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A HIGHER ORDER MONOTONE ITERATIVE SCHEME FOR NONLINEAR NEUMANN BOUNDARY VALUE PROBLEMS

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ABSTRACT. The generalized quasilinearization technique has been employed to obtain a sequence of approximate solutions converging monotonically and rapidly to a solution of the nonlinear Neumann boundary value problem.

1. Introduction

The method of generalized quaslinearization introduced by Lakshmikantham ([4, 5]) has been successfully employed to obtain a sequence of approximate solutions converging monotonically to a solution of the nonlinear problem, see, for example, [1-3, 6-10]. In this paper, we continue the study of nonlinear Neumann problems addressed in [1] and improve the convergence of a sequence of approximate solutions converging monotonically to a solution of the nonlinear Neumann boundary value problem. In fact, we establish the convergence of order $k(k \geq 2)$.

2. Some basic results

We know that the linear Neumann boundary value problem

$$-u''(t) = \lambda u(t), \qquad t \in J = [0, \pi]$$
$$u'(0) = 0, \qquad u'(\pi) = 0,$$

has a nontrivial solution if and only if $\lambda = m^2, m = 1, 2, \ldots$ and thus, for $\lambda \neq m^2$ and $\xi(t) \in C[0, \pi]$, the corresponding nonhomogeneous problem

$$-u''(t) - \lambda u(t) = \xi(t), \qquad t \in J$$

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$$u'(0) = 0, \qquad u'(\pi) = 0,$$

has a unique solution

$$u(t) = \int_0^\pi G_\lambda(t,s)\xi(s)ds,$$

where G_λ is the Green's function of the associated homogeneous problem and is given by

$$G_\lambda = \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}\pi}\begin{cases} \cos[\sqrt{\lambda}(\pi - s)]\cos[\sqrt{\lambda}t], & \text{if } 0 \le t \le s \le \pi \\ \cos[\sqrt{\lambda}s)]\cos[\sqrt{\lambda}(\pi - t)], & \text{if } 0 \le s \le t \le \pi \end{cases}$$

for $\lambda > 0$,

$$G_\lambda = \frac{1}{\sqrt{-\lambda}\sinh\sqrt{-\lambda}\pi}$$
$$\begin{cases} \cosh[\sqrt{-\lambda}(\pi - s)]\cosh[\sqrt{-\lambda}t], & \text{if } 0 \le t \le s \le \pi \\ \cosh[\sqrt{-\lambda}s]\cosh[\sqrt{-\lambda}(\pi - t)], & \text{if } 0 \le s \le t \le \pi \end{cases}$$

for $\lambda < 0$. We observe that $G_\lambda \ge 0$ for $\lambda < 0$. Now, we consider the nonlinear Neumann problem

$$-u''(t) = f(t, u(t)), \qquad t \in J \tag{1}$$
$$u'(0) = 0, \qquad u'(\pi) = 0,$$

where $f : J \times R \to R$ is continuous. The problem (1) is equivalent to the integral equation

$$u(t) = u(0) - \int_0^t (t - s)f(s, u(s))ds \tag{2}$$

with

$$\int_0^t f(s, u(s))ds = 0. \tag{3}$$

We shall say that $\alpha(t) \in C^2[J]$ is a lower solution of (1) if

$$-\alpha''(t) \le f(t, \alpha(t)), \qquad t \in J$$
$$\alpha'(0) \ge 0, \qquad \alpha'(\pi) \le 0,$$

and $\beta \in C^2[J]$ is an analogue upper solution of (1) if

$$-\beta''(t) \ge f(t, \beta(t)), \qquad t \in J$$
$$\beta'(0) \le 0, \qquad \beta'(\pi) \ge 0.$$

The following theorem plays a crucial role in the forthcoming analysis and for its proof, see reference [11].

THEOREM 1. *Let $\alpha, \beta \in C^2[J, R]$ be lower and upper solutions of* (1) *respectively such that $\alpha(t) \le \beta(t)$ on J. Then there exists a solution $u(t)$ of* (1) *such that $\alpha(t) \le u(t) \le \beta(t)$, $t \in J$.*

3. Higher order monotone iterative scheme

THEOREM 2. *Assume that*

(B_1) *$\alpha, \beta \in C^2[J, R]$ such that $\alpha(t) \le \beta(t)$ on J are lower and upper solutions of (1) respectively.*

$(B_2)]$ *$\frac{\partial^i f}{\partial x^i}(t,u)$, $i = 1, 2, 3, ..., k$ exist and are continuous on $\Omega = \{(t,u) \in J \times R\}$ such that $\frac{\partial f}{\partial u}(t,u) < 0$, $\frac{\partial^k}{\partial u^k}(f(t,u) + \phi(t,u)) \ge 0$ for some function $\phi \in C^{0,k}[J \times R, R]$ such that $\frac{\partial^k \phi}{\partial u^k}(t,u) \ge 0$.*
Then there exists a monotone nondecreasing sequence $\{\mu_n\}$ of solutions which converges uniformly to a solution of (1) *with the order of convergence k $(k \ge 2)$.*

Proof. Set

$$\phi(t,u) = F(t,u) - f(t,u), \qquad t \in J.$$

Using (B_2) and generalized mean value theorem, we have

$$f(t,u) \ge \sum_{i=0}^{k-1} \frac{\partial^i F}{\partial u^i}(t,v)\frac{(u-v)^i}{(i)!} - \phi(t,u),$$

where $\alpha(t) \le v(t) \le u(t) \le \beta(t)$. Now, we define

$$\begin{aligned} K(t,u,v) &= \sum_{i=0}^{k-1} \frac{\partial^i F}{\partial u^i}(t,v)\frac{(u-v)^i}{(i)!} - \phi(t,u), \\ &= \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(t,v)\frac{(u-v)^i}{(i)!} - \frac{\partial^k \phi}{\partial u^k}(t,\xi)\frac{(u-v)^k}{(k)!}, \end{aligned}$$

where $v \le \xi \le u$, and $\alpha \le v \le u \le \beta$ on J.

Observe that

$$K(t,u,v) \le f(t,u), \qquad K(t,u,u) = f(t,u). \tag{4}$$

Now, set $\mu_o = \alpha$ and consider the problem

$$\begin{aligned} -u''(t) &= K(t, u(t), \mu_o(t)), \qquad t \in J \\ u'(0) &= 0, \qquad u'(\pi) = 0. \end{aligned} \tag{5}$$

Using (B_1) and (4), we get

$$-\mu_o''(t) \leq f(t, \mu_o(t)) = K(t, \mu_o(t), \mu_o(t)), \qquad t \in J$$

$$\mu_o'(0) \geq 0, \qquad \mu_o'(\pi) \leq 0,$$

and

$$-\beta''(t) \geq f(t, \beta(t)) \geq K(t, \beta, \mu_o), \qquad t \in J$$

$$\beta'(0) \leq 0, \qquad \beta'(\pi) \geq 0,$$

which imply that μ_o and β are lower and upper solution of (5) respectively. Hence, by Theorem 1, there exists a solution μ_1 of (5) such that $\mu_o \leq \mu_1 \leq \beta$ on J. Next, we consider the problem

$$-u''(t) = K(t, u(t), \mu_1(t)), \qquad t \in J \tag{6}$$

$$u'(0) = 0, \qquad u'(\pi) = 0.$$

Employing the earlier arguments, it can be shown that there exists a solution μ_2 of (6) such that $\mu_1 \leq \mu_2 \leq \beta$ on J, where μ_1 and β are lower and upper solution of (6) respectively. Continuing this process successively, we obtain a monotone sequence $\{\mu_n\}$ of solutions satisfying

$$\mu_o \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots \leq \mu_{n-1} \leq \mu_n \leq \beta,$$

on J, where the element μ_n of the sequence is the solution of the problem

$$-u''(t) = K(t, u(t), \mu_{n-1}(t)), \qquad t \in J$$

$$u'(0) = 0, \qquad u'(\pi) = 0.$$

Since the sequence$\{\mu_n\}$ is monotone, it follows that it has a pointwise limit μ. To show that μ is in fact a solution of (1), we observe that μ_n is the solution of the following Neumann problem

$$-u''(t) = f_n(t), \qquad t \in J$$

$$u'(0) = 0, \qquad u'(\pi) = 0.$$

where $f_n(t) = K(t, \mu_n(t), \mu_{n-1}(t))$. Since $f_n(t)$ is continuous on Ω and $\alpha \leq \mu_n \leq \beta$ on Ω for $n = 1, 2, \ldots$, it follows that the sequence $\{f_n(t)\}$ is bounded in $C[J, R]$. This together with the the monotonicity of $\{\mu_n\}$ implies that the sequence $\{\mu_n\}$ uniformly converges to μ. Letting $n \rightarrow \infty$, and using the uniform convergence of $\{\mu_n\}$, we find that μ satisfies the integral equation (2) and (3) and hence μ is a solution of (1).

To show that the convergence of the sequence is of order k $(k \geq 2)$, we set $e_n = \mu - \mu_n$, $a_n = \mu_{n+1} - \mu_n$, $n = 1, 2, 3, \cdots$. Clearly, $a_n \geq 0$,

$e_n \geq 0,\ e_n - a_n = e_{n+1},\ a_n \leq e_n$ and $a_n^k \leq e_n^k$. Using the mean value theorem repeatedly, we have

$$\begin{aligned}
-e_n''(t) &= \mu_n''(t) - \mu''(t) \\
&= \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(t, \mu_{n-1}) \frac{(e_{n-1}^i - a_{n-1}^i)}{(i)!} \\
&\quad + \frac{\partial^k f}{\partial u^k}(t, \zeta(t)) \frac{e_{n-1}^k}{k!} + \frac{\partial^k \phi}{\partial u^k}(t, \zeta(t)) \frac{a_{n-1}^k}{k!} \\
&\leq \left(\sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(t, \mu_{n-1}) \frac{1}{(i)!} \sum_{j=0}^{i-1} e_{n-1}^{i-j-1} a_{n-1}^j\right) e_n \\
&\quad + \left[\frac{\partial^k f}{\partial u^k}(t, \zeta(t)) + \frac{\partial^k \phi}{\partial u^k}(t, \zeta(t))\right] \frac{e_{n-1}^k}{k!} \\
&\leq q_n(t) e_n + N e_{n-1}^k,
\end{aligned}$$

where

$$q_n(t) = \sum_{i=1}^{k-1} \frac{\partial^i}{\partial u^i} f(t, \mu_{n-1}) \frac{1}{(i)!} \sum_{j=0}^{i-1} e_{n-1}^{i-1-j} a_{n-1}^j,$$

and $N > 0$ provides bound for $\frac{\partial^k F}{\partial u^k}(t, \zeta(t))$ on Ω. As $lim_{n\to\infty} q_n(t) = f_u(t, \mu) < 0$, we can choose $\lambda < 0$ and $n_o \in N$ such that for $n \geq n_o$, $q_n(t) < \lambda$, we have

$$-e_n''(t) - \lambda(t) e_n(t) \leq (q_n(t) - \lambda) e_n(t) + N e_{n-1}^k \leq N e_{n-1}^k,$$
$$e_n'(0) = 0, \qquad e_n'(\pi) = 0,$$

whose solution is

$$e_n(t) = \int_0^\pi G_\lambda(t, s) N e_{n-1}^k ds, \qquad t \in J.$$

Taking maximum over $[0, \pi]$, we obtain

$$||e_n|| \leq C ||e_{n-1}||^k,$$

where C provides a bound on $N \int_0^\pi G_\lambda(t, s) ds$.
This completes the proof. □

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