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# SOME RESULTS ON THE STABILITY OF THE HOMOGENEOUS FUNCTIONS

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Abstract. In this paper we present result on Pexider type of stability and superstability of homogeneous mappings. Some consequences for quadratic mappings and linear operators are also established.

## 1. Introduction

A question about the stability of the linear mappings was posed by S. Ulam (1940) and answered by D. H. Hyers in [4]. His result reads as follows.

Theorem. *Let $X$ and $Y$ be Banach spaces and $\varepsilon > 0$ given real number. Assume that a map $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon \text{ for all } x, y \in X.$$

*Then there exists exactly one additive mapping $a : X \to Y$ such that*

$$\|f(x) - a(x)\| \leqslant \varepsilon \text{ holds for all } x \in X.$$

Since then many generalizations have been obtained both for linear and other types of mappings (see e.g. [2], [3]).

The problem of the stability of homogeneous mappings has been discussed by Prof. L. Székelyhidi, Prof. J. Tabor [6] and the author of this paper [1] during the 4th International Conference on Functional Equations and Inequalities held in Krynica (Poland, 1993), where some result have been presented, (see also [5]).

In the our paper we shall prove some new results concerning that problem.

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## 2. Main results

LEMMA 1. *Let $E$ be a real linear space and $E_1$ a real normed space. Let $f : E \to E_1$, $g : E \to E_1$, $\varphi : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \times E \to \mathbb{R}_+$ be given functions. Suppose that*

$$\|f(\alpha x) - \varphi(\alpha)g(x)\| \leqslant h(\alpha, x) \tag{1}$$

*for all $(\alpha, x) \in \mathbb{R} \times E$ and $\varphi(1) = 1$. Then the following inequalities hold true for all $(\alpha, x) \in \mathbb{R} \times E$*

$$\|f(\alpha x) - \varphi(\alpha)f(x)\| \leqslant H(\alpha, x),$$

$$\|g(\alpha x) - \varphi(\alpha)g(x)\| \leqslant G(\alpha, x),$$

$$\|f(\alpha^n x) - \varphi^n(\alpha)f(x)\| \leqslant \sum_{s=1}^{n} |\varphi(\alpha)|^{s-1} H(\alpha, \alpha^{n-s}x), \tag{2}$$

$$\|g(\alpha^n x) - \varphi^n(\alpha)g(x)\| \leqslant \sum_{s=1}^{n} |\varphi(\alpha)|^{s-1} G(\alpha, \alpha^{n-s}x), \tag{3}$$

*where $H(\alpha, x) := h(\alpha, x) + |\varphi(\alpha)|h(1, x)$, $G(\alpha, x) := h(\alpha, x) + h(1, \alpha x)$.*

*Proof.* Appylying (1) we get for all $(\alpha, x) \in \mathbb{R} \times E$

$$\begin{aligned}&\|f(\alpha x) - \varphi(\alpha)f(x)\| \\ &\leqslant \|f(\alpha x) - \varphi(\alpha)g(x)\| + \|\varphi(\alpha)g(x) - \varphi(\alpha)f(x)\| \\ &\leqslant h(\alpha, x) + |\varphi(\alpha)|h(1, x) = H(\alpha, x).\end{aligned}$$

This means that (2) is satisfied for $n = 1$. Now we have:

$$\begin{aligned}&\|f(\alpha^{n+1}x) - \varphi^{n+1}(\alpha)f(x)\| \\ &\leqslant \|f(\alpha^{n+1}x) - \varphi(\alpha)f(\alpha^n x)\| + \|\varphi(\alpha)f(\alpha^n x) - \varphi^{n+1}(\alpha)f(x)\| \\ &\leqslant H(\alpha, \alpha^n x) + |\varphi(x)| \cdot \|f(\alpha^n x) - \varphi^n(\alpha)f(x)\| \\ &\leqslant H(\alpha, \alpha^n x) + \sum_{s=1}^{n} |\varphi(\alpha)|^s H(\alpha, \alpha^{n-s}x) \\ &= \sum_{s=1}^{n+1} |\varphi(\alpha)|^{s-1} H(\alpha, \alpha^{n+1-s}x).\end{aligned}$$

Therefore, by the induction principle, the inequality (2) is true for all natural numbers $n$. The inequality (3) one can verify by the same way. □

Now we shall prove the following.

THEOREM 1. *Let the assumptions of Lemma 1 be satisfied and let $E_1$ be a Banach space. Suppose that there exists $\beta \in \mathbb{R}$ such that $\varphi(\beta) \neq 0$ and the series*

$$\sum_{n=1}^{\infty} |\varphi(\beta)|^{-n} H(\beta, \beta^n x) \tag{4}$$

*pointwise converges for all $x \in E$ and*

$$\liminf_{n\to\infty} |\varphi(\beta)|^{-n} H(\alpha, \beta^n x) = 0 \tag{5}$$

*for all $(\alpha, x) \in \mathbb{R} \times E$.*

*Then there exists exactly one $\varphi$-homogeneous function $F : E \to E_1$:*

$$F(\alpha x) = \varphi(\alpha) F(x), \quad (\alpha, x) \in \mathbb{R} \times E \tag{6}$$

*such that*

$$\|F(x) - f(x)\| \leqslant \sum_{n=1}^{\infty} |\varphi(\beta)|^{-n} H[\beta, \beta^{n-1} x], \tag{7}$$

$$\|F(x) - g(x)\| \leqslant \sum_{n=1}^{\infty} |\varphi(\beta)|^{-n} G[\beta, \beta^{n-1} x] \tag{8}$$

*for $x \in E$.*

*Proof.* Let's consider the following Hyers' sequence

$$F_n(x) := \varphi^{-n}(\beta) f(\beta^n x), \quad x \in E, \quad n \in \mathbb{N} \text{ (natural numbers)}. \tag{9}$$

Then, by (2), we get the estimation

$$\|F_n(x) - f(x)\| \leqslant \sum_{s=1}^{n} |\varphi(\beta)|^{-(n-s)-1} H(\beta, \beta^{n-s} x),$$

which means that

$$\|F_n(x) - f(x)\| \leqslant \sum_{s=1}^{n} |\varphi(\beta)|^{-s} H(\beta, \beta^{s-1} x) \tag{10}$$

for $n \in \mathbb{N}$ and $x \in X$.

Now we shall verify that $\{F_n(x)\}$ is a Cauchy sequence for every $x \in E$. Indeed, in view of (2), we have the inequalities for all natural $n, m$, $n > m$ and $x \in E$

$$\begin{aligned} \|F_n(x) - F_m(x)\| &\leqslant |\varphi(\beta)|^{-n} \cdot \|f(\beta^n x) - \varphi(\beta)^{(n-m)} f(\beta^m x)\| \\ &\leqslant \sum_{s=m+1}^{n} |\varphi(\beta)|^{-s} H(\beta, \beta^{s-1} x). \end{aligned}$$

Consequently, since that series (4) converges, we proved our statement.

Denote

$$F(x) := \lim_{n\to\infty} F_n(x) \quad \text{for all} \quad x \in E.$$

The function $F$ satisfies the condition (6). To this and consider the expression

$$F(\alpha x) - \varphi(\alpha)F(x) = \lim_{n\to\infty} \{\varphi^{-n}(\beta)[f(\alpha\beta^n x) - \varphi(\alpha)f(\beta^n x)]\}.$$

Because

$$|\varphi(\beta)|^{-n} \cdot \|f(\alpha\beta^n x) - \varphi(\alpha)f(\beta^n x)\| \leqslant |\varphi(\beta)|^{-n} H(\alpha, \beta^n x),$$

then appyling the assumption (5), we see that

$$F(\alpha x) - \varphi(\alpha)F(x) = 0 \quad \text{for} \quad (\alpha, x) \in \mathbb{R} \times E.$$

From (10) we get imediately the estimation (7).

Similarly, define

$$G_n(x) := \varphi^{-n}(\beta)g(\beta^n x), \quad \text{for} \quad x \in E.$$

Then, as for the sequence (9), we can prove that there exists the limit

$$G(x) := \lim_{n\to\infty} G_n(x), \quad x \in E$$

and taking into account (3), we get the inequality

$$\|G_n(x) - g(x)\| \leqslant \sum_{s=1}^{n} |\varphi(\beta)|^{-s} G(\beta, \beta^{s-1}x), \quad x \in E. \tag{11}$$

Furthermore, we have

$$\begin{aligned} \|F_n(x) - G_n(x)\| &= |\varphi(\beta)|^{-n} \cdot \|f(\beta^n x) - g(\beta^n x)\| \leqslant |\varphi(\beta)|^{-n} h(1, \beta^n x) \\ &= \frac{1}{2}|\varphi(\beta)|^{-n} H(1, \beta^n x), \end{aligned}$$

hence, by (5) we get the equality

$$F(x) = G(x) \quad \text{for all} \quad x \in E.$$

Therefore, the inequality (8), follows from (11).

To finish the proof, let's assume that there exist two $\varphi$-homogeneous functions $F_s : E \to E_1$, $s = 1, 2$ such that (7) and (8) hold for $F = F_s$, $s = 1, 2$.

If $F_1$ and $F_2$ are zero functions, then $F_1 = F_2$. So let's assume that $F_1 \neq 0$, i.e. there exists $x_0 \in E$ such that $F_1(x_0) \neq 0$. Then we have for $\alpha, \beta \in \mathbb{R}$,

$$F_1(\alpha\beta x) = \varphi(\alpha\beta)F_1(x_0) = \varphi(\alpha)\varphi(\beta)F_1(x_0)$$

and consequently

$$\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta) \quad \text{for} \quad \alpha, \beta \in \mathbb{R}. \tag{12}$$

Now, from (7) we get for all $n \in \mathbb{N}$ and all $x \in E$

$$\begin{aligned} \|F_1(x) - F_2(x)\| &= |\varphi(\beta^n)|^{-1} \cdot \|F_1(\beta^n x) - F_2(\beta^n x)\| \\ &\leqslant |\varphi(\beta)|^{-n}\{\|F_1(\beta^n x) - f(\beta^n x)\| \\ &\quad + \|F_2(\beta^n x) - f(\beta^n x)\|\} \\ &\leqslant 2 \sum_{s=n+1}^{\infty} |\varphi(\beta)|^{-s} H(\beta, \beta^{s-1}x). \end{aligned}$$

Since the series (4) converges, then the last inequality implies that $F_1 = F_2$ for all $x \in E$ and the proof of the theorem is completed. □

REMARK 1. From the proof of the theorem we guess that if $F$ is not identically equal to the zero function, then the function $\varphi$ must be multiplicative i.e. satisfies condition (12).

REMARK 2. The estimations (7) and (8) may be different.

COROLLARY 1. *Let $E$ be a real linear space and $E_1$ a real Banach space. Let $f : E \to E_1$ and $g : E \to E_1$ satisfy the inequality*

$$\|f(\alpha x) - |\alpha|^v g(x)\| \leqslant \delta + |\alpha|^v \varepsilon \quad \textit{for all} \quad x \in E \quad \textit{and} \quad \alpha \in \mathbb{R}, \tag{13}$$

*where $v > 0$, $\delta \geqslant 0$, $\varepsilon \geqslant 0$ are given real numbers. Then there exists exactly one absolutly $v$-homogeneous function $F : E \to E_1$ (i.e. $F(\alpha x) = |\alpha|^v F(x)$ for $x \in E$, $\alpha \in \mathbb{R}$) such that*

$$\|F(x) - f(x)\| \leqslant \delta + 2\varepsilon, \tag{14}$$

$$\|F(x) - g(x)\| \leqslant \varepsilon \quad \textit{for } x \in E. \tag{15}$$

*Proof.* From (2) and (3) for $n = 1$, we get

$$\|f(\alpha x) - |\alpha|^v f(x)\| \leqslant \delta + |\alpha|^v(\delta + 2\varepsilon),$$

$$\|g(\alpha x) - |\alpha|^v g(x)\| \leqslant 2\delta + \varepsilon + |\alpha|^v \varepsilon, \quad \text{for } \alpha \in \mathbb{R}, \ x \in E.$$

Now, from Theorem 1, for every $2 \leqslant \beta = m \in \mathbb{N}$ there exists $v$-homogeneous function

$$F_m(x) := \lim_{n \to \infty} m^{-nv} f(m^n x), \quad x \in E$$

satisfying two inequalities

$$\|F_m(x) - f(x)\| \leqslant [\delta + m^v(\delta + 2\varepsilon)](m^v - 1)^{-1},$$

$$\|F_m(x) - g(x)\| \leqslant (2\delta + \varepsilon + m^v\varepsilon)(m^v - 1)^{-1},$$

for $x \in E$.

To finish the proof one can use the same method as for the proof of Corollary 1 in [1]. □

COROLLARY 2. *Let the assumptions of Corollary 1 be fulfilled. If moreover, $\delta = 0$, then $f$ is $v$-homogeneous and*

$$\|f(x) - g(x)\| \leqslant \varepsilon, \quad \text{for} \quad x \in E; \tag{16}$$

*or $\varepsilon = 0$, then $g$ is $v$-homogeneous and*

$$\|f(x) - g(x)\| \leqslant \delta, \quad \text{for} \quad x \in E. \tag{17}$$

*Proof.* The statement that $f$ or $g$ is $v$-homogeneous can be proved similarly as has it been done for Corollary 2 in [1]. From Corollary 1 we guess that if $\delta = 0$, then $F = f$ and (16) follows direct from (15) and in the case $\varepsilon = 0$ we get $F = g$ and (17) follows from (14). This completes the proof. □

DEFINITION. A function $f : E \to E_1$ is called a quadratic function iff for all $x, y \in E$

$$f(x + y) + f(x - y) = 2f(x) + 2f(y).$$

COROLLARY 3. *Let $E$ be a real linear space and $E_1$ a real Banach space. Let $f, g, h : E \to E_1$ be given functions and $\varepsilon > 0$ a given number. Then we have $f = g + h$, where $g$ is a quadratic 2-homogeneous function and $\|h(x)\| \leqslant \varepsilon$ for $x \in E_1$ if and only if*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leqslant 6\varepsilon, \tag{18}$$

$$\|f(\alpha x) - \alpha^2 f(x)\| \leqslant \varepsilon + \alpha^2\varepsilon \tag{19}$$

*for all $x, y \in E$ and $\alpha \in \mathbb{R}$.*

*Proof.* Assume that $f = g+h$, where $g$ is quadratic and 2-homogeneous and $h$ bounded by $\varepsilon$.
Then we have for all $x, y \in E$ and $\alpha \in \mathbb{R}$

$$\begin{aligned}\|f(x + y)+&f(x - y) - 2f(x) - 2f(y)\| \\ &= \|h(x + y) + h(x - y) - 2h(x) - 2h(y)\| \\ &\leqslant 6\varepsilon\end{aligned}$$

and

$$\|f(\alpha x) - \alpha^2 f(x)\| = \|h(\alpha x) - \alpha^2 h(x)\| \leqslant \varepsilon + \alpha^2\varepsilon,$$

i.e., (18) and (19) hold true.

Now assume that (18) and (19) are satisfied. From Corollary 1 there exists 2-homogeneous function $g : E \to E_1$ such that

$$\|f(x) - g(x)\| \leqslant \varepsilon \quad \text{for} \quad x \in E.$$

Define $f - g = h$, then $f = g + h$ and $\|h(x)\| \leqslant \varepsilon$ for $x \in E$.
In view of (18) we have

$$\begin{aligned}&\|g[\alpha(x+y)] + g[\alpha(x-y)] - 2g(\alpha x) - 2g(\alpha y)\\ &\qquad + h[\alpha(x+y)] + h[\alpha(x-y)] - 2h(\alpha x) - 2h(\alpha y)\|\\ &\qquad \leqslant 6\varepsilon,\end{aligned}$$

hence for $\alpha \neq 0$ we get

$$\begin{aligned}&\|g(x+y) + g(x-y) - 2g(x) - 2g(y)\\ &\qquad + \alpha^{-2}[h(\alpha(x+y)) + h(\alpha(x-y)) - 2h(\alpha x) - 2h(\alpha y)]\|\\ &\qquad \leqslant 6\varepsilon\alpha^{-2}.\end{aligned}$$

Therefore, letting $\alpha \to \infty$, we obtain

$$\|g(x+y) + g(x-y) - 2g(x) - 2g(y)\| = 0$$

for all $x, y \in E$, which means that $g$ is also a quadratic function and proof is completed. □

Corollary 4. *Let $f : \mathbb{K} \to \mathbb{K}$, where $\mathbb{K}$ is a set of real or complex numbers satisfy the inequality* (19) *for all $x \in E$ and $\alpha \in \mathbb{K}$. Then consequently* (18) *holds true.*

*Proof.* From (19) by Corollary 1 it follows that there exists 2-homogeneous function $F : \mathbb{K} \to \mathbb{K}$ such that

$$\|F(x) - f(x)\| \leqslant \varepsilon \quad \text{for} \quad x \in E.$$

Therefore, we obtain for all $x, y \in \mathbb{K}$

$$\begin{aligned}&|f(x+y) + f(x-y) - 2f(x) - 2f(y)\\ &\qquad - F(x+y) - F(x-y) + 2F(x) + 2F(y)|\\ &\qquad \leqslant 6\varepsilon\end{aligned}$$

and since $F$ is 2-homogeneous function on $\mathbb{K}$, we obtain our assertion. □

Now we shall present the following.

Theorem 2. *Let $E$ be a normed space over $\mathbb{K}$ (the set of real or complex numbers) and $k : E \to \mathbb{R}_+$ be a mapping with the property*

$$k(Ax) \leqslant \|A\|^{p_2} k(x), \quad x \in E \quad \text{and} \quad A \in L(E). \tag{20}$$

*If $f, g : E \to E$ satisfy the inequality*

$$\|f(Ax) - Ag(x)\| \leqslant \|A\|^{p_1} k(x), \quad x \in E, \quad A \in L(x) \tag{21}$$

*and $p_1 > p_2$, $p_2 < 1$ or $p_1 < p_2$, $p_2 > 1$, $(p_1, p_2 \in \mathbb{R})$, then there exists $\alpha \in \mathbb{K}$ such that*

$$f(x) = \alpha x \tag{22}$$

$$\|f(x) - g(x)\| \leqslant k(x) \tag{23}$$

*for all $x \in E$ ($L(E)$ denotes the space of linear continous operators on, $E$ into $E$).*

*Proof.* The inequality (23) is a direct consequence of (21). From (21) one can get easily that for $A \in L(E)$, $x \in E$

$$\|f(Ax) - Af(x)\| \leqslant h(A, x)$$

where

$$h(A, x) = \|A\|^{p_1} k(x) + \|A\| k(x).$$

Now we take

$$A_n = \alpha_n I,$$

where $I$ denotes the identity operator, $\alpha_n = \frac{1}{n}$ if $p_1 > p_2$ or $\alpha_n = n$ if $p_1 < p_2$. Then

$$\begin{aligned} h(AA_n, A_n^{-1}x) &= \|AA_n\|^{p_1} k(A_n^{-1}x) + \|AA_n\| k(A_n^{-1}x) \\ &\leqslant \|A\|^{p_1} \alpha_n^{p_1 - p_2} k(x) + \|A\| \alpha_n^{1-p_2} k(x) \to 0, \quad n \to \infty. \end{aligned}$$

Therefore, by Corollary 1 from the paper [5] we get our statement. □

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