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SOME ESTIMATES FOR COMPACT HYPERSURFACES IN HYPERBOLIC SPACE

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Abstract. Let M be a compact hypersurface in hyperbolic space and let A be the area of M and V be the volume of the compact domain bounded by M. In this paper, we find a lower bound for $\frac{A}{V}$ in two cases, M has constant scalar curvature and M has constant mean curvature.

1. Introduction

In this section we review some basic facts that are needed in the next section. Let $i : M \to H^{n+1}$ be an embedding of n-dimensional compact hypersurface M to hyperbolic space H^{n+1}. Since M is compact, it is the boundary of a compact domain N of H^{n+1}, i.e., $M = \partial N$. Let ν be the *interior* unit normal field of M, $\{e_i\}_{i=1}^n$ be an orthonormal frame, and σ the second fundamental form, both with respect to the normal field ν, i.e. $\sigma(e_i, e_j) = - < \nu_*(e_i), e_j >$. Let S and $H = \frac{1}{n}\sum_i \sigma(e_i, e_i)$ be the scalar curvature of M and mean curvature of the immersion, respectively.
For a function f defined everywhere on H^{n+1} we use the notations $\overline{\Delta} f$ and $\overline{\nabla}^2 f$ for the Laplacian and Hessian of f, respectively, and if $g = f|_M$, then ∇g and Δg denote the gradient and Laplacian of g in M, respectively.

The important formula in this paper is due to Reily [2] that for use of it, we need to introduce some related notation.

Suppose that B is a symmetric linear operator on an m-dimensional inner product space W. For an integer r, $0 \le r \le m$, the *r-th invariant*

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of B, $S_r(B)$, is defined by the condition:

$$det\ (I+\lambda B) = S_0(B) + \lambda S_1(B) + \ldots + \lambda^m S_m(B).$$

The r-th Newton operator associated with B is defined as follows.

$$T_r(B) = S_r(B)I - S_{r-1}(B) \cdot B + \ldots + (-1)^r B^r.$$

In this paper we need two cases:
a) B is the shape operator for the embedding i from M to N, i.e., $m=n$, then we write K_r and T_r instead of $S_r(B)$ and $T_r(B)$, respectively.
b) B is the Hessian of a function f on M , i.e. $m=n+1$, then we write $S_r(f)$ and $T_r(f)$ instead of $S_r(B)$ and $T_r(B)$, respectively.

It is clear that $K_r = (-1)^n \binom{n}{r} H_r$, where H_r is the r-th mean curvature of M in N and the coefficient $(-1)^n$ follows from the choice of the interior unit normal vector field ν to M in N. Also we have $S_1(f) = \overline{\Delta} f$, where $\overline{\Delta}$ is the Laplacian of f in H^{n+1} and $\frac{1}{n}K_1 = -H$.
We have also the Newton inequality as follows (for example, see [2]).

$$(\overline{\Delta} f)^2 = (S_1(f))^2 \geq \frac{2(n+1)}{n} S_2(f)$$

or

$$\frac{n}{n+1}(S_1(f))^2 \geq 2S_2(f). \tag{1}$$

In the other word by definition of scalar curvature we have :

$$S = 2\sum_{i<j} K(e_i, e_j).$$

Therefore, Gauss equation for immersed hypersurface M^n in H^{n+1} implies:

$$S + n(n-1) = n^2H^2 - |\sigma|^2, \tag{2}$$

and by Schwarz inequality

$$S \leq n(n-1)H^2. \tag{3}$$

2. Main results

Let V and A be the volume of N and the area of M, respectively. Let dV and dA be the canonical measures on H^{n+1} and M, respectively.

THEOREM 2.1. *Let M^n be an n-dimensional compact hypersurface embedded in $(n+1)$-dimensional hyperbolic space H^{n+1}. If the scalar curvature S be positive constant, then $(\frac{A}{V})^2 \geq S$.*

Proof. Since $S > 0$, by (3) H is positive everywhere, so we have:

$$\sqrt{S} \leq \sqrt{n(n-1)}H, \tag{4}$$

enddocument Integration on M yields

$$\sqrt{S}A \leq \sqrt{n(n-1)} \int_M H dA. \tag{5}$$

Thus,

$$SA^2 \leq n(n-1)(\int_M H dA)^2. \tag{6}$$

Let f be the solution of the Dirichlet problem,

$$\overline{\Delta} f = 1 \text{ on } N \quad \text{and } g = 0 \text{ on } M.$$

From divergence theorem we have:

$$V = \int_N \overline{\Delta} f dV = -\int_M U dA. \tag{7}$$

Now, by equation (14) of [2],

$$\begin{aligned} \int_N 2S_2(f)dV = &\int_M \{(\Delta g - UK_1)U - (\nabla g, \nabla U) - \sigma(\nabla g, \nabla g)\} dA \\ &+ \int_N \text{Ric}(\text{grad}\, f, \text{grad}\, f)dV, \end{aligned} \tag{8}$$

where $U =< \text{grad}\, f, \nu >: M \to R$, i.e., normal derivative of f on M.

Substituting the condition of f in (8) and using the fact $-K_1 = nH$ imply that:

$$\int_N 2S_2(f)dV = \int_M nU^2 H dA + \int_N \text{Ric}(\text{grad}\, f, \text{grad}\, f)dV.$$

Thus by (1) we have:

$$\int_N \frac{n}{n+1}(S_1(f))^2 dV \geq \int_M nU^2 H dA + \int_N \text{Ric}(\text{grad}\, f, \text{grad}\, f)dV. \tag{9}$$

Since H^{n+1} is a symmetric space of non compact type, its Ricci curvature and its Ricci tensor are negative everywhere.
So $\int_N \text{Ric}\ (\text{grad}\, f, \text{grad}\, f)dV$ is negative. This fact beside $S_1(f) = \overline{\Delta} f$ and equation (9) yield:

$$
\begin{aligned}
V &= \int_N (\overline{\Delta} f)^2 dx_1 \cdots dx_{n+1} \\
&= \int_N (S_1(f))^2 dx_1 \cdots dx_{n+1} \geq (n+1) \int_M U^2 H dA.
\end{aligned} \tag{10}
$$

Thus, we have:

$$
\frac{V}{n+1} \geq \int_M U^2 H dA, \tag{11}
$$

By Schwarz inequality for U, (5) and (11) we conclude that:

$$
\begin{aligned}
\int_M U^2 H dA &\geq \sqrt{\frac{S}{n(n-1)}} \int_M U^2 dA \\
&\geq \frac{\sqrt{S}}{\sqrt{n(n-1)}A} [\int_M U dA]^2 \\
&= \sqrt{\frac{S}{n(n-1)}} \frac{U^2}{A^2} .
\end{aligned}
$$

Substitution in (11) gives us:

$$
\frac{V}{n+1} \geq \sqrt{\frac{S}{n(n-1)}} \frac{V^2}{A} ,
$$

or

$$
\frac{A^2}{V^2} \geq \frac{S(n+1)^2}{n(n-1)} .
$$

Since

$$
\frac{(n+1)^2}{n(n-1)} \geq 1
$$

thus,

$$
(\frac{A}{V})^2 \geq S .
$$

□

THEOREM 2.2. *Let M and N be the same as in Theorem 2.1 . If M has positive constant mean curvature H_c , then $\frac{A}{V} \geq nH_c$.*

Proof. Let $f : N \to R$ be the solution of the Poisson equation on N, i.e.,

$$
\overline{\Delta} f = 1 \text{ on } N \text{ and } f = 0 \text{ on } M.
$$

Thus, by the fact $Ric < 0$ on H^{n+1} in (8) we have,

$$\int_N 2S_2(f)dV \geq \int_M nU^2 H_c = nH_c \int_M U^2 dA.$$

Using (1) in definition of V conclude that:

$$V \geq nH_c \int_M U^2 dA. \tag{12}$$

On the other hand Schwarz inequality implies:

$$nH_c \int_M U^2 dA \geq \frac{nH_c}{A}(\int_M UdA)^2. \tag{13}$$

Since $U = \frac{\partial f}{\partial \nu}$, by Stoke's theorem,

$$(\int_M UdA)^2 = (\int_N \overline{\Delta} f dx_1 \cdots dx_n)^2 = V^2. \tag{14}$$

Therefore, (12), (13) and (14) imply,

$$V \geq \frac{nH_c}{A} V^2,$$

or

$$\frac{A}{V} \geq nH_c.$$

□

Following example is an application of Theorem 2.2 .

EXAMPLE 2.3. Let $i : M^2 \to H^3$ be an isometric immersion from a compact, stable surface M^2 with constant mean curvature $H_0 > 1$ to three dimensional hyperbolic space H^3. If A be the area of M^2, by [1] we have, $(H_0^2 - 1)A = 4\pi$. Let N^3 be the compact domain in H^3 that its boundary is M^2 and let V be the volume of N^3. Now, Theorem 2.2 implies the following upper bound for V.

$$V \leq \frac{4\pi}{2H_0(H_0^2 - 1)} \leq \frac{2\pi}{H_0^2 - 1}.$$

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