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ON EXCHANGE *qb*-IDEALS

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ABSTRACT. In this paper, we establish necessary and sufficient conditions for an exchange ideal to be a *qb*-ideal. It is shown that an exchange ideal I of a ring R is a *qb*-ideal if and only if whenever $a\widetilde{\approx}b$ via I, there exists $u \in I_q^{-1}$ such that $a = ubu_q^{-1}$ and $b = u_q^{-1}au$. This gives a generalization of the corresponding result of exchange QB-rings.

In [3], Ara et al. discovered a new class of rings, the QB-rings. We say that R is a QB-ring if $aR + bR = R$ with $a, b \in R$ implies that $a + by \in R_q^{-1}$ for a $y \in R$. An ideal I of a ring R is a QB-ideal in case $aR+bR = R$ with $a \in 1+I, b \in R$ implies that $a+by \in R_q^{-1}$ for a $y \in R$, where $R_q^{-1} = \{u \in R \mid \exists a, b \in R \text{ such that } (1-ua)\perp(1-bu)\}$. In this paper, we introduce the notation of *qb*-ideal as a natural generalization of that of QB-ideal. Let I be an ideal of a ring R. We say that I is a *qb*-ideal of R in case $aR + bR = R$ with $a \in I, b \in R$ implies that $a + by \in I_q^{-1}$ for a $y \in R$. Recall that R is an exchange ring if for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus_{i\in I} A_i$, where $M_R \cong R$ and the index set I is finite, then there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus (\bigoplus_{i\in I} A_i')$. An ideal I of a ring R is an exchange ideal provided that for every $x \in I$ there exist an idempotent $e \in I$ and elements $r, s \in I$ such that $e = xr = x + s - xs$ (cf. [1] and [11]). Clearly, every ideal of an exchange ring is an exchange ideal.

The main purpose of this paper is to establish necessary and sufficient conditions for an exchange ideal of a ring to be a *qb*-ideal. We prove that an exchange ideal I of a ring R is a *qb*-ideal if and only if whenever $a\widetilde{\approx}b$

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via I, there exists $u \in I_q^{-1}$ such that $a = ubu_q^{-1}$ and $b = u_q^{-1}au$. This gives a generalization of the corresponding result of exchange QB-rings.

Throughout, all rings are associative with identity. An element $x \in R$ is regular provided that $x = xyx$ for a $y \in R$. We say that $x, y \in R$ are centrally orthogonal, in symbols $x \perp y$, if $xRy = 0$ and $yRx = 0$. We use R_q^{-1} to denote $\{u \in R \mid \exists a, b \in R \text{ such that } (1-ua)\perp(1-bu)\}$. If $u \in R_q^{-1}$, we use u_q^{-1} to denote some fixed $v \in R$ with $(1-uv)\perp(1-vu)$. Set $I_q^{-1} = R_q^{-1} \cap (1+I)$.

LEMMA 1. *Let I be an exchange ideal of a ring R. Then the following are equivalent:*

(1) I is a qb-ideal.

(2) Whenever $ax + b = 1$ with $a \in I, x, b \in R$, there exists $y \in R$ such that $a + by \in I_q^{-1}$.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Suppose that $aR + bR = R$ with $a \in I, b \in 1 + I$. Then $ax + by = 1$ for some $x, y \in R$. So we have $z \in R$ such that $a + byz \in I_q^{-1}$, as required. □

THEOREM 2. *Let I be an exchange ideal of a ring R. Then the following are equivalent:*

(1) I is a qb-ideal.

(2) For any regular $x \in I$, there exists $u \in I_q^{-1}$ such that $x = xux$.

Proof. (1) $\Rightarrow$ (2) Given any regular $x \in I$, we have a $y \in R$ such that $x = xyx$. Hence $x = xzx$ and $z = yxy \in I$. From $zx + (1 - zx) = 1$ with $z \in I$, we can find a $w \in R$ such that $z + (1 - zx)w = u \in I_q^{-1}$. Therefore $x = xzx = x\big(z + (1 - zx)\big)x = xux$.

(2) $\Rightarrow$ (1) Suppose that $ax + b = 1$ with $a \in I, x, b \in R$. Then $b \in 1 + I$. Since I is an exchange ideal of R, there exists an idempotent $e \in R$ such that $e = bs$ and $1-e = (1-b)t$ for some $s, t \in R$ by [1,Lemma 1.1]. So $(1 - e)axt + e = 1$; hence, $(1 - e)axt(1 - e)a = (1 - e)a$. This infers that $(1 - e)a \in I$ is regular, and then we have a $u \in I_q^{-1}$ such that $(1 - e)a = (1 - e)au(1 - e)a$. Set $u(1 - e)a = f$. Clearly, $f \in I$ and $fxt + ue = u$, so $f(x + ue) + (1 - f)ue = u$. As $u \in R_q^{-1}$, we have $a, b \in R$ such that $(1 - ua)\perp(1 - bu) = 0$. Then we can take $u_q^{-1} = a = b - aub$. Let $g = (1 - f)ueu_q^{-1}(1 - f)$. From $(1 - f)ue = (1 - f)u$, we have $(1 - f)ue = (1 - f)uu_q^{-1}(1 - f)u = g(1 - f)u$ and $g^2 = (1 - f)uu_q^{-1}(1 - f)uu_q^{-1}(1 - f) = g$. Clearly, $fg = gf = 0$; hence, $f(x + ue) = fu$ and $g(1 - f)u = gu$. Therefore $(f + g)u = u$. That

is, $u\big(a+bs(v(1-f)-a)\big)\big(1+fuev(1-f)\big)u=u\big((1-e)a+bsv(1-f)\big)\big(1+fuev(1-f)\big)u=\big(u(1-e)a+ueu_q^{-1}(1-f)\big)\big(1+fueu_q^{-1}(1-f)\big)u=\big(f+ueu_q^{-1}(1-f)(1-fueu_q^{-1}(1-f))\big)\big(1+fueu_q^{-1}(1-f)\big)u=\big(f+(1-f)ueu_q^{-1}(1-f)\big)u=u$. Let $y=s\big(u_q^{-1}(1-f)-a\big)$ and $w=\big(1+fueu_q^{-1}(1-f)\big)u$. Then $w(a+by)w=w$ with $w\in R_q^{-1}$. As $u\in 1+I$, we deduce that $w\in 1+I$. On the other hand, $b\in 1+I$, we have $y\in 1+I$. Therefore $a+by\in 1+I$. Similarly to [6, Theorem 1], we deduce that $a+by\in R_q^{-1}$, as required. □

COROLLARY 3. *I be an exchange ideal of a ring R. Then the following are equivalent:*

(1) I is a qb-ideal.

(2) For any regular $x\in I$, there exists $u\in I_q^{-1}$ such that $ux\in I$ is an idempotent.

Proof. (1) $\Rightarrow$ (2) is clear by Theorem 2.

(2) $\Rightarrow$ (1) Suppose that $ax+b=1$ with $a\in I, x,b\in R$. Since I is an exchange ideal of R, by [1, Lemma 1.1], we have an idempotent $e\in 1+I$ such that $e=bs$ and $1-e=(1-b)t$ for some $s,t\in R$. Analogously to Theorem 2, we claim that $(1-e)a\in I$ is regular. Hence there exists $u\in I_q^{-1}$ such that $u(1-e)a=f$ is an idempotent of R. So $fxt+ue=u$, whence $f(x+ue)+(1-f)ue=u$. Let $g=(1-f)ueu_q^{-1}(1-f)$. Similarly to Theorem 2, we have $w(a+by)w=w$ with $w\in I_q^{-1}$, where $y=s\big(u_q^{-1}(1-f)-a\big)$ and $w=\big(1+fueu_q^{-1}(1-f)\big)u$. Clearly, $y\in 1+I$, and then $a+by\in 1+I$. Therefore $a+by\in I_q^{-1}$. It follows by Lemma 1 that I is a qb-ideal. □

COROLLARY 4. *Let I be an exchange ideal of a ring R. Then the following are equivalent:*

(1)I is a qb-ideal.

(2)Whenever $Ra+Rb=R$ with $a\in I, b\in R$ implies that $a+zb\in I_q^{-1}$ for a $z\in R$.

(3)Whenever $ax+b=1$ with $a,x\in I, b\in R$, there exists $z\in R$ such that $x+zb\in I_q^{-1}$.

Proof. (1) $\Rightarrow$ (2) In view of Theorem 2, I^{op} is a qb-deal of R^{op}. Therefore $Ra+Rb=R$ with $a\in I, b\in R$ implies that $a+zb\in I_q^{-1}$ for a $z\in R$. (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (1) Given any regular $x\in I$, analogously to Theorem 2, we have a $y\in I$ such that $x=xyx$ and $y=yxy$. From $xy+(1-xy)=1$, there exists a $z\in R$ such that

$y + z(1 - xy) = u \in I_q^{-1}$. Then $x = xyx = x\big(y + z(1 - xy)\big)x = xux$. It follows by Theorem 2 that I is a qb-ideal. □

THEOREM 5. *Let I be an exchange ideal of a ring R. Then the following are equivalent:*

(1) I is a qb-ideal.

(2) For any regular $x \in I$, there exist idempotent $e \in I$ and $u \in I_q^{-1}$ such that $x = eu$.

Proof. (1) $\Rightarrow$ (2) For any regular $x \in I$, there exists a $y \in R$ such that $x = xyx$ and $y = yxy$. Similarly to Theorem 2, we see that $y \in I$. From $xy + (1 - xy) = 1$, there exists $s \in R$ such that $x + (1 - xy)s = u \in I_q^{-1}$. Set $e = xy$. Then $x = xyx = x\big(x + (1 - xy)s\big) = eu$, as required.

(2) $\Rightarrow$ (1) Suppose that $ax + b = 1$ with $a \in I, x, b \in R$. Since I is an exchange ideal, there exists an idempotent $e \in R$ such that $e = bs$ and $1-e = (1-b)t$ for some $s, t \in R$. Analogously to Theorem 2, $(1-e)a \in I$ is regular. So we have a $u \in R_q^{-1}$ and an idempotent $f \in I$ such that $(1-e)a = fu$. Hence $fuxt + e = 1$. Thus $fuxt(1-f) + e(1-f) = 1-f$, which shows that $a + bs\big((1 - f)u - a\big) = (1 - e)a + e(1 - f)u = fu + e(1 - f)u = \big(1 - fuxt(1 - f)\big)u \in R_q^{-1}$. Since $f \in I$ and $u \in 1 + I$, we have $a + bs\big((1 - f)u - a\big) \in 1 + I$. Thus $a + bs\big((1 - f)u - a\big) \in I_q^{-1}$. According to Lemma 1, we complete the proof. □

COROLLARY 6. *Let I be an exchange ideal of a ring R. Then the following are equivalent:*

(1) I is a qb-ideal.

(2) For any regular $x \in I$, there exist idempotent $e \in I$ and $u \in I_q^{-1}$ such that $x = ue$.

Proof. In view of Theorem 2, I is qb-ideal of R if and only if I^{op} is a qb-ideal of R^{op}. Therefore we get the result by Theorem 5. □

THEOREM 7. *Let I be an ideal of a regular ring. If eRe is a QB-ring for all idempotents $e \in I$, then I is a qb-ideal of R.*

Proof. Given any regular $x \in I$, by [10, Lemma 1.1], there exists an idempotent $e \in I$ such that $x \in eRe$. Since eRe is a regular QB-ring, by Corollary 6, we have an idempotent $efe \in eRe$ and an element $eue \in (eRe)_q^{-1}$ such that $x = (eue)(exe) = (eue + 1 - e)(exe)$. Clearly, $exe \in R$ is an idempotent. As $eue \in (eRe)_q^{-1}$, we have $\big(e - eue(eue)_q^{-1}\big)(eRe)\big(e - (eue)_q^{-1}(eue)\big) = 0$; and then $\big(1 - (eue + 1 - e)((eue)_q^{-1} + 1 - e)\big)R\big(1 -$

$((eue)_q^{-1}+1-e)((eue)+1-e)\big)=0$. Likewise, we have $\big(1-((eue)_q^{-1}+1-e)(eue+1-e)\big)R\big(1-(eue+1-e)((eue)_q^{-1}+1-e)\big)=0$. This means that $eue+1-e\in R_q^{-1}$. Moreover, $eue+1-e\in 1+I$. Therefore we complete the proof by Corollary 6 again. □

Recall that an ideal I of a ring R is strongly π-regular provided that for any $x\in I$ there exist $n(x)\in\mathbb{N}$ and $y\in R$ such that $x^{n(x)}=x^{n(x)}y$.

COROLLARY 8. *Every strongly π-regular ideal of a regular ring is a qb-ideal.*

Proof. Let I be a strongly π-regular ideal of a regular ring R. Given any idempotent $e\in I$, then eRe is a strongly π-regular ring. So eRe has stable range one; hence, eRe is a QB-ring. Therefore we get the result by Theorem 7. □

We say that $a\overset{\sim}{\to}b$ via I if there exist $x,y,z\in I$ such that $a=zbx, b=xay, x=xyx=xzx$.

LEMMA 9. *Let I be an exchange ideal of a ring R. Then the following are equivalent:*

(1) $a\overset{\sim}{\to}b$ via I.

(2) There exist some $x,y\in I$ such that $a=xby, b=yax, x=xyx$ and $y=yxy$.

Proof. (2)⇒(1) is trivial.

(1)⇒(2) Since $a\overset{\sim}{\to}b$ via I, there are $x,y,z\in I$ such that $b=xay, zbx=a$ and $x=xyx=xzx$. By replacing y with yxy and z with zxz, we can assume $y=yxy$ and $z=zxz$. One directly checks that $xazxy=xzbxzxy=xzbxy=xay=b, zxybx=zxyxayx=zxayx=zbx=a, zxy=zxyxzxy$ and $x=xzxyx$. Since $x=xyx=xzx$ and $x\in I$. Clearly, $y,z\in I$. Therefore $zxy\in I$, as required. □

THEOREM 10. *Let I be an exchange ideal of a ring R. Then the following are equivalent:*

(1) I is a qb-ideal.

(2) Whenever $a\overset{\sim}{\to}b$ via I, then there exists $u\in I_q^{-1}$ such that $a=ubu_q^{-1}$ and $b=u_q^{-1}au$.

Proof. (1)⇒(2) Suppose that $a\overset{\sim}{\to}b$ via I. By Lemma 9, there exist $x,y\in I$ such that $a=xby, b=yax, x=xyx$ and $y=yxy$. Using

Theorem 2, we can find a $u \in I_q^{-1}$ such that $y = yuy$. Similarly to Theorem 2, we have $u_q^{-1} \in R$ such that $(1 - uu_q^{-1})\perp(1 - u_q^{-1}u)$ and $u = uu_q^{-1}u$. Set $w = y + (1 - yu)u_q^{-1}(1 - uy)$. Then $uwu = u\big(y + (1 - yu)u_q^{-1}(1 - uy)\big)u = uyu + (1 - uy)uu_q^{-1}u(1 - yu) = uyu + u(1 - yu) = u$. In addition, we have $yuw = yu\big(y + (1 - yu)u_q^{-1}(1 - uy)\big) = yuy = y$ and $w \in 1 + I$. Clearly, $1 - uw = (1 - uy)(1 - uu_q^{-1})$ and $1 - wu = (1 - u_q^{-1}u)(1 - yu)$. Hence $(1 - uw)R(1 - wu) = 0$ and $(1 - wu)R(1 - uw) = 0$. Let $k = (1 - xy - uy)u(1 - yx - yu)$ and $l = (1 - yx - yu)w(1 - xy - uy)$. Then $klk = (1 - xy - uy)u(1 - yx - yu)(1 - yx - yu)w(1 - xy - uy)(1 - xy - uy)u(1 - yx - yu) = (1 - xy - uy)uwu(1 - yx - yu) = k$. Furthermore, we deduce that $1 - kl = (1 - xy - uy)(1 - uw)(1 - xy - uy)$ and $k = (1 - yx - yu)(1 - wu)(1 - xy - uy)$. Since $(1 - uw)R(1 - wu) = 0$ and $1 - xy - uy, 1 - yx - yu \in U(R)$, we have $(1 - kl)R(1 - lk) = 0$ and $(1 - lk)R(1 - kl) = 0$. In addition, $kbl = (1 - xy - uy)u(1 - yx - yu)b(1 - yx - yu)w(1 - xy - uy) = (1 - xy - uy)(u - uyx - uyu)by = xyuby = xby = a$. By a similar rout, we have $lak = b$. One easily checks that $l, k \in 1 + I$. So we have $u \in I_q^{-1}$ such that $a = ubu_q^{-1}$ and $b = u_q^{-1}au$.

(2)$\Rightarrow$(1) Given any regular $x \in I$, there exists $y \in R$ such that $x = xyx$ and $y = yxy$. Since xy and yx are both idempotents, we have $R = yxR \oplus (1 - yx)R = xyR \oplus (1 - xy)R$. Clearly, we have an isomorphism $\eta : xyR = xR \cong yxR$ given by $\eta(xr) = yxr$ for any $r \in R$. Similarly to Theorem 2, we see that $y \in I$. It is easy to verify that $xy \widetilde{\sim} yx$ via I. Therefore we can find $u \in I_q^{-1}$ such that $yx = uxyu_q^{-1}$ and $xy = u_q^{-1}yxu$. Similarly to [6, Theorem 7], we construct maps $\alpha : (1 - xy)R \to (1 - yx)R$ given by $(1 - xy)r \to (1 - yx)u(1 - xy)r$ for any $r \in R$ and $\beta : (1 - yx)R \to (1 - xy)R$ given by $(1 - yx)r \to (1 - xy)u_q^{-1}(1 - yx)r$ for any $r \in R$. Define $\phi : R = xR \oplus (1 - xy)R \to yxR \oplus (1 - yx)R$ given by $\phi(x_1 + x_2) = \eta(x_1) + \alpha(x_2)$ for any $x_1 \in xR, x_2 \in (1 - xy)R$ and $\psi : R = yxR \oplus (1 - yx)R \to xR \oplus (1 - xy)R = R$ given by $\psi(y_1 + y_2) = \eta^{-1}(y_1) + \beta(y_2)$ for any $y_1 \in yxR, y_2 \in (1 - yx)R$. Analogously [6, Theorem 7], we deduce that $(1 - \psi\phi)\perp(1 - \phi\psi)$; hence, $\phi \in R_q^{-1}$. Obviously, $x = x\phi x$. In addition, $\phi \in 1 + I$. It follows by Theorem 2 that I is a qb-ideal. $\square$

COROLLARY 11. *Let I be an exchange ideal of a ring R. Then the following are equivalent:*

(1) I is a qb-ideal.

(2) For any idempotents $e, f \in I$, $eR \cong fR$ implies that there exists $u \in I_q^{-1}$ such that $e = ufu_q^{-1}$ and $f = u_q^{-1}eu$.

Proof. (1)$\Rightarrow$(2) Suppose that $eR \cong fR$ via I. Then there exist $a, b \in R$ such that $e = ab$ and $f = ba$, where $a \in eRf, b \in fRe$. Clearly, $e = afb, f = bea, a = aba, b = bab$ and $a, b \in I$. That is, $e \overset{\sim}{\approx} f$ via I. By Theorem 10, we have $u \in I_q^{-1}$ such that $e = ufu_q^{-1}$ and $f = u_q^{-1}eu$.

(2)$\Rightarrow$(1) is obtained by the proof of "(2) $\Rightarrow$ (1)" in Theorem 10. □

[5, Lemma 3] shows that every exchange QB-ideal of a ring is a qb-ideal. It is well known that $M_n(I)$ is an exchange QB-ideal of $M_n(R)$ in case I is an exchange QB-ideal of R. We naturally end this paper by asking a question: If I is an exchange qb-ideal of a ring R, is $M_n(I)$ an exchange qb-ideal of $M_n(R)$?

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