x) - f(-x)]$, and $B = f(0)$ for all $x \in X$. Then $Q(0) = 0$, $Q(-x) = Q(x)$, $A(0) = 0$, $A(-x) = -A(x)$,

$$(3) \quad \begin{aligned} & m^2 Q\left(\frac{x+y+z}{m}\right) + Q(x) + Q(y) + Q(z) \\ &= n^2 \left[Q\left(\frac{x+y}{n}\right) + Q\left(\frac{y+z}{n}\right) + Q\left(\frac{z+x}{n}\right) \right], \end{aligned}$$

and

$$(4) \quad \begin{aligned} & m^2 A\left(\frac{x+y+z}{m}\right) + A(x) + A(y) + A(z) \\ &= n^2 \left[A\left(\frac{x+y}{n}\right) + A\left(\frac{y+z}{n}\right) + A\left(\frac{z+x}{n}\right) \right] \end{aligned}$$

for all $x, y, z \in X$.

First we claim that Q is quadratic. That is,

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in X$. Putting $z = 0$ and $y = -x$ in (3) yields

$$(5) \quad Q(x) = n^2 Q\left(\frac{x}{n}\right)$$

for all $x \in X$. Putting $y = z = 0$ in (3) yields

$$(6) \quad m^2 Q\left(\frac{x}{m}\right) = Q(x)$$

for all $x \in X$. By (3), (5) and (6), we have

$$(7) \quad \begin{aligned} & Q(x+y+z) + Q(x) + Q(y) + Q(z) \\ &= Q(x+y) + Q(y+z) + Q(z+x) \end{aligned}$$

for all $x, y, z \in X$. By replacing $z = -x$ in (7) we get

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in X$.

Secondly we claim that A is additive. Putting $y = z = 0$ in (4) yields

$$(8) \quad m^2 A\left(\frac{x}{m}\right) + A(x) = 2n^2 A\left(\frac{x}{n}\right)$$

for all $x \in X$. Putting $y = x$ and $z = -x$ in (4) yields

$$(9) \quad m^2 A\left(\frac{x}{m}\right) + A(x) = n^2 A\left(\frac{2x}{n}\right)$$

for all $x \in X$. By (8) and (9) we have $2A(x) = A(2x)$ for all $x \in X$. Putting $y = z = 0$ and replacing x by $2x$ in (4) we get

$$(10) \quad 2m^2 A\left(\frac{x}{m}\right) + 2A(x) = 4n^2 A\left(\frac{x}{n}\right)$$

for all $x \in X$. Replacing y by $2x$ and z by $-x$ in (4) we have

$$(11) \quad 2m^2 A\left(\frac{x}{m}\right) + 2A(x) = n^2 A\left(\frac{3x}{n}\right) + A\left(\frac{x}{n}\right)$$

for all $x \in X$. By (10) and (11) we have $3A(x) = A(3x)$ for all $x \in X$.

Suppose that $A((k-1)x) = (k-1)A(x)$ and $A(kx) = kA(x)$ for some integer k . Putting $y = z = 0$ and replacing x by kx in (4) we get

$$(12) \quad km^2 A\left(\frac{x}{m}\right) + kA(x) = 2kn^2 A\left(\frac{x}{n}\right)$$

for all $x \in X$. Replacing y by kx and z by $-x$ in (4) we have

$$(13) \quad km^2 A\left(\frac{x}{m}\right) + kA(x) = n^2 A\left(\frac{(k+1)}{n}x\right) + (k-1)n^2 A\left(\frac{x}{n}\right)$$

for all $x \in X$. By (12) and (13), we have $n^2 A\left(\frac{(k+1)x}{n}\right) = (k+1)A\left(\frac{x}{n}\right)$ and so $(k+1)A(x) = A((k+1)x)$ for all $x \in X$. An induction argument implies that

$$(14) \quad A(kx) = kA(x)$$

for all integer k and all $x \in X$. By (4) and (14) we get

$$(15) \quad \begin{aligned} & mA(x+y+z) + A(x) + A(y) + A(z) \\ &= n[A(x+y) + A(y+z) + A(z+x)] \end{aligned}$$

for all $x, y, z \in X$ and $m+1 = 2n$.

In the case $n = m = 1$, if we replace $z = -x - y$ in (15), we have $A(x+y) = A(x) + A(y)$ for all $x, y \in X$. Let $n \neq 1$. Putting $z = 0$ and $m = 2n - 1$ yields $(n-1)A(x+1) = (n-1)[A(x) + A(y)]$ for all $x \in X$. Therefore A is additive.

(Sufficiency) This is obvious. □

3. Hyers-Ulam-Rassias stability of the equation (1)

Throughout this section X and Y will be a normal linear space and a real Banach space, respectively. Let $m, n \neq 0, \pm 1$ be fixed integers with $m + 1 = 2n$ and let $\varphi : X \times X \times X \rightarrow [0, \infty)$ be a mapping satisfying one of the condition (a), (b) and one of the condition (c), (d) :

$$\begin{aligned}
 \text{(a)} \quad \Phi_1(x, y, z) &= \sum_{k=1}^{\infty} \frac{1}{n^{2k}} \varphi(n^k x, n^k y, n^k z) < \infty, \\
 \text{(b)} \quad \Phi_2(x, y, z) &= \sum_{k=1}^{\infty} \frac{1}{m^{2k}} \varphi(m^k x, m^k y, m^k z) < \infty, \\
 \text{(c)} \quad \Phi_3(x, y, z) &= \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x, 2^k y, 2^k z) < \infty, \\
 \text{(d)} \quad \Phi_4(x, y, z) &= \sum_{k=1}^{\infty} 2^{k-1} \varphi\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right) < \infty
 \end{aligned}$$

for all $x, y, z \in X$. One of the conditions (a), (b) will be needed to derive a quadratic function and one of the conditions (c), (d) will be needed to derive an additive function in the following theorem.

THEOREM 3.1. *If the function $f : X \rightarrow Y$ satisfies*

$$\begin{aligned}
 &\|m^2 f\left(\frac{x+y+z}{m}\right) + f(x) + f(y) + f(z) \\
 &\quad - n^2\left[f\left(\frac{x+y}{n}\right) + f\left(\frac{y+z}{n}\right) + f\left(\frac{z+x}{n}\right)\right]\| \\
 \text{(16)} \quad &\leq \varphi(x, y, z)
 \end{aligned}$$

for all $x, y, z \in X$, then there exist a unique quadratic function $Q : X \rightarrow Y$, a unique additive function $A : X \rightarrow Y$ and a unique element $B \in Y$ such that

$$\begin{aligned}
 \|f(x) - Q(x) - A(x) - B\| &\leq \epsilon_i(x) + \delta_i(x), \\
 \left\| \frac{f(x) + f(-x)}{2} - Q(x) - B \right\| &\leq \epsilon_i(x),
 \end{aligned}$$

and

$$\left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| \leq \delta_i(x)$$

for all $x \in X$, $i = 1$ or 2 , where

$$\begin{aligned}\epsilon_1(x) &= \frac{1}{4}[\Phi_1(x, -x, 0) + \Phi_1(-x, x, 0)] + \frac{\varphi(0, 0, 0)}{2n^2(n^2 - 1)}, \\ \epsilon_2(x) &= \frac{1}{2}[\Phi_2(x, 0, 0) + \Phi_2(-x, 0, 0) + \Phi_2(x, -x, 0) + \Phi_2(-x, x, 0)] \\ &\quad + \frac{\varphi(0, 0, 0)}{2m^2(m^2 - 1)}, \\ \delta_1(x) &= \frac{1}{4n^2}[\Phi_3(nx, 0, 0) + \Phi_3(-nx, 0, 0) + \Phi_3(nx, nx, -nx) \\ &\quad + \Phi_3(-nx, -nx, nx)], \\ \delta_2(x) &= \frac{1}{2n^2}[\Phi_4\left(\frac{nx}{2}, 0, 0\right) + \Phi_4\left(\frac{-nx}{2}, 0, 0\right) + \Phi_4\left(\frac{nx}{2}, \frac{nx}{2}, \frac{-nx}{2}\right) \\ &\quad + \Phi_4\left(\frac{-nx}{2}, \frac{-nx}{2}, \frac{nx}{2}\right)]\end{aligned}$$

for all $x \in X$. The function Q , A and the element B are given by

$$\begin{aligned}Q(x) &= \begin{cases} \lim_{k \rightarrow \infty} \frac{f(n^k x) + f(-n^k x)}{2 \cdot n^{2k}} & \text{if } \varphi \text{ satisfies (a),} \\ \lim_{k \rightarrow \infty} \frac{f(m^k x) + f(-m^k x)}{2 \cdot m^{2k}} & \text{if } \varphi \text{ satisfies (b),} \end{cases} \\ A(x) &= \begin{cases} \lim_{k \rightarrow \infty} \frac{f(2^k x) + f(-2^k x)}{2 \cdot 2^k} & \text{if } \varphi \text{ satisfies (c),} \\ \lim_{k \rightarrow \infty} 2^{k-1} [f(\frac{x}{2^k}) - f(\frac{-x}{2^k})] & \text{if } \varphi \text{ satisfies (d),} \end{cases}\end{aligned}$$

for all $x \in X$ and $B = f(0)$.

Proof. Let $f_1 : X \rightarrow Y$ be the function defined by $f_1(x) := \frac{(f(x) + f(-x))}{2} - f(0)$ for all $x \in X$. Then $f_1(0) = 0$, $f_1(x) = f_1(-x)$, and

$$\begin{aligned}(17) \quad & \left\| m^2 f_1\left(\frac{x+y+z}{m}\right) + f_1(x) + f_1(y) + f_1(z) \right. \\ & \quad \left. - n^2 \left[f_1\left(\frac{x+y}{n}\right) + f_1\left(\frac{y+z}{n}\right) + f_1\left(\frac{z+x}{n}\right) \right] \right\| \\ & \leq \frac{1}{2} [\varphi(x, y, z) + \varphi(-x, -y, -z)] + \varphi(0, 0, 0)\end{aligned}$$

for all $x, y, z \in X$. Putting $z = 0$ and $y = -x$ in (17) and dividing by 2 yields

$$\begin{aligned}(18) \quad & \left\| f_1(x) - n^2 f_1\left(\frac{x}{n}\right) \right\| \\ & \leq \frac{1}{2^2} [\varphi(x, -x, 0) + \varphi(-x, x, 0)] + \frac{1}{2} \varphi(0, 0, 0).\end{aligned}$$

Replacing x by nx in (18) and dividing by n^2 we have

$$(19) \quad \begin{aligned} & \left\| f_1(x) - \frac{f_1(nx)}{n^2} \right\| \\ & \leq \frac{1}{4n^2} [\varphi(nx, -nx, 0) + \varphi(-nx, nx, 0)] + \frac{1}{2n^2} \varphi(0, 0, 0) \end{aligned}$$

for all $x \in X$. Assume that φ satisfies the condition (a). Replacing x by $n^{k-1}x$ and dividing by $(n^2)^{k-1}$ in (19) we have

$$(20) \quad \begin{aligned} & \left\| \frac{f_1(n^{k-1}x)}{(n^2)^{k-1}} - \frac{f_1(n^kx)}{(n^2)^k} \right\| \\ & \leq \frac{1}{4n^{2k}} [\varphi(n^kx, n^kx, 0) + \varphi(-n^kx, n^kx, 0)] + \frac{1}{2n^{2k}} \varphi(0, 0, 0) \end{aligned}$$

for all $k \in N$ and $x \in X$. An induction argument implies

$$(21) \quad \begin{aligned} & \left\| f_1(x) - \frac{f_1(n^kx)}{n^{2k}} \right\| \\ & \leq \frac{1}{4} \sum_{i=1}^k \frac{1}{n^{2i}} [\varphi(n^i x, -n^i x, 0) + \varphi(-n^i x, n^i x, 0)] \\ & \quad + \frac{\varphi(0, 0, 0)}{2} \sum_{i=1}^k \frac{1}{n^{2i}} \end{aligned}$$

for all $k \in N$ and $x \in X$. Hence

$$(22) \quad \begin{aligned} & \left\| \frac{f_1(n^kx)}{n^{2k}} - \frac{f_1(n^lx)}{n^{2l}} \right\| \\ & \leq \frac{1}{4} \sum_{i=l+1}^k \frac{1}{n^{2i}} [\varphi(n^i x, -n^i x, 0) + \varphi(-n^i x, n^i x, 0)] \\ & \quad + \frac{\varphi(0, 0, 0)}{2} \sum_{i=l+1}^k \frac{1}{n^{2i}} \end{aligned}$$

for all $k, l \in N$ with $k > l$ and $x \in X$. This shows that $\left\{ \frac{f_1(n^kx)}{n^{2k}} \right\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $Q : X \rightarrow Y$ by

$$Q(x) = \lim_{k \rightarrow \infty} \frac{f_1(n^kx)}{n^{2k}} \quad \text{for all } x \in X.$$

Then $Q(0) = 0$, $Q(-x) = Q(x)$, and $Q(nx) = n^2Q(x)$ for all $x \in X$. By (17) we have

$$\begin{aligned} & \left\| m^2 Q\left(\frac{x+y+z}{m}\right) + Q(x) + Q(y) + Q(z) \right. \\ & \quad \left. - n^2 \left[Q\left(\frac{x+y}{n}\right) + Q\left(\frac{y+z}{n}\right) + Q\left(\frac{z+x}{n}\right) \right] \right\| \\ & \leq \frac{1}{2} \lim_{k \rightarrow \infty} \left[\frac{1}{n^{2k}} [\varphi(n^k x, n^k y, n^k z) + \varphi(-n^k x, -n^k y, -n^k z)] \right. \\ & \quad \left. + \frac{1}{n^{2k}} \varphi(0, 0, 0) \right] \\ & = 0 \end{aligned}$$

Thus we get

$$\begin{aligned} & m^2 Q\left(\frac{x+y+z}{m}\right) + Q(x) + Q(y) + Q(z) \\ & = n^2 \left[Q\left(\frac{x+y}{n}\right) + Q\left(\frac{y+z}{n}\right) + Q\left(\frac{z+x}{n}\right) \right] \end{aligned}$$

for all $x, y, z \in Y$. By the same method as that of the proof of Theorem 2.1, we have

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in X$ and so Q is quadratic. Taking the limit in (21) as $k \rightarrow \infty$, we have

$$(23) \quad \begin{aligned} & \|f_1(x) - Q(x)\| \\ & \leq \frac{1}{4} [\Phi_1(x, -x, 0) + \Phi_1(-x, x, 0)] + \frac{\varphi(0, 0, 0)}{2n^2(n^2 - 1)} \end{aligned}$$

for all $x \in X$.

If Q' is another quadratic function satisfying (23), then $Q'(0) = 0$, $Q'(2x) = 4Q'(x)$, and $Q'(-x) = Q'(x)$ for all $x \in X$. Replacing y by $2x$ in $Q'(x+y) + Q'(x-y) = 2Q'(x) + 2Q'(y)$ we have $Q'(3x) + Q'(-x) = 2Q'(x) + 2Q'(2x)$ and so $Q'(3x) = 9Q'(x)$ for all $x \in X$. An induction argument implies $Q'(kx) = k^2Q'(x)$ for all $k \in N$.

Thus we have

$$\begin{aligned}
& \|Q(x) - Q'(x)\| \\
& \leq \left\| \frac{Q(n^k x)}{n^{2k}} - \frac{f_1(n^k x)}{n^{2k}} \right\| + \left\| \frac{f_1(n^k x)}{n^{2k}} - \frac{Q'(n^k x)}{n^{2k}} \right\| \\
& \leq \frac{1}{2n^{2k}} [\Phi_1(n^k x, -n^k x, 0) + \Phi_1(-n^k x, n^k x, 0)] + \frac{\varphi(0, 0, 0)}{n^2(n^2 - 1)n^{2k}} \\
& = \frac{1}{2} \sum_{i=k+1}^{\infty} \frac{1}{n^{2k}} [\varphi(n^i x, -n^i x, 0) + \varphi(-n^i x, n^i x, 0)] + \frac{\varphi(0, 0, 0)}{n^2(n^2 - 1)n^{2k}}
\end{aligned}$$

for all $k \in N$ and $x \in X$. Therefore we can conclude that $Q(x) = Q'(x)$ for all $x \in X$.

Assume that φ satisfies condition (b). Putting $y = z = 0$ in (17)

$$\begin{aligned}
& \left\| m^2 f_1\left(\frac{x}{m}\right) + f_1(x) - 2n^2 f_1\left(\frac{x}{n}\right) \right\| \\
(24) \quad & \leq \frac{1}{2} [\varphi(x, 0, 0) + \varphi(-x, 0, 0)] + \varphi(0, 0, 0)
\end{aligned}$$

for all $x \in X$. Hence

$$\begin{aligned}
& \left\| m^2 f_1\left(\frac{x}{m}\right) - f_1(x) \right\| \\
& \leq \left\| m^2 f_1\left(\frac{x}{m}\right) + f_1(x) - 2n^2 f_1\left(\frac{x}{n}\right) \right\| + 2 \left\| f_1(x) - n^2 f_1\left(\frac{x}{n}\right) \right\| \\
(25) \quad & \leq \frac{1}{2} [\varphi(x, 0, 0) + \varphi(-x, 0, 0) + \varphi(x, -x, 0) + \varphi(-x, x, 0)] \\
& \quad + \frac{1}{2} \varphi(0, 0, 0)
\end{aligned}$$

for all $x \in X$. Replacing x by mx in (25) and dividing by m^2 we have

$$\begin{aligned}
& \left\| f_1(x) - \frac{f_1(mx)}{m^2} \right\| \\
& \leq \frac{1}{2m^2} [\varphi(mx, 0, 0) + \varphi(-mx, 0, 0) + \varphi(mx, -mx, 0) + \varphi(-mx, mx, 0)] \\
& \quad + \frac{1}{2m^2} \varphi(0, 0, 0)
\end{aligned}$$

for all $x \in X$. By the same proof as that of the case (a), we can define a function $Q : X \rightarrow Y$ by

$$Q(x) = \lim_{k \rightarrow \infty} \frac{f_1(m^k x)}{m^{2k}} \quad \text{for all } x \in X.$$

and also we easily have that Q is a unique quadratic function such that

$$\begin{aligned}
 & \|f_1(x) - Q(x)\| \\
 (27) \quad & \leq \frac{1}{2}[\Phi_2(x, 0, 0) + \Phi_2(-x, 0, 0) + \Phi_2(x, -x, 0) + \Phi_2(-x, x, 0)] \\
 & \quad + \frac{\varphi(0, 0, 0)}{2m^2(m^2 - 1)}
 \end{aligned}$$

for all $x \in X$.

Now let $f_2 : X \rightarrow Y$ be the function defined by $f_2(x) := \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$. Then $f_2(0) = 0$, $f_2(-x) = -f_2(x)$ and

$$\begin{aligned}
 & \|m^2 f_2\left(\frac{x+y+z}{m}\right) + f_2(x) + f_2(y) + f_2(z) \\
 (28) \quad & \quad - n^2 \left[f_2\left(\frac{x+y}{n}\right) + f_2\left(\frac{y+z}{n}\right) + f_2\left(\frac{z+x}{n}\right) \right] \| \\
 & \leq \frac{1}{2}[\varphi(x, y, z) + \varphi(-x, -y, -z)]
 \end{aligned}$$

for all $x, y, z \in X$. Putting $y = z = 0$ in (28) we get

$$\begin{aligned}
 & \|m^2 f_2\left(\frac{x}{m}\right) + f_2(x) - 2n^2 f_2\left(\frac{x}{n}\right)\| \\
 (29) \quad & \leq \frac{1}{2}[\varphi(x, 0, 0) + \varphi(-x, 0, 0)]
 \end{aligned}$$

for all $x \in X$. Putting $y = x$ and $z = -x$ in (28) we get

$$\begin{aligned}
 & \|m^2 f_2\left(\frac{x}{m}\right) + f_2(x) - n^2 f_2\left(\frac{2x}{n}\right)\| \\
 (30) \quad & \leq \frac{1}{2}[\varphi(x, x, -x) + \varphi(-x, -x, x)]
 \end{aligned}$$

for all $x \in X$. By (29) and (30) we have

$$\begin{aligned}
 & \|2n^2 f_2\left(\frac{x}{n}\right) - n^2 f_2\left(\frac{2x}{n}\right)\| \\
 (31) \quad & \leq \frac{1}{2}[\varphi(x, 0, 0) + \varphi(-x, 0, 0) + \varphi(x, x, -x) + \varphi(-x, -x, x)]
 \end{aligned}$$

for all $x \in X$. Assume that φ satisfies condition (c). Replacing x by nx and dividing by $2n^2$ in (31) we have

$$\begin{aligned}
 & \left\| f_2(x) - \frac{f_2(2x)}{2} \right\| \\
 (32) \quad & \leq \frac{1}{4n^2}[\varphi(nx, 0, 0) + \varphi(-nx, 0, 0) \\
 & \quad + \varphi(nx, nx, -nx) + \varphi(-nx, -nx, nx)]
 \end{aligned}$$

for all $x \in X$. Replacing x by $2^{k-1}x$ and dividing by 2^{k-1} we obtain

$$(33) \quad \begin{aligned} & \left\| \frac{f_2(2^{k-1}x)}{2^{k-1}} - \frac{f_2(2^k x)}{2^k} \right\| \\ & \leq \frac{1}{4n^2 2^{k-1}} [\varphi(2^{k-1}nx, 0, 0) + \varphi(-2^{k-1}x, 0, 0) \\ & \quad + \varphi(2^{k-1}nx, 2^{k-1}nx, -2^{k-1}nx) \\ & \quad + \varphi(-2^{k-1}nx, -2^{k-1}nx, 2^{k-1}nx)] \end{aligned}$$

for all $x \in X$. An induction argument implies

$$(34) \quad \begin{aligned} & \left\| f_2(x) - \frac{f_2(2^k x)}{2^k} \right\| \\ & \leq \frac{1}{4n^2} \sum_{i=l}^{k-1} \frac{1}{2^i} [\varphi(2^i nx, 0, 0) + \varphi(-2^i nx, 0, 0) \\ & \quad + \varphi(2^i nx, 2^i nx, -2^i nx) + \varphi(-2^i nx, -2^i nx, 2^i nx)] \end{aligned}$$

for all $x \in X$ and $k \in N$. Hence

$$\begin{aligned} & \left\| \frac{f_2(2^k x)}{2^k} - \frac{f_2(2^l x)}{2^l} \right\| \\ & \leq \frac{1}{4n^2} \sum_{i=l}^{k-1} \frac{1}{2^i} [\varphi(2^i nx, 0, 0) + \varphi(-2^i nx, 0, 0) \\ & \quad + \varphi(2^i nx, 2^i nx, -2^i nx) + \varphi(-2^i nx, -2^i nx, 2^i nx)] \end{aligned}$$

for all $k, l \in N$ with $k > l$ and $x \in X$. This shows that $\{\frac{f_2(2^k x)}{2^k}\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $A : X \rightarrow Y$ by

$$A(x) = \lim_{n \rightarrow \infty} \frac{f_2(2^n x)}{2^n} \quad \text{for all } x \in X.$$

Note that $A(0) = 0$, $A(-x) = -A(x)$, and $A(2x) = 2A(x)$ for all $x \in X$. By (28) we have

$$\begin{aligned} & \left\| m^2 A\left(\frac{x+y+z}{m}\right) + A(x) + A(y) + A(z) \right. \\ & \quad \left. - n^2 \left[A\left(\frac{x+y}{n}\right) + A\left(\frac{y+z}{n}\right) + A\left(\frac{z+x}{n}\right) \right] \right\| \\ & \leq \frac{1}{2^k} [\varphi(2^k x, 2^k y, 2^k z) + \varphi(-2^k x, -2^k y, -2^k z)] \\ & = 0 \end{aligned}$$

Thus we get

$$\begin{aligned} m^2 A\left(\frac{x+y+z}{m}\right) + A(x) + A(y) + A(z) \\ = n^2 \left[A\left(\frac{x+y}{n}\right) + A\left(\frac{y+z}{n}\right) + A\left(\frac{z+x}{n}\right) \right] \end{aligned}$$

for all $x, y, z \in X$. By the same method as the that of the proof of Theorem 2.1, we have $A(x+y) = A(x) + A(y)$ for all $x, y \in X$ and so A is additive.

Taking the limit in (34) as $k \rightarrow \infty$ we obtain

$$\begin{aligned} (35) \quad & \|f_2(x) - A(x)\| \\ & \leq \frac{1}{4n^2} [\Phi_3(nx, 0, 0) + \Phi_3(-nx, 0, 0) + \Phi_3(nx, nx, -nx) \\ & \quad + \Phi_3(-nx, -nx, nx)] \end{aligned}$$

for all $x \in X$. If A' is another additive mapping satisfying (35) then we have

$$\begin{aligned} & \|A(x) - A'(x)\| \\ & \leq \left\| \frac{A(2^k x)}{2^k} - \frac{f_2(2^k x)}{2^k} \right\| + \left\| \frac{f_2(2^k x)}{2^k} - \frac{A'(2^k x)}{2^k} \right\| \\ & \leq \frac{1}{2n^2 \cdot 2^k} [\Phi_3(2^k nx, 0, 0) + \Phi_3(-2^k nx, 0, 0) \\ & \quad + \Phi_3(2^k nx, 2^k nx, -2^k nx) + \Phi_3(-2^k nx, -2^k nx, 2^k nx)] \\ & = \frac{1}{2n^2} \sum_{i=k}^{\infty} \frac{1}{2^i} [\varphi(2^i nx, 0, 0) + \varphi(-2^i nx, 0, 0) \\ & \quad + \varphi(2^i nx, 2^i nx, -2^i nx) + \varphi(-2^i nx, -2^i nx, 2^i nx)] \end{aligned}$$

for all $k \in \mathbb{N}$ and $x \in X$. Therefore we can conclude that $A(x) = A'(x)$ for all $x \in X$.

Assume that φ satisfies that the condition (d). Dividing by n^2 and replacing x by $\frac{nx}{2}$ in (31) we have

$$\begin{aligned} (36) \quad & \|2f_2\left(\frac{x}{2}\right) - f_2(x)\| \\ & \leq \frac{1}{2n^2} [\varphi\left(\frac{nx}{2}, 0, 0\right) + \varphi\left(-\frac{nx}{2}, 0, 0\right) \\ & \quad + \varphi\left(\frac{nx}{2}, \frac{nx}{2}, -\frac{nx}{2}\right) + \varphi\left(-\frac{nx}{2}, -\frac{nx}{2}, \frac{nx}{2}\right)] \end{aligned}$$

for all $x \in X$. Replacing x by $\frac{x}{2^{k-1}}$ and multiplying 2^{k-1} in (36) we have

$$(37) \quad \begin{aligned} & \|2^k f_2(\frac{x}{2^k}) - 2^{k-1} f_2(\frac{x}{2^{k-1}})\| \\ & \leq \frac{2^{k-1}}{2n^2} [\varphi(\frac{nx}{2^k}, 0, 0) + \varphi(-\frac{nx}{2^k}, 0, 0) \\ & \quad + \varphi(\frac{nx}{2^k}, \frac{nx}{2^k}, -\frac{nx}{2^k}) + \varphi(-\frac{nx}{2^k}, -\frac{nx}{2^k}, \frac{nx}{2^k})] \end{aligned}$$

for all $k \in N$ and $x \in X$. An induction argument implies

$$(38) \quad \begin{aligned} & \|2^k f_2(\frac{x}{2^k}) - f_2(x)\| \\ & \leq \frac{1}{2n^2} \sum_{i=1}^k 2^{i-1} [\varphi(\frac{nx}{2^i}, 0, 0) + \varphi(-\frac{nx}{2^i}, 0, 0) \\ & \quad + \varphi(\frac{nx}{2^i}, \frac{nx}{2^i}, -\frac{nx}{2^i}) + \varphi(-\frac{nx}{2^i}, -\frac{nx}{2^i}, \frac{nx}{2^i})] \end{aligned}$$

for all $i \in N$ and $x \in X$. Hence

$$\begin{aligned} & \|2^k f_2(\frac{x}{2^k}) - 2f_2(\frac{x}{2^l})\| \\ & \leq \frac{1}{2n^2} \sum_{i=l+1}^k 2^{i-1} [\varphi(\frac{nx}{2^i}, 0, 0) + \varphi(-\frac{nx}{2^i}, 0, 0) \\ & \quad + \varphi(\frac{nx}{2^i}, \frac{nx}{2^i}, -\frac{nx}{2^i}) + \varphi(-\frac{nx}{2^i}, -\frac{nx}{2^i}, \frac{nx}{2^i})] \end{aligned}$$

for all $k, l \in N$ with $k > l$ and $x \in X$. This shows that $\{2^k f_2(\frac{x}{2^k})\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $A : X \rightarrow Y$ by

$$A(x) = \lim_{k \rightarrow \infty} 2^k f_2(\frac{x}{2^k}) \quad \text{for all } x \in X.$$

Then $A(0) = 0$, $A(-x) = -A(x)$, $A(2x) = 2A(x)$ for all $x \in X$. By the same proof as that of case (c) we have

$$\begin{aligned} & m^2 A(\frac{x+y+z}{m}) + A(x) + A(y) + A(z) \\ & = n^2 [A(\frac{x+y}{n}) + A(\frac{y+z}{n}) + A(\frac{z+x}{n})] \end{aligned}$$

for all $x, y, z \in X$, and so A is additive. Taking the limit in (38) we obtain

$$\begin{aligned} & \|f_2(x) - A(x)\| \\ & \leq \frac{1}{2n^2} [\Phi_4(\frac{nx}{2}, 0, 0) + \Phi_4(-\frac{nx}{2}, 0, 0) + \Phi_4(\frac{nx}{2}, \frac{nx}{2}, -\frac{nx}{2}) \\ & \quad + \Phi_4(-\frac{nx}{2}, -\frac{nx}{2}, \frac{nx}{2})] \end{aligned}$$

for all $x \in X$. Also we easily have that A is unique. Let $B = f(0)$. Since $f(x) = f_1(x) + f_2(x) + f(0)$ for all $x \in X$, it follows that

$$\begin{aligned} \|f(x) - Q(x) - A(x) - B\| & \leq \|f_1(x) - Q(x)\| + \|f_2(x) - A(x)\| \\ & \leq \epsilon_i(x) + \delta_i(x) \end{aligned}$$

for all $x \in X$, $i = 1$ or 2 , and $j = 1$ or 2 . We complete the proof. \square

REMARK 3.2. If $m^2 + 3 = 3n^2$ in Theorem 3.1, then $m = 3$ and $n = 2$. In this case, the equation (1) equals the equation (2) and thus the stability of this equation is proved by Theorem 3.1 in [16].

COROLLARY 3.3. Let $p \in (0, 1) \cup (1, 2)$ and $\theta > 0$. Suppose that the function $f : X \rightarrow Y$ satisfies

$$\begin{aligned} & \|m^2 f(\frac{x+y+z}{m}) + f(x) + f(y) + f(z) \\ & \quad - n^2 [f(\frac{x+y}{n}) + f(\frac{y+z}{n}) + f(\frac{z+x}{n})]\| \\ & \leq \theta (\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned}$$

for all $x, y, z \in X$. Then there exist a unique quadratic function $Q : X \rightarrow Y$, a unique additive function $A : X \rightarrow Y$, and a unique element $B \in Y$ such that

$$\begin{aligned} & \|f(x) - Q(x) - A(x) - B\| \\ & \leq \theta \|x\|^p \cdot (\min\{\frac{n^p}{n^2 - np}, \frac{3m^p}{m^2 - mp}\} + \frac{4}{n^2 |2^p - 2|}) \\ & \quad \cdot \|\frac{f(x) - f(-x)}{2} - Q(x) - B\| \\ & \leq \theta \|x\|^p \cdot \min\{\frac{n^p}{n^2 - np}, \frac{3m^p}{m^2 - mp}\} \end{aligned}$$

and

$$\left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| \leq \frac{4}{n^2} \theta \|x\|^p \frac{1}{|2^p - 2|}$$

for all $x \in X$.

Proof. Let $\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$. Then

$$\varphi(0, 0, 0) = 0, \quad \varphi(sx, sx, -sx) = \varphi(-sx, -sx, sx) = 3\theta s^p \|x\|^p$$

for all $x \in X$ and for some $s \in N$. If $0 < p < 2$ then we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{s^{2k}} \varphi(s^k x, s^k y, s^k z) &= \sum_{k=1}^{\infty} 3\theta \|x\|^p s^{k(p-2)} \\ &= 3\theta \|x\|^p \frac{sp}{s^2 - sp} \end{aligned}$$

for all $x \in X$ and for some integer s . If $0 < p < 1$, then we have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x, 2^k y, 2^k z) &= \sum_{k=0}^{\infty} 3\theta \|x\|^p 2^{k(p-1)} \\ &= 3\theta \|x\|^p \frac{2}{2 - 2p} \end{aligned}$$

for all $x \in X$ and for some integer s . If $1 < p < 2$, then we have

$$\begin{aligned} \sum_{k=1}^{\infty} 2^{k-1} \varphi\left(\frac{x}{2^k}, \frac{x}{2^k}, \frac{x}{2^k}\right) &= \sum_{k=1}^{\infty} \frac{3}{2} \theta \|x\|^p 2^{k(1-p)} \\ &= \frac{3}{2} \theta \|x\|^p \frac{2}{2p - 2} \end{aligned}$$

for all $x \in X$. Thus

$$\begin{aligned} \epsilon_1(x) &= \theta \|x\|^p \frac{n^p}{n^2 - n^p} \quad \text{if } 0 < p < 2, \\ \epsilon_2(x) &= 3\theta \|x\|^p \frac{m^p}{m^2 - m^p} \quad \text{if } 0 < p < 2, \\ \delta_1(x) &= \frac{2}{n^2} \theta \|x\|^p \frac{2^p}{2 - 2^p} \quad \text{if } 0 < p < 1, \\ \delta_2(x) &= \frac{2}{n^2} \theta \|x\|^p \frac{2}{2^p - 2} \quad \text{if } 0 < p < 1 \end{aligned}$$

for all $x \in X$. □

COROLLARY 3.4. *Let $\theta > 0$ be a real number. If the function $f : X \rightarrow Y$ satisfies*

$$\begin{aligned} & \|m^2 f(\frac{x+y+z}{m}) + f(x) + f(y) + f(z) \\ & \quad - n^2 [f(\frac{x+y}{n}) + f(\frac{y+z}{n}) + f(\frac{z+x}{n})]\| \\ & \leq \theta \end{aligned}$$

for all $x, y, z \in X$, then there exist a unique quadratic function $Q : X \rightarrow Y$, a unique additive function $A : X \rightarrow Y$ and a unique element $B \in Y$ such that

$$\begin{aligned} & \|f(x) - Q(x) - A(x) - B\| \\ & \leq \theta \cdot (\min\{\frac{n^2 + 1}{2(n^2 - 1)n^2}, \frac{4m^2 + 1}{2(m^2 - 1)m^2}\} + \frac{1}{n^2}) \\ & \quad \|\frac{f(x) - f(-x)}{2} - Q(x) - B\| \\ & \leq \theta \cdot \min\{\frac{n^2 + 1}{2(n^2 - 1)n^2}, \frac{4m^2 + 1}{2(m^2 - 1)m^2}\} \end{aligned}$$

and

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \leq \frac{2\theta}{n^2}$$

□

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DEPARTMENT OF COMPUTER AND INFORMATION SECURITY, DAEJEON UNIVERSITY,
DAEJEON 300-716, KOREA
E-mail: ywlee@dju.ac.kr