

ON (α, δ) -SKEW ARMENDARIZ RINGS

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ABSTRACT. For a ring endomorphism α and an α -derivation δ , we introduce (α, δ) -skew Armendariz rings which are a generalization of α -rigid rings and Armendariz rings, and investigate their properties. A semiprime left Goldie ring is α -weak Armendariz if and only if it is α -rigid. Moreover, we study on the relationship between the Baerness and p.p. property of a ring R and these of the skew polynomial ring $R[x; \alpha, \delta]$ in case R is (α, δ) -skew Armendariz. As a consequence we obtain a generalization of [11], [14] and [16].

Throughout this paper R denotes an associative ring with unity, $\alpha : R \rightarrow R$ is an endomorphism and δ an α -derivation of R , that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. We denote $S = R[x; \alpha, \delta]$ the Öre extension whose elements are the polynomials over R , the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$. A ring R is called *Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j . Recall that a ring R is *reduced* if R has no nonzero nilpotent elements. Observe that reduced rings are *abelian* (i.e., all idempotents are central).

According to Krempa[15], an endomorphism α of a ring R is called to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. We call a ring R α -rigid if there exists a rigid endomorphism α of R . Note that any rigid endomorphism of a ring is a monomorphism and α -rigid rings are reduced rings by Hong et al.[10]. Properties of α -rigid rings have been studied in Krempa[15], Hong[10], and Hirano[8].

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In [9] Hong et al. defines a ring R with an endomorphism α to be α -skew Armendariz if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x; \alpha]$ satisfy $f(x)g(x) = 0$, then $a_i\alpha^i(b_j) = 0$ for each i, j . Motivated by results in Armendariz[2], Anderson and Camillo[1], Tsiu-Kwen Lee and Tsai-Lien Wong[16], and Hong et al.[9], we investigate a generalization of α -rigid rings and Armendariz rings which we call an (α, δ) -skew Armendariz ring.

DEFINITION 1. Let α be an endomorphism and δ be an α -derivation of a ring R . We say that R is an (α, δ) -skew Armendariz (or simply, (α, δ) -Armendariz) ring, if for polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ in $R[x; \alpha, \delta]$, $f(x)g(x) = 0$ implies $a_ix^i b_j x^j = 0$ for each i, j .

Note that each α -skew Armendariz ring is (α, δ) -skew Armendariz, where δ is the zero mapping. As it is mentioned in Hong et al. [9], if R is an Armendariz ring, then it is an I_R -Armendariz ring, where I_R is the identity endomorphism of R and thus every reduced ring R is I_R -Armendariz. However, there exists an I_R -Armendariz ring R which is not reduced. For example, $R = \mathbb{Z}_{n^2}$, where \mathbb{Z}_{n^2} is the ring of integers modulo n^2 and $n \geq 2$ is a positive integer, is a commutative Armendariz ring which is not reduced. Thus R is an I_R -Armendariz ring, but it is not I_R -rigid.

Clearly every subring of an (α, δ) -Armendariz ring is (α, δ) -Armendariz. Also, every α -rigid ring is (α, δ) -Armendariz, but the converse does not hold [9, Example 1].

DEFINITION 2. Let α be an endomorphism and δ be an α -derivation of R . We say that R is an (α, δ) -skew weak Armendariz (or simply (α, δ) -weak Armendariz) ring, if for linear polynomials $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x$ in $R[x; \alpha, \delta]$, $f(x)g(x) = 0$ implies $a_ix^i b_j x^j = 0$ for $i, j \in \{0, 1\}$.

It is clear that every (α, δ) -Armendariz ring is (α, δ) -weak Armendariz and that every subring of an (α, δ) -weak Armendariz ring is still (α, δ) -weak Armendariz. However, an (α, δ) -weak Armendariz ring is not necessarily (α, δ) -Armendariz in general [16, Example 3.2]. But, weak Armendariz von Neumann regular rings are reduced [1, Theorem 6]. In this paper we give an affirmative answer to a question of Hong et al. [9, page 115] and show that, a ring R with a monomorphism α and α -derivation δ , is α -rigid if and only if it is reduced and (α, δ) -weak Armendariz. Moreover we provide several examples of non semiprime

(hence non α -rigid) rings which are (α, δ) -weak Armendariz. Furthermore we prove that, for a semiprime left Goldie ring R with a monomorphism α , R is an α -rigid ring if and only if R is an α -weak Armendariz ring if and only if its classical left quotient ring $Q(R)$ is $\tilde{\alpha}$ -weak Armendariz. Finally, we show that, if R is an (α, δ) -Armendariz ring, then (1) $R[x; \alpha, \delta]$ is an abelian ring, (2) R is a Baer (resp. p.p.-) ring if and only if $R[x; \alpha, \delta]$ is a Baer (resp. p.p.-) ring.

The following results provide some properties of (α, δ) -Armendariz rings, which is needed in the sequel:

LEMMA 3. *Let R be an (α, δ) -weak Armendariz ring and $ab = 0$. Then $\alpha(a)\delta(b) = \delta(a)b = 0$.*

Proof. Since $ab = 0$, $\delta(a)b + \alpha(a)\delta(b) = 0$. Put $f(x) = \delta(a) + \alpha(a)x$ and $g(x) = b + bx$ in $R[x; \alpha, \delta]$. Since $f(x)g(x) = 0$, we have $\delta(a)b = \alpha(a)\delta(b) = 0$. \square

LEMMA 4. *Let R be an (α, δ) -weak Armendariz ring. Then for each idempotent element $e \in R$, we have $\alpha(e) = e$ and $\delta(e) = 0$.*

Proof. Since $e = e^2$, we have $\delta(e) = \delta(e)e + \alpha(e)\delta(e)$. Let $f(x) = \delta(e) + \alpha(e)x$ and $g(x) = (e - 1) + (e - 1)x$. Then $f(x)g(x) = \delta(e)e - \delta(e) + \alpha(e)\delta(e) + (\delta(e)e - \delta(e) + \alpha(e)\delta(e))x = 0$. Since R is (α, δ) -Armendariz, $\delta(e)e = \delta(e)$ and hence $\alpha(e)\delta(e) = 0$. Now suppose that, $h(x) = \delta(e) - (1 - \alpha(e))x$ and $k(x) = e + ex$. Then $h(x)k(x) = (\delta(e)e - \delta(e) + \alpha(e)\delta(e)) + (\delta(e)e - (1 - \alpha(e))\delta(e))x = 0$. Hence $\delta(e)e = 0$ and so $\delta(e) = \delta(e)e = 0$.

Now take $p(x) = (1 - e) + (1 - e)\alpha(e)x$ and $q(x) = e + (e - 1)\alpha(e)x$. Then $p(x)q(x) = 0$, since $\delta(e) = \delta(\alpha(e)) = 0$. Hence $\alpha(e) = e\alpha(e)$. Now suppose that $t(x) = e + e(1 - \alpha(e))x$ and $s(x) = (1 - e) - e(1 - \alpha(e))x$. Then $t(x)s(x) = -e(1 - \alpha(e))(\delta(e) - \delta(e\alpha(e)))x = 0$, since $\delta(e) = \delta(e\alpha(e)) = 0$. Hence $e(1 - \alpha(e)) = 0$, since R is (α, δ) -Armendariz. Therefore $e = e\alpha(e) = \alpha(e)$. \square

In the following lemma we employ the same method in the proof of [9, Lemma 19]:

LEMMA 5. *Let R be an (α, δ) -Armendariz ring. If $e^2 = e \in R[x; \alpha, \delta]$, where $e = e_0 + e_1x + \dots + e_nx^n$, then $e = e_0$.*

Proof. Since $e(1 - e) = 0 = (1 - e)e$, we have $(e_0 + e_1x + \dots + e_nx^n)((1 - e_0) - e_1x - \dots - e_nx^n) = 0$ and $((1 - e_0) - e_1x - \dots - e_nx^n)(e_0 + e_1x + \dots + e_nx^n) = 0$. Hence, $e_0(1 - e_0) = 0$, $e_0e_i = 0$ and $(1 - e_0)e_i = 0$ for $1 \leq i \leq n$, since R is (α, δ) -Armendariz. Thus $e_i = 0$ for $1 \leq i \leq n$, and so $e = e_0 = e_0^2$. \square

Now we give an affirmative answer to a question of Hong et al. [9, Page 115]:

THEOREM 6. *A ring R with a monomorphism α , is an α -rigid ring if and only if it is a reduced (α, δ) -weak Armendariz ring.*

Proof. It is clear that an α -rigid ring is reduced and by [10, Proposition 6] R is (α, δ) -weak Armendariz. Now, suppose that R is a reduced ring and $\alpha\alpha(a) = 0$ for $a \in R$. By Lemma 3, $\delta(a)\alpha(a) = \alpha(a)\delta(\alpha(a)) = 0$. Now, let $h(x) = \alpha(a) - \alpha(a)x$ and $k(x) = a + \alpha(a)x$. Then $h(x)k(x) = 0$. Since R is (α, δ) -weak Armendariz, we have $\alpha(a)\alpha(a) = 0$. Hence $a = 0$, since R is reduced and α is a monomorphism. \square

Next we provide several examples of (α, δ) -weak Armendariz rings:

Let R be a ring and let

$$T_n(R) := \left\{ \left(\begin{array}{cccccc} a_1 & a_2 & a_3 & \dots & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & 0 & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_1 \end{array} \right) \mid a_i \in R \right\},$$

with $n \geq 2$. We can denote elements of $T_n(R)$ by (a_1, a_2, \dots, a_n) . Then $T_n(R)$ is a ring with addition point-wise and multiplication given by $(a_1, a_2, \dots, a_n)(b_1, b_2, \dots, b_n) = (a_1b_1, a_1b_2 + a_2b_1, \dots, a_1b_n + a_2b_{n-1} + \dots + a_nb_1)$, for each $a_i, b_j \in R$. For an endomorphism α and an α -derivation δ of R , the natural extension $\bar{\alpha} : T_n(R) \rightarrow T_n(R)$ defined by $\bar{\alpha}((a_i)) = (\alpha(a_i))$ is an endomorphism of $T_n(R)$ and $\bar{\delta} : T_n(R) \rightarrow T_n(R)$ defined by $\bar{\delta}((a_i)) = (\delta(a_i))$, is an $\bar{\alpha}$ -derivation of $T_n(R)$.

LEMMA 7. *Let α be an endomorphism and δ an α -derivation of a ring R . Let R be an α -rigid ring, $A = (a_1, a_2, a_3, \dots, a_{n+1})$ and $B = (b_1, b_2, b_3, \dots, b_{n+1}) \in T_{n+1}(R)$. If $AB = 0$, then $A\bar{\delta}(B) = 0$. Also $AB = 0$ if and only if $A\bar{\alpha}(B) = 0$.*

Proof. Let $A = (a_1, a_2)$ and $B = (b_1, b_2) \in T_2(R)$ with $AB = 0$. Then $a_1b_1 = a_1b_2 + a_2b_1 = 0$. Since R is reduced, $a_1b_2 = a_2b_1 = 0$. Now suppose that $A = (a_1, a_2, a_3, \dots, a_{n+1})$ and $B = (b_1, b_2, b_3, \dots, b_{n+1}) \in T_{n+1}(R)$ with $AB = 0$. Then we have the following system of equations:

$$\begin{aligned} a_1b_1 &= 0; \\ a_1b_2 + a_2b_1 &= 0; \\ a_1b_3 + a_2b_2 + a_3b_1 &= 0; \end{aligned}$$

\vdots
 $a_1b_n + a_2b_{n-1} + \dots + a_nb_1 = 0;$
 $a_1b_{n+1} + a_2b_n + \dots + a_{n+1}b_1 = 0.$
 Hence $a_1b_j = 0$ for all $1 \leq j \leq n + 1$, since R is reduced. Thus
 $a_2b_1 = 0;$
 $a_2b_2 + a_3b_1 = 0;$
 \vdots
 $a_2b_{n-1} + \dots + a_nb_1 = 0;$
 $a_2b_n + \dots + a_{n+1}b_1 = 0,$
 and hence $(a_2, a_3, \dots, a_{n+1})(b_2, b_3, \dots, b_{n+1}) = 0$. By induction hypothesis, we have $a_ib_j = 0$ for all $2 \leq i \leq n + 1$ and $2 \leq j \leq n + 1 - i + 1$. Thus $a_ib_j = 0$ for all $1 \leq i \leq n + 1$ and $1 \leq j \leq n - i + 1$. Hence $a_i\alpha(b_j) = a_i\delta(b_j) = 0$ for all $1 \leq i \leq n + 1$ and $1 \leq j \leq n - i + 1$, since R is α -rigid. Therefore $A\bar{\alpha}(B) = A\bar{\delta}(B) = 0$. By a similar way one can show that, $A\bar{\alpha}(B) = 0$ implies $AB = 0$. \square

THEOREM 8. *Let α be a monomorphism and δ an α -derivation of a ring R . Then R is an α -rigid ring if and only if $T_n(R)$ is an $(\bar{\alpha}, \bar{\delta})$ -weak Armendariz ring, for each $n \geq 2$.*

Proof. One can show that the map $\varphi : T_n(R)[x; \bar{\alpha}, \bar{\delta}] \rightarrow T_n(R[x; \alpha, \delta])$, given by $\varphi(A_0 + A_1x + \dots + A_kx^k) = (g_1, g_2, g_3, \dots, g_n)$ with $A_i = (a_{i1}, a_{i2}, a_{i3}, \dots, a_{in})$ and $g_j = a_{0j} + a_{1j}x + \dots + a_{kj}x^k$, for all $0 \leq i \leq k$ and $1 \leq j \leq n$, is an isomorphism. Now suppose that $f = A_0 + A_1x + \dots + A_t x^t$ and $g = B_0 + B_1x + \dots + B_m x^m$ are polynomials in $T_n(R)[x; \bar{\alpha}, \bar{\delta}]$ with $fg = 0$, where $A_i = (a_{i1}, a_{i2}, a_{i3}, \dots, a_{in})$ and $B_j = (b_{j1}, b_{j2}, b_{j3}, \dots, b_{jn})$ for all $0 \leq i \leq t$ and $0 \leq j \leq m$. By Lemma 7, $f_i g_j = 0$ where $f_i = a_{0i} + a_{1i}x + \dots + a_{ti}x^t$ and $g_j = b_{0j} + b_{1j}x + \dots + b_{mj}x^m$ for $1 \leq i \leq n$ and $1 \leq j \leq n - i + 1$. Since R is α -rigid, we have $a_{ri}b_{sj} = 0$ for $1 \leq i \leq n$, $1 \leq j \leq n - i + 1$, $0 \leq r \leq t$ and $0 \leq s \leq m$. Hence $A_r B_s = 0$ for $0 \leq r \leq t$ and $0 \leq s \leq m$. Thus $A_r \bar{\alpha}(B_s) = A_r \bar{\delta}(B_s) = 0$, by Lemma 7. Therefore $T_n(R)$ is $(\bar{\alpha}, \bar{\delta})$ -Armendariz.

Conversely, suppose that $T_n(R)$ is $(\bar{\alpha}, \bar{\delta})$ -weak Armendariz. Hence R is (α, δ) -weak Armendariz as a subring. Let $r\alpha(r) = 0$, for $r \in R$. Then $\delta(r)\alpha(r) = \alpha(r)\delta\alpha(r) = 0$, by Lemma 3. Take $h(x) = (0, 0, 1, \dots, 0) - (0, \alpha(r), 0, \dots, 0)x$ and $k(x) = (0, 0, \dots, 1, 0) + (0, \dots, \alpha(r), 0, 0)x \in T_n(R)[x; \bar{\alpha}, \bar{\delta}]$. Then $h(x)k(x) = 0$. Since $T_n(R)[x; \bar{\alpha}, \bar{\delta}]$ is $(\bar{\alpha}, \bar{\delta})$ -weak Armendariz, $(0, 0, 1, \dots, 0)(0, \dots, \alpha(r), 0, 0) = 0$ and hence $\alpha(r) = 0$. Therefore $r = 0$, since α is monomorphism.

By [10, Proposition 6], every α -rigid ring is (α, δ) -Armendariz. The

following is an example of a non semiprime (hence non α -rigid) ring which is (α, δ) -Armendariz:

EXAMPLE 9. Let α be an endomorphism and δ be an α -derivation of R . Let R be an α -rigid ring and

$$R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

be a subring of the upper triangular matrix ring over a ring. The endomorphism α of R is extended to the endomorphism $\bar{\alpha} : R_3 \rightarrow R_3$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$ and the α -derivation δ of R is also extended to $\bar{\delta} : R_3 \rightarrow R_3$ defined by $\bar{\delta}((a_{ij})) = (\delta(a_{ij}))$. We can easily see that $\bar{\delta}$ is an $\bar{\alpha}$ -derivation of R_3 . We show that, (i) for $A, B \in R_3$, $AB = 0$ if and only if $A\bar{\alpha}(B) = 0$, also $AB = 0$ implies that $A\bar{\delta}(B) = 0$, (ii) R_3 is not $\bar{\alpha}$ -rigid, (iii) R_3 is $(\bar{\alpha}, \bar{\delta})$ -Armendariz.

(i) Suppose that $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \bar{\alpha} \left(\begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \right) = 0$. Then

we have the following equations:

- (1) $a_1\alpha(a_2) = 0$
- (2) $a_1\alpha(b_2) + b_1\alpha(a_2) = 0$
- (3) $a_1\alpha(c_2) + b_1\alpha(d_2) + c_1\alpha(a_2) = 0$
- (4) $a_1\alpha(d_2) + d_1\alpha(a_2) = 0$.

Since R is reduced, from Eq.(1), we have $\alpha(a_2)a_1 = 0$. Multiplying $\alpha(a_2)$ to Eq.(2) from the left-hand side, then we have $b_1\alpha(a_2) = \alpha(a_2)b_1 = 0$, since R is reduced. Hence $a_1\alpha(b_2) = \alpha(b_2)a_1 = 0$. Multiplying $\alpha(a_2)$ to Eq.(3) from the left-hand side, then we have $c_1\alpha(a_2) = \alpha(a_2)c_1 = 0$, since R is reduced. Hence Eq.(3) becomes

$$(5) \quad a_1\alpha(c_2) + b_1\alpha(d_2) = 0.$$

Multiplying $\alpha(a_2)$ to Eq.(4) from left-hand side, then $d_1\alpha(a_2) = \alpha(a_2)d_1 = 0$ and $a_1\alpha(d_2) = \alpha(d_2)a_1 = 0$, since R is reduced. Multiplying a_1 to Eq.(5) from the right-hand side, then we have $a_1\alpha(c_2) = \alpha(c_2)a_1 = 0$ and $b_1\alpha(d_2) = \alpha(d_2)b_1 = 0$. Thus $a_1a_2 = a_1b_2 = a_1c_2 = a_1d_2 = b_1a_2 = b_1d_2 = c_1a_2 = d_1a_2 = 0$, since R is α -rigid.

Hence
$$\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} = 0.$$

Now assume that
$$\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} = 0.$$

Then by a similar argument, we have $a_1\alpha(a_2) = a_1\alpha(b_2) = b_1\alpha(a_2) = c_1\alpha(a_2) = d_1\alpha(a_2) = a_1\alpha(d_2) = a_1\alpha(c_2) = b_1\alpha(d_2) = 0$.

Hence
$$\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \bar{\alpha} \left(\begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \right) = 0.$$

Assume that
$$\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} = 0.$$

Then $a_1a_2 = a_1b_2 = a_1c_2 = a_1d_2 = b_1a_2 = b_1d_2 = c_1a_2 = d_1a_2 = 0$. Hence $a_1\delta(a_2) = a_1\delta(b_2) = a_1\delta(c_2) = a_1\delta(d_2) = b_1\delta(a_2) = b_1\delta(d_2) = c_1\delta(a_2) = d_1\delta(a_2) = 0$, since R is α -rigid.

Thus
$$\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \bar{\delta} \left(\begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \right) = 0.$$

(ii) Since R_3 is not reduced, R_3 is not $\bar{\alpha}$ -rigid.

(iii) Suppose that $p = \sum_{i=0}^m \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} x^i$ and

$q = \sum_{j=0}^n \begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a_j & d_j \\ 0 & 0 & a_j \end{pmatrix} x^j$ are polynomials in $R_3[x; \bar{\alpha}, \bar{\delta}]$ such that

$pq = 0$. By Hong et al. [10, Proposition 6], $R[x; \alpha, \delta]$ is reduced. So that by Hong et al. [9, Proposition 17],

$$\begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} \bar{\alpha}^i \left(\begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a_j & d_j \\ 0 & 0 & a_j \end{pmatrix} \right) = 0, \text{ for all } i, j. \text{ Hence,}$$

$$\begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} \begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a_j & d'_j \\ 0 & 0 & a_j \end{pmatrix} = 0 \text{ for all } i, j. \text{ Therefore by (i) } R_3$$

is $(\bar{\alpha}, \bar{\delta})$ -Armendariz.

Suppose that R is a semiprime left Goldie ring and α is a monomorphism of R . Let $\mathcal{C} = \mathcal{C}_R(0)$ be the regular elements of R . By Goldie's Theorem [5, Theorem 5.10], R has a semisimple Artinian quotient ring $Q = \mathcal{C}^{-1}R$. By [12, Proposition 2.4], we have $\alpha^{-1}(\mathcal{C}_R(0)) = \mathcal{C}_R(0)$. We extend α to Q , with $\tilde{\alpha}(c^{-1}r) = \alpha(c)^{-1}\alpha(r)$, for $c \in \mathcal{C}_R(0)$ and $r \in R$. Now we consider the classical left quotient ring $Q = \mathcal{C}^{-1}R$ of an α -Armendariz semiprime left Goldie ring R , in the following:

THEOREM 10. *Let R be a semiprime left Goldie ring and α be a monomorphism of R such that $\alpha(1) = 1$. Then R is an α -Armendariz (resp. α -weak Armendariz) ring if and only if $Q = \mathcal{C}^{-1}R$ is an $\tilde{\alpha}$ -Armendariz (resp. $\tilde{\alpha}$ -weak Armendariz) ring.*

Proof. Let $f(x) = c_0^{-1}a_0 + c_1^{-1}a_1x + \dots + c_n^{-1}a_nx^n$ and $g(x) = s_0^{-1}b_0 + s_1^{-1}b_1x + \dots + s_m^{-1}b_mx^m$ be polynomials in $Q[x; \tilde{\alpha}]$ such that $f(x)g(x) = 0$. Since \mathcal{C} is a left denominator set, there exist $c, s \in \mathcal{C}$ and $a'_i, b'_j \in R$ such that $c_i^{-1}a_i = c^{-1}a'_i$ and $s_j^{-1}b_j = s^{-1}b'_j$ for $i = 0, 1, \dots, n, j = 0, 1, \dots, m$. Then $(a'_0 + a'_1x + \dots + a'_nx^n)s^{-1}(b'_0 + b'_1x + \dots + b'_mx^m) = 0$. Thus $(a'_0s^{-1} + a'_1\alpha(s)^{-1}x + \dots + a'_n\alpha^n(s)^{-1}x^n)(b'_0 + b'_1x + \dots + b'_mx^m) = 0$. There exist $s' \in \mathcal{C}$ and $a''_i \in R$ such that $a'_i\alpha^i(s)^{-1} = s'^{-1}a''_i$ for $i = 0, 1, \dots, n$. Hence $(a''_0 + a''_1x + \dots + a''_nx^n)(b'_0 + b'_1x + \dots + b'_mx^m) = 0$. Since R is α -Armendariz, $a''_i\alpha^i(b'_j) = 0$ for $i = 0, 1, \dots, n, j = 0, 1, \dots, m$. Hence $s'^{-1}a''_i\alpha^i(b'_j) = 0$ for $i = 0, 1, \dots, n, j = 0, 1, \dots, m$. Thus $a'_i\alpha^i(s^{-1}b'_j) = a'_i\alpha^i(s)^{-1}\alpha^i(b'_j) = 0$ for $i = 0, 1, \dots, n, j = 0, 1, \dots, m$ and hence $c_i^{-1}a_i\alpha^i(s_j^{-1}b_j) = c^{-1}a'_i\alpha^i(s^{-1}b_j) = 0$, for $i = 0, 1, \dots, n, j = 0, 1, \dots, m$. Therefore Q is $\tilde{\alpha}$ -Armendariz. The converse is clear.

In the above argument by replacing α -Armendariz with α -weak Armendariz, the result follows. \square

Now we extend [14, Proposition 18] and [16, Theorem 3.3], to the more general case as in the following:

THEOREM 11. *Let R be a semiprime left Goldie ring and α be a monomorphism of R such that $\alpha(1) = 1$. Then R is an α -weak Armendariz ring if and only if R is an α -rigid ring.*

Proof. By [9, Corollary 4], every α -rigid ring is α -weak Armendariz. Conversely, suppose that R is a semiprime left Goldie α -weak Armendariz ring. By Theorem 6, it is enough to show that R is reduced. Let $Q = \mathcal{C}^{-1}R$ be the left Goldie quotient ring of R . Then $\tilde{\alpha} : Q \rightarrow Q$ defined by $\tilde{\alpha}(s^{-1}r) = \alpha(s)^{-1}\alpha(r)$ is a monomorphism of Q . By Theorem 10, Q is $\tilde{\alpha}$ -weak Armendariz. Since R is semiprime left Goldie, Q is semisimple Artinian by [5, Theorem 5.10], and hence $Q \simeq M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$, where D_i are division rings and n_i are positive integers for $1 \leq i \leq k$. We claim that $n_i = 1$ for $1 \leq i \leq k$. First we show that $n_1 = 1$. Since Q is $\tilde{\alpha}$ -weak Armendariz, by Lemma 4, $\tilde{\alpha}(e) = e$ for each $e^2 = e \in Q$. Let E_{ij} be the matrix units in M_{n_1} for $1 \leq i \leq n, 1 \leq j \leq n$. Hence $\tilde{\alpha}((E_{ii}, 0, \dots, 0)) = (E_{ii}, 0, \dots, 0)$ for $1 \leq i \leq n$. Since $E_{1n}E_{nn} = E_{1n}$ and $E_{11}E_{1n} = E_{1n}$, we have $\tilde{\alpha}((E_{1n}, 0, \dots, 0))(E_{nn}, 0, \dots, 0) = \tilde{\alpha}((E_{1n}, 0, \dots, 0))$ and $(E_{11}, 0, \dots, 0)\tilde{\alpha}((E_{1n}, 0, \dots, 0)) = \tilde{\alpha}((E_{1n}, 0, \dots, 0))$. Thus there exists $a_{1n} \in D_1$ such that $\tilde{\alpha}((E_{1n}, 0, \dots, 0)) = (a_{1n}E_{1n}, 0, \dots, 0)$. Since $E_{n1}E_{11} = E_{n1}$ and $E_{nn}E_{n1} = E_{n1}$, we have $\tilde{\alpha}((E_{n1}, 0, \dots, 0))(E_{11}, 0, \dots, 0) = \tilde{\alpha}((E_{n1}, 0, \dots, 0))$ and $(E_{nn}, 0, \dots, 0)\tilde{\alpha}((E_{n1}, 0, \dots, 0)) = \tilde{\alpha}((E_{n1}, 0, \dots, 0))$. Thus there exists b_{n1} in D_1 such that $\tilde{\alpha}((E_{n1}, 0, \dots, 0)) = (b_{n1}E_{n1}, 0, \dots, 0)$. Since $E_{1n}E_{n1} = E_{11}$ and $E_{n1}E_{1n} = E_{nn}$, we have $\tilde{\alpha}((E_{n1}, 0, \dots, 0))\tilde{\alpha}((E_{1n}, 0, \dots, 0)) = (E_{nn}, 0, \dots, 0)$ and $\tilde{\alpha}((E_{1n}, 0, \dots, 0))\tilde{\alpha}((E_{n1}, 0, \dots, 0)) = (E_{11}, 0, \dots, 0)$. Thus $a_{1n}b_{n1} = 1 = b_{n1}a_{1n}$. Now suppose that $f(x) = \tilde{\alpha}((E_{1n}, 0, \dots, 0)) - (E_{11}, 0, \dots, 0)x$ and $g(x) = (E_{1n}, 0, \dots, 0) + (E_{nn}, 0, \dots, 0)x$. Then $f(x)g(x) = 0$. Since Q is $\tilde{\alpha}$ -weak Armendariz, $\tilde{\alpha}((E_{1n}, 0, \dots, 0))(E_{nn}, 0, \dots, 0) = 0$. Hence $a_{1n} = 0$. This is a contradiction. Thus $n_1 = 1$. By a similar argument one can show that $n_i = 1$ for $2 \leq i \leq k$. Hence $Q \simeq D_1 \times \cdots \times D_k$ and so Q is reduced. Therefore R is reduced. \square

By [10, Proposition 9], a ring R is α -rigid if and only if $R[x; \alpha, \delta]$ is reduced. Next we extend this result and [9, Proposition 20], to the more general setting, in the following:

THEOREM 12. *Let R be an (α, δ) -Armendariz ring. Then $R[x; \alpha, \delta]$ is an abelian ring.*

Proof. By Lemma 4, for each idempotent $e \in R$, $\alpha(e) = e$ and $\delta(e) = 0$. By Lemma 5, the set of idempotent elements of $R[x; \alpha, \delta]$ is a subset of the set of idempotent elements of R . So it is enough to show that R is abelian. Let A be the set of idempotents of R . For each $e, f \in A$, we claim that $efR \cap (1 - f)(1 - e)A = 0$. Suppose that $ef(-t) = (1 - f)(1 - e)s \in efR \cap (1 - f)(1 - e)A$ for some $t \in R$ and

$s \in A$. Let $g(x) = e + (1 - f)x$ and $h(x) = (1 - e)s + ftx$. Then $g(x)h(x) = [(1 - f)\delta(f)t + (1 - f)\alpha(f)\delta(t)]x = 0$, since $\delta(1 - e) = \delta(f) = 0$, $\alpha(f) = f$ and $ef(-t) = (1 - f)(1 - e)s$. Hence $eft = 0$, since R is (α, δ) -Armendariz. Thus $efR \cap (1 - f)(1 - e)A = 0$. Now suppose that $fe = 0$. So $-ef = (1 - f)(1 - e)f \in efR \cap (1 - f)(1 - e)A = 0$. Hence $ef = 0$. Next, take $k = e + er(1 - e)$ and $l = (1 - e) + (1 - e)re$, for $r \in R$. It is clear that $k^2 = k, l^2 = l$ and $(1 - e)k = el = 0$. Hence $k(1 - e) = le = 0$. Thus $er = ere = re$. Therefore R is an abelian ring. \square

Now we turn our attention to the relationship between the Baerness and p.p. property of a ring R and these of the skew polynomial ring $R[x; \alpha, \delta]$ in case R is (α, δ) -Armendariz.

Recall that R is a Baer ring if the right annihilator of every non-empty subset of R is generated by an idempotent of R . These definitions are left-right symmetric. A ring R is called a *right* (resp. *left*) *p.p.-ring* if every principal right (resp. left) ideal is projective (equivalently, if the right (resp. left) annihilator of an element of R is generated (as a right (resp. left) ideal) by an idempotent of R). R is called a p.p.-ring if it is both right and left p.p.

Now we extend [9, Theorem 21] and [14, Theorem 10], in the following:

THEOREM 13. *Let α be a monomorphism and δ be an α -derivation of R . If R is an (α, δ) -Armendariz ring, then R is a Baer ring if and only if $R[x; \alpha, \delta]$ is a Baer ring.*

Proof. Since R is (α, δ) -Armendariz, R is abelian by Theorem 12. But every abelian Baer ring is reduced, so R is α -rigid by Theorem 6. Therefore by [10, Theorem 11], the result follows. \square

By a similar proof as Theorem 13, we can obtain:

THEOREM 14. *Let α be a monomorphism and δ be an α -derivation of R . If R is an (α, δ) -Armendariz ring then, R is a p.p.-ring if and only if $R[x; \alpha, \delta]$ is a p.p.-ring.*

Note that Theorem 14, is a generalization of [9, Theorem 22] and [14, Theorem 9].

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