Bull. Korean Math. Soc. **42** (2005), No. 1, pp. 23–27

ROTATION SURFACES WITH 1-TYPE GAUSS MAP

ATHOUMANE NIANG

ABSTRACT. In this paper, we study rotation surfaces in a Euclidean space with pointwise 1-type Gauss map and obtain by the use of the concept of pointwise finite type Gauss map, a characterization theorem for rotation surfaces of constant mean curvature.

1. Introduction

Recently in 2000, and in the framework of the theory of finite type submanifolds (see [2], [3]), the authors of [7] raising the following problem : classify all submanifolds in an $m-$Euclidean space $\mathbb{E}^m$(or in the Minkowski space $\mathbb{E}_1^m$) satisfying the following equation

$$\Delta G = fG, \tag{1.1}$$

where Δ in the Laplacian of the induced metric and G the Gaus map of the submanifold,and for some function f on the submanifold.

The authors of [7] have studied ruled surfaces in 3−dimensional Minko wski space $\mathbb{E}_1^3$ with pointwise 1-type Gauss map, and obtain a classification theorem for them. Also, submanifolds in pseudo-Euclidean space with finite type Gauss map are studied (cf [1], [5] among others).

In the paper [6], a characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map in 3−dimensional Euclidean space is obtained.

On the other hand, Chen and Piccini [4] made a general study on submanifolds of Euclidean space with finite type Gauss map and classified the compact surfaces of 1-type Gauss map.

In this paper we use the concept of pointwise 1-type Gauss map intoduced in [7] to obtain the following theorem.

Received April 9, 2003.
2000 Mathematics Subject Classification: 53B25, 53C40.
Key words and phrases: Laplacian, Gauss map, pointwise 1-type.

THEOREM 1.1. *A rotation surface M in 3−Euclidean space $\mathbb{E}^3$ is pointwise 1-type Gauss map if and only if its mean curvature is a contant.*

Throughout this paper, we assume that all surfaces are connected and all objects are at least of class C^3.

2. Preliminaries

Let $(x(s), y(s)), s \in I$ be any smooth curve parametrised by the arc lenght and the domain of definition I is any open interval of the set of real numbers.

We define a surface of revolution M in $\mathbb{E}^3$ by an isometric immersion X defined :

$$X(s,\theta) = (x(s), y(s)\cos(\theta), y(s)\sin(\theta)); \; s \in I, \; 0 \leq \theta \leq 2\pi.$$

We will assume that $y(s) > 0$.
The first and the second fundamental forms of M are $ds^2 + y^2(s)d\theta^2$ and $(x''(s)y'(s) - x'(s)y''(s))ds^2 + x'(s)y(s)d\theta^2$, respectively. Then one can easily get that its mean curvature H is given by

$$2H = (x''(s)y'(s) - x'(s)y''(s)) + \frac{x'(s)}{y(s)}. \tag{2.1}$$

The Gauss map G of M is given by

$$G = (y(s), -x'(s)\cos(\theta), -x'(s)\sin(\theta)), \tag{2.2}$$

which is obtained from the classical formula $G = \frac{X_s \times X_\theta}{|X_s \times X_\theta|}$, where $\times$ denotes the cross-product in $\mathbb{E}^3$.

To obtain the Laplacian Δ of M we apply the following formula :

$$\Delta = -\frac{1}{\sqrt{|\det(g_{ij})|}} \Sigma \frac{\partial}{\partial x^i}(\sqrt{|\det(g_{ij})|} g^{ij} \frac{\partial}{\partial x^j}). \tag{2.3}$$

So by using 2.2 and 2.3, and the first fondamental form given above, one gets by an easy computaton the Laplacian ΔG of the Gauss map G of M :

$$\Delta G = \begin{pmatrix} -y''' - \frac{y'}{y}y'' \\ (x''' + \frac{y'}{y}x'' - \frac{1}{y^2}x')\cos\theta \\ (x''' + \frac{y'}{y}x'' - \frac{1}{y^2}x')\sin\theta \end{pmatrix}. \tag{2.4}$$

3. Proof of the theorem

Now we consider a rotation surface M given as in preliminaries, that satisfies moreover the equation : $\Delta G = fG$, for some function f in M. That is M pointwise 1-type Gauss map; and this is also equivalent to the following condition:

$$\Delta G - \langle \Delta G, G\rangle G = 0 \tag{3.1}$$

where $\langle , \rangle$ denotes the inner product of $\mathbb{E}^3$.

STEP1. We first compute the function $f := \langle \Delta G, G\rangle$, by using the relations 2.2 and 2.4, and obtain that

$$f = y'\{(-y''' - \frac{y'}{y}y'')\} - x'\{(x''' + \frac{y'}{y}x'' - \frac{1}{y^2}x')\}.$$

It is convenient to make the notations:

$$A = y''' + \frac{y'}{y}y'' \text{ and } B = x''' + \frac{y'}{y}x'' - \frac{1}{y^2}x',$$

then the function f becomes $f = -y'A - x'B$. With these notations we have

$$\Delta G = (-A, B\cos\theta, B\sin\theta).$$

Then the condition in 3.1 becomes

$$\begin{cases} -A + y'(y'A + x'B) = 0, \\ B - x'(y'A + x'B). \end{cases}$$

Since the curve $(x(s), y(s)), s \in I$ is parametrised by the arc length, then we have

$$x'^2 + y'^2 = 1. \tag{3.2}$$

And by using this equation, the above conditon becomes

$$\begin{cases} -x'^2A + x'y'B = 0 \\ -x'y'A + y'^2B = 0. \end{cases} \tag{3.3}$$

STEP2. We assume that neither x' nor y' is the zero function in a subinterval J of the interval I. Then the condition given by 3.3 becomes

$$-x'A + y'B = 0.$$

By using the relation 3.2 and its derivative, the condition above is now :

$$(y'x''' - x'y''') + (\frac{x'}{y})' = 0. \tag{3.4}$$

Now we take the derivative of the formula 2.1 for mean curvature H to obtain that

$$(y'x''' - x'y''') = 2H' - (\frac{x'}{y})'.$$

Inserting the left member of this relation in 3.4, we see that H' is zero on M, and conclude that the mean curvature is a contant.

STEP3. Now we assume either one the functions x' or y' is the zero function in subinterval J of the interval I. If we assume x' is the zero function on J then the function y' is a nonzero contant on J and vice versa. We might also assume the interval J to an open interval.
Assume x' is the zero function on an open interval J. Then one see easily that the open set $U = \{(s, \theta); s \in I,\ 0 \leq \theta \leq 2\pi\}$ of M is planar and have zero mean curvature.

Since the manifold M is connected, its mean curvature H cannot jump between the zero value and constant nonzero values. And this proof the one part of the theorem.

STEP4. Conversely assume the mean curvature H be constant. We just have to show how to obtain the conditon 3.3 which is equivalent to fact that M is pointwise $1-$type Gauss map.
As we seen above we have :

$$(y'x''' - x'y''') = 2H' - (\frac{x'}{y})'.$$

And therefore we get

$$(a):\ (y'x''' - x'y''') = -(\frac{x'}{y})'.$$

On the other hand we have

$$(b):\ -x'A + y'B = (y'x''' - x'y''') + (\frac{x'}{y})'.$$

From (a) and (b) one we get the following equation

$$(c):\ -x'A + y'B = 0.$$

To get the two equations condition 3.3, we multiply the equation (c) by x' and by y', rspectively. This proves the theorem.

ACKNOWLEDGMENTS. During my preparation of the thesis at Montpellier University II, the collaboration with the professor Lafontaine is decided, I would like to acknowledge him.

References

[1] L. J. Alías, A. Ferrández, P. Lucas, and M. A. Meroñno, *On the Gauss map of B-scrolls*, Tsukuba J. Math. **22** (1998), 317–377.
[2] B. -Y. Chen, *On submanifolds of finite type*, Soochow J. Math. **9** (1987), 65–81.
[3] ———, *Total Mean Curvature and Submanifolds of finite Type*, World Scientific, Singapore, 1984.
[4] B. -Y. Chen and P. Piccinni, *Submanifolds with finite type Gauss map*, Bull. Austral. Math. Soc. **35** (1987), 161–186.
[5] S. M. Choi, *On the Gauss map of ruled surfaces in a three-dimensional Minkowski space*, Tsukuba J. math. **19** (1995), 285–304.
[6] M. K. Choi and Y. H. Kim, *Characterisation of the helicoid as ruled surfaces with pointwise* 1*-type Gauss map*, Bull. Korean Math. Soc. **38** (2001), no. 4, 753–761.
[7] Y. H. Kim and D. W. Yoon, *Ruled surfaces with pointwise* 1*-type Gauss map*, J. Geom. Phys. **34** (2000), 191–205.

Recherches en Mathematiques Laboratoires G.T.A, Université Montpellier II, 34000 Montpellier, France.
E-mail: niang@math.univ-montp2.fr