J. Korean Math. Soc. **42** (2005), No. 2, pp. 335–351

A THEOREM OF CLIFFORD TYPE FOR LINEAR SYSTEMS ON CURVES

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ABSTRACT. This paper concerns the relation between the degree and the projective dimension of linear systems on curves. We generalize Clifford's theorem and its improvement by Coppens and G. Martens and classify the special curves for our problem, and estimate their gonality.

0. Introduction

The well-known classical Clifford's Theorem (Theorem 2.1) is the starting point of the study of special divisors on curves. It gives a relation between the degree and the projective dimension of linear systems on curves. About fifteen years ago Coppens and G. Martens obtained an improvement of it in their famous paper [5] (Theorem 2.2).

We would like to refine these theorems in this paper. For this purpose we introduce a new notion of l-Clifford curves (Definition 2.5), which are the exceptional curves for our analogy of Clifford's inequality. So our problem is translated to the classification of l-Clifford curves. In our words, the result of Coppens and G. Martens is the determination of 2-Clifford curves.

In Section 2 we obtain a rough description of l-Clifford curves for any $l \geq 2$ (Theorem 2.7). To put it simply, we can say that any l-Clifford curve is nearly extremal (see Definition 2.4) or admits a covering of another curve of mapping degree $\leq l$. The next step is finding criteria for l-Clifford curves. So we investigate their gonality in the rest of the section. For example, we show that the gonality is not greater than $2l$ for l-Clifford curves of certain type (Theorem 2.11).

Received December 17, 2003.
2000 Mathematics Subject Classification: 14H51.
Key words and phrases: linear system, special divisor, Clifford's theorem, gonality.

In Section 3 we restrict ourselves to 3-Clifford curves and classify them in detail (Theorem 3.1, Theorem 3.3). We also determine their gonality (Corollary 3.4).

The author would like to express my sincere gratitude to my teacher K. Konno. The author is also grateful to Prof. C. Keem, Prof. A. Ohbuchi and the referee for many useful suggestions and all people I met during my stay in Korea.

Notation and Conventions

A variety (curve, surface, etc.) means a reduced, irreducible and projective one over the complex numbers $\mathbf{C}$ unless otherwise mentioned. Everywhere in this paper C stands for a smooth curve of genus g. The gonality of C is the minimal degree of surjective morphisms from C to $\mathbb{P}^1$.

A g_d^r is an r-dimensional linear system of degree d on C. It does not need to be complete nor free from base points, but it will be both mostly in this paper. If it is free from base points, then it gives a morphism from C onto a non-degenerate (possibly singular) curve in $\mathbf{P}^r$. It will be denoted by $\Phi_{g_d^r}$. When $r \geq 2$, the g_d^r is said to be simple if $\Phi_{g_d^r}$ is birational onto its image.

Assume that the g_d^r is not simple (then it is also said to be compounded) and let C' be the normalization of its image curve. Then the induced morphism $\varphi : C \to C'$ is a non-trivial covering map of some degree $n \geq 2$. A linear system g_e^s on C is said to be induced by φ (or by C') if there exists a $g_{e/n}'^s$ on C' such that $g_e^s = \varphi^*(g_{e/n}'^s)$. For example, our g_d^r is induced by φ.

Let D and D' be two divisors on a variety. We will write $D \sim D'$ if they are linearly equivalent.

For a smooth variety X, K_X denotes the canonical divisor.

For a real number x, $[x]$ denotes the greatest integer not exceeding x.

1. Preliminary results

We will use several known results.

LEMMA 1.1. *The degree of a non-degenerate surface S in $\mathbf{P}^r$ is not less than $r-1$. If it becomes equality, then S is one of the following:*

- *a rational normal surface scroll; or*
- *a Veronese surface (in this case $r = 5$).*

In the former case, if S is singular then it is a cone over a smooth rational curve.

LEMMA 1.2. *Let S be a surface scroll (possibly singular) and $C \subset S$ be a curve on S not lying in the singular locus of S. Let α denote the degree of the map from C to a general hyperplane section H of S induced by the ruling on S. Then the arithmetic genus of C is computed as follows:*

$$p_a(C) = (\alpha - 1)\left(\deg C - 1 - \frac{1}{2}\alpha \deg S\right) + \alpha g_0,$$

where g_0 is the sectional genus of S, i. e. $g_0 = p_g(H)$.

THEOREM 1.3 (Castelnuovo's bound [2], [6]). *Assume that C admits a simple linear system $g^r_d (r \geq 2)$ without base points. Then*

$$g \leq \pi_0(d, r) = \binom{m}{2}(r-1) + m\varepsilon,$$

where $m := \left[\frac{d-1}{r-1}\right]$ and $\varepsilon := d - 1 - m(r-1)$.

Recall that a smooth curve C of genus g which has a simple $g^r_d (r \geq 2)$ is said to be extremal with respect to the g^r_d if its genus is equal to Castelnuovo's bound, i. e. $g = \pi_0(d, r)$. Then the g^r_d is very ample and C is identified with the image curve of $\Phi_{g^r_d}$, which is also said to be extremal.

THEOREM 1.4 ([1], [2]). *Let d and r be integers such that $r \geq 3$, $d \geq 2r + 1$. Set $m := \left[\frac{d-1}{r-1}\right]$ and $\varepsilon := d - 1 - m(r-1)$. Then any extremal curve C in $\mathbf{P}^r$ lies on a surface of minimal degree and it is one of the following:*

(i) *The image of a smooth plane curve of degree d' under the Veronese embedding $\mathbf{P}^2 \to \mathbf{P}^5$. In this case $r = 5$, $d = 2d'$ and $\operatorname{gon}(C) = d' - 1$; or*
(ii) *A smooth member of the linear system $|mH + L|$ on a rational normal scroll. In this case $\varepsilon = 0$ and $\operatorname{gon}(C) = m$; or*
(iii) *A smooth member of the linear system $|(m+1)H - (r - \varepsilon - 2)L|$ on a rational normal scroll and $\operatorname{gon}(C) = m + 1$,*

where H (resp. L) is the class of a hyperplane section of the scroll (resp. of a line of the ruling).

THEOREM 1.5 ([6]). *Assume that C admits a simple $g_d^r (r \geq 2)$ without base points. Let C_0 be the image of the map $\Phi_{g_d^r} : C \to \mathbf{P}^r$ and let p_a denote the arithmetic genus of C_0. Set*

$$\pi_1(d,r) = \binom{m_1}{2} r + m_1(\varepsilon_1 + 1) + \mu_1,$$

where

$$m_1 := \left[\frac{d-1}{r}\right], \quad \varepsilon_1 := d - 1 - m_1 r$$

$$\text{and} \quad \mu_1 := \begin{cases} 1 & (\text{if } \varepsilon_1 = r-1) \\ 0 & (\text{otherwise}). \end{cases}$$

Then

(1) *If $p_a > \pi_1(d,r)$ and $d \geq 2r+1$, then C_0 lies on a surface of degree $r-1$.*
(2) *If $p_a = \pi_1(d,r)$ and $d \geq 2r+3$, then C_0 lies on a surface of degree r or less.*

REMARK 1.6 ([2], [5]). Let C be a smooth curve of genus g. The gonality of C, denoted by k, is bounded by g, i. e. $k \leq \left[\frac{g+3}{2}\right]$.

PROPOSITION 1.7 (Castelnuovo-Severi inequality). *Let C, B_1 and B_2 be smooth curves of respective genera g, g_1 and g_2. Assume that $\varphi_i : C \to B_i \quad (i = 1,2)$ is a surjective morphism of degree d_i. If $\psi := \varphi_1 \times \varphi_2 : C \to B_1 \times B_2$ is birational onto its image, then $g \leq (d_1 - 1)(d_2 - 1) + d_1 g_1 + d_2 g_2$.*

REMARK 1.8. We would like to point out a simple but useful fact concerning this proposition. Assume that ψ is not birational onto its image, i. e. $n := \deg \psi \geq 2$ and let C' be the normalization of its image curve. Then C' has two surjective morphisms $\varphi'_i : C' \to B_i \quad (i = 1,2)$ of degree d_i/n. Applying Castelnuovo-Severi inequality to C', we obtain the following inequality:

$$g(C') \leq \left(\frac{d_1}{n} - 1\right)\left(\frac{d_2}{n} - 1\right) + \frac{d_1}{n} g_1 + \frac{d_2}{n} g_2.$$

2. A theorem of Clifford type

In this section we generalize Clifford's theorem.

THEOREM 2.1 (Clifford's Theorem). *Let g_d^r be a linear system on C with $0 \leq d \leq 2g$. Then $2r \leq d$.*

Coppens and G. Martens obtained a refinement of this theorem.

THEOREM 2.2 ([5]). *Assume that C admits a complete linear system g_d^r without base points satisfying $0 \leq d \leq g-1$ and $3r > d$. Then one of the following occurs:*

(i) *$r \geq 2$, $d = 3r-1$ and g_d^r embeds C in $\mathbf{P}^r$ as an extremal curve with $g = 3r$. Furthermore, if $r \neq 5$ then C is tetragonal, i. e.* $\mathrm{gon}(C) = 4$. *If $r = 5$ and $d = 14$, there is a further possibility that C is hexagonal, i. e.* $\mathrm{gon}(C) = 6$*; or*

(ii) *C is a double covering of another smooth curve C' (of genus g') with $g \geq 6g'+3$. In this case the gonality of C and C', denoted by k and k' respectively, satisfy $k = 2k'$ and $2r \leq d - 2(k-3)$.*

REMARK 2.3. The converse of this result is also true. In fact, it is clear for (i). For the case (ii) we consider any linear system of degree $3g'+1$ on C'. Its dimension is $2g'+1$ since it is non-special. Thus the pull-back of the linear system is a $g_{2(3g'+1)}^{2g'+1}$ on C. Then $2(3g'+1) \leq g-1$ and $3 \cdot (2g'+1) > 2(3g'+1)$.

Clifford's Theorem and Theorem 2.2 tell us that

if $0 \leq d \leq 2g-2$ then $2r \leq d$ for any g_d^r on any curve,
if $0 \leq d \leq g-1$ then $3r \leq d$ for any g_d^r on C
unless C is one of the curves described in Theorem 2.2.

We would like to generalize this result. That is to say, we expect to have a better inequality for linear systems on almost all curves under some stronger condition and classify the rest curves.

First of all we introduce some notions for describing our results.

DEFINITION 2.4. Let C be a smooth curve of genus g which admits a simple $g_d^r (r \geq 2, d \geq 2r+1)$ and let C_0 be the image curve of $\Phi_{g_d^r}$. Then C is said to be nearly extremal with respect to the g_d^r if the arithmetic genus of C_0 is greater than $\pi_1(d,r)$. In this case C_0 is said to be nearly extremal, too.

Note that any nearly extremal curve lies on a surface of degree $r-1$ in $\mathbf{P}^r$ by Theorem 1.5.

DEFINITION 2.5. Let l be an integer ≥ 1. Consider the condition:

$(\star_l)$ if $0 \leq d \leq \frac{2}{l}(g-1)$ then $(l+1)r \leq d$ for any g_d^r.

We will say that C is an *l-Clifford curve* if it does not satisfy $(\star_l)$, i. e. there exists a g^r_d on C such that $0 \leq d \leq \frac{2}{l}(g-1)$ and $(l+1)r > d$.

From our point of view, Clifford's Theorem exactly states that any smooth curve satisfies $(\star_1)$, i. e. there exist no 1-Clifford curves. The result of Coppens and G. Martens (Theorem 2.2) is the determination of 2-Clifford curves. In fact, the notion of l-Clifford curves is motivated by these interpretations.

EXAMPLE 2.6. Examples of l-Clifford curves:

(1) Any smooth plane quintic or any smooth hyperelliptic curve of genus ≥ 3 is a 2-Clifford curve. Any smooth plane sextic or any smooth trigonal curve of genus ≥ 6 is a 3-Clifford curve.
(2) Let Γ (resp. Δ) be the fiber of the first (resp. second) projection from $\mathbb{P}^1 \times \mathbb{P}^1$. Then any smooth and irreducible member of $|5\Delta + 6\Gamma|$ is a 3-Clifford curve since its genus is twenty and it admits a g^3_{11}.

Roughly speaking, we can say that "any l-Clifford curve is nearly extremal or admits a covering of another curve of mapping degree $\leq l$". This is described more precisely in Theorem 2.7.

It might seem that the assumption for the degree in $(\star_l)$ is too strong, but it will turn out to be reasonable. We will discuss it after proving our main result (see Example 2.8).

Here we would like to give a simple remark. If a curve possesses a linear system violating $(\star_l)$ then in fact it admits a complete and base point free one as such. So it suffices to consider complete linear systems without base points in our argument. Then we obtain a description of l-Clifford curves for any $l \geq 2$.

THEOREM 2.7. *Let C be a smooth l-Clifford curve of genus g, where l is a fixed integer ≥ 2. Let g^r_d be a complete linear system without base points on C satisfying $0 \leq d \leq \frac{2}{l}(g-1)$ and $(l+1)r > d$. Then one of the following occurs:*

(i) *C is nearly extremal with respect to the g^r_d; or*
(ii) *C is a covering of another smooth curve C' (of genus g') of degree n $(2 \leq n \leq l)$. If $n \geq \frac{l+1}{2}$, then*

$$g \geq \frac{ln\{(l+1)g'+1\}}{2(l+1-n)} + 1$$

holds. Furthermore if $r \geq 2$, then the gonality of C and C', denoted by k and k' respectively, satisfy $k = nk'$ and the inequality $nr \leq d - 2k + 3n$ holds.

Proof. There are two cases:
(i) $r \geq 2$ and g^r_d is simple. We make our way by induction on l. The case $l = 2$ is directly follows from Theorem 2.2. So assume that $l \geq 3$. If C admits another simple linear system of dimension ≥ 2 which breaks $(\star_i)$ for some $i < l$, then the hypothesis of the induction implies the conclusion. Hence we may assume that C satisfies $(\star_i)$ for any $i < l$. In particular, since the condition $d \leq \frac{2}{l}(g-1)$ for our g^r_d implies $d \leq \frac{2}{l-1}(g-1)$, we obtain $d \geq lr$ by $(\star_{l-1})$.

Let C_0 be the image curve of the morphism $\Phi_{g^r_d} : C \to \mathbb{P}^r$. If $d > lr$ then $m_1 := [(d-1)/r] = l$, so we have

$$\pi_1(d,r) = \binom{l}{2} r + l(d-1-lr+1) = ld - \frac{1}{2}l(l+1)r.$$

Hence

$$g - \pi_1(d,r) > \frac{1}{2}ld - \pi_1(d,r) = \frac{1}{2}l\{(l+1)r - d\} > 0,$$

so C_0 is nearly extremal. It is similar in the case $d = lr$.
(ii) $r = 1$ or g^r_d is compounded. Then $n := \deg \Phi_{g^r_d} \geq 2$. Let C' be the normalization of the image of the morphism $\Phi_{g^r_d} : C \to \mathbb{P}^r$ and let g' be its genus. Then C has a covering $\varphi : C \to C'$ of degree n and C' possesses a $g'^r_{d/n}$ such that $\varphi^*(g'^r_{d/n}) = g^r_d$. We may assume that the covering map φ never factors through another curve.

Assume that $n \geq \frac{l+1}{2}$. Then the $g'^r_{d/n}$ is non-special, i. e. $r = \frac{d}{n} - g'$, since otherwise we have a contradiction $2r \leq \frac{d}{n} < \frac{l+1}{n}r$ by Clifford's Theorem. Since $d \leq (l+1)r - 1$, we obtain

$$0 \leq g' = \frac{d}{n} - r \leq \frac{1}{n}\{(l+1-n)r - 1\},$$

which shows $n \leq l$. We also have

$$r \geq \frac{ng'+1}{l+1-n}, \quad 2g-2 \geq ld = ln(g'+r) \geq \frac{ln\{(l+1)g'+1\}}{l+1-n}.$$

Finally, further assume that $r \geq 2$. Let us denote by k (resp. k') the gonality of C (resp. C'). It remains to show that $k = nk'$. Suppose, to the contrary, that $k < nk'$. The assumption for φ and Castelnuovo-Severi inequality (Proposition 1.7) imply

$$(*) \qquad g \leq (n-1)(k-1) + ng'.$$

Suppose that $g' = 0$. Then $k' = 1$, $k < n$ and $g < (n-1)^2$. On the other hand, $d = nr \geq 2n$ and $g > \frac{1}{2}ld \geq ln \geq n^2$ from our assumption.

This is a contradiction, hence we obtain $g' > 0$. It follows from $(*)$ and $k < nk' \leq n \cdot \frac{g'+3}{2}$ that

$$g < (n-1) \cdot \frac{n(g'+3)}{2} + ng'$$
$$= \frac{1}{2}n\{(n+1)g' + 3(n-1)\}$$

On the other hand, we have already obtained that $g \geq \frac{ln\{(l+1)g'+1\}}{2(l+1-n)}$. Hence

$$l\{(l+1)g'+1\} < (l+1-n)\{(n+1)g' + 3(n-1)\}.$$

It follows that

$$\{l(l+1) - (n+1)(l+1-n)\}g' \leq 3(n-1)(l+1-n) - l - 1.$$

It is easy to check that the left side is positive and we have

$$\{l(l+1) - (n+1)(l+1-n)\} \leq 3(n-1)(l+1-n) - l - 1,$$

which implies that

$$(l+1)^2 \leq 2(2n-1)(l+1-n).$$

But the maximal value of the right side as a function of n is $\left(l+\frac{1}{2}\right)^2$ (it is attained when $n = \frac{2l+3}{4}$), which is a contradiction. Hence $k = nk'$. In particular $k \leq n(g'+3)/2 = (d - nr + 3n)/2$. Thus we complete the proof. □

We should remark on our definition of l-Clifford curves, so we construct an example which shows its validity.

EXAMPLE 2.8. Let S be a smooth surface scroll over an elliptic curve in $\mathbf{P}^r (r \geq 3)$ and let H (resp. f) denote its hyperplane section (resp. its fiber of the ruling). Note that $H^2 = r$, $H.f = 1$ and $K_S \equiv -2H + rf$. Choose a smooth irreducible element C of the linear system $|5H - (r+1)f|$. Then its degree and genus are

$$d = C.H = 4r - 1,$$
$$g = \frac{1}{2}C.(C + K_S) + 1 = 6r - 3 = \pi_1(4r-1, r),$$

respectively. Hence they satisfy

$$d = \left[\frac{2}{3}(g-1)\right] + 2 \quad \text{and} \quad 4r > d,$$

but C is not nearly extremal. Thus we know that the notion of l-Clifford curves is valid if we hope a clear description of special curves for our problem.

DEFINITION 2.9. An l-Clifford curve C is said to be *of type I* (resp. *of type II*) if (i) (resp. (ii)) in Theorem 2.7 holds.

In the rest of this section, let l denote a fixed integer not less than 3. We shall estimate the gonality of l-Clifford curves.

First of all we note an easy fact:

LEMMA 2.10. *Let C be a smooth l-Clifford curve and assume that C satisfies $(\star_i)$ for any $i < l$. Then $k := gon(C) \geq l$.*

Proof. By definition C has a pencil g_k^1. Suppose that $k < l$. Let g_d^r be a linear system satisfying $0 \leq d \leq \frac{2}{l}(g-1)$ and $(l+1)r > d$. Note that $d \geq lr$ since otherwise the g_d^r breaks $(\star_{l-1})$. Hence $k < l \leq d$. So $k \cdot k < ld \leq 2g - 2$, which implies that $k < \frac{2}{k}(g-1)$. Applying $(\star_k)$ to the g_k^1 we have a contradiction $(k+1) \cdot 1 \leq k$. □

From now on we consider l-Clifford curves of type I. Let C be a smooth l-Clifford curve with a simple $g_d^r (r \geq 2)$ satisfying $0 \leq d \leq \frac{2}{l}(g-1)$ and $(l+1)r > d$. If $r = 2$ then C has a plane model of degree $d \leq 2l + 1$, which implies that $\mathrm{gon}(C) \leq 2l$.

So assume that $r \geq 3$. Theorem 2.7 tells us that the image curve C_0 of the morphism $\Phi_{g_d^r}$ lies on a surface S of degree $r - 1$ in $\mathbf{P}^r$. By Lemma 1.1 S is a rational normal surface scroll or a Veronese surface.

In the latter case, $r = 5$, $5l \leq d \leq 5l + 4$ and d is even. Furthermore C_0 is isomorphic to a plane curve of degree $d/2$. In particular $\mathrm{gon}(C) \leq \frac{d}{2} - 1 \leq \frac{5}{2}l + 1$.

In the following we restrict ourselves to the case that S is a rational normal surface scroll.

THEOREM 2.11. *Let C be a smooth l-Clifford curve with a simple $g_d^r (r \geq 3)$ satisfying $0 \leq d \leq \frac{2}{l}(g-1)$ and $(l+1)r > d$. Assume that the image curve C_0 of the morphism $\Phi_{g_d^r}$ lies on a rational normal scroll S. Let α denote the degree of the map from C_0 to a general hyperplane section of S induced by the ruling. Then*

(1) *The gonality k of C is not more than α and*

$$\alpha \leq \begin{cases} 2l & (r \geq 4) \\ (5l-1)/2 & (r = 3) \end{cases}$$

Furthermore if $r \geq 6$, or $l \geq 5$ and $r \geq 5$, then $\alpha < 2l$.

(2) *Assume that* C *satisfies* $(\star_i)$ *for any* $i < l$. *If*

$$\alpha < 2l \quad \textit{and} \quad g > \max\{(\alpha-1)(\alpha-2), 2(\alpha-1)(\alpha-6)+11\},$$

then $k = \alpha$. *In particular, if* $r \geq 16$ *then* $k = \alpha$.

Proof. (1) It is clear that $k \leq \alpha$. We can compute the arithmetic genus of C_0 by Lemma 1.2:

$$p_a(C_0) = (\alpha - 1)\left\{d - 1 - \frac{1}{2}\alpha(r-1)\right\}.$$

Note that $g_0 = 0$ since S is a rational normal scroll now. From our assumption we have $p_a(C_0) \geq g \geq ld/2 + 1$ and it follows that

$$f(\alpha) := (r-1)\alpha^2 - (2d + r - 3)\alpha + (l+2)d \leq 0.$$

Suppose that $r \geq 4$. Then a straightforward calculation tells us that

$$\begin{aligned} f(2l+1) &= (r-1)(2l+1)^2 - (2d+r-3)(2l+1) + (l+2)d \\ &= (2l+1)\{(2l+1)(r-1) - (r-3)\} - 3ld \\ &\geq (2l+1)(2lr - 2l + 2) - 3l\{(l+1)r - 1\} \\ &= (l^2 - l)r - 4l^2 + 5l + 2 \\ &\geq l + 2. \end{aligned}$$

Hence $f(2l+1) > 0$. On the other hand the quadratic function $f(\alpha)$ attains its minimal value at

$$\alpha_0 := \frac{2d + r - 3}{2(r-1)} = \frac{d-1}{r-1} + \frac{1}{2}.$$

It is easy to check that $\alpha_0 < 2l + 1$ if $r \geq 3$. Hence our inequality $f(\alpha) \leq 0$ implies that $\alpha \leq 2l$. It is similar in the other cases.

(2) First of all, note that $d \geq lr$ since C satisfies $(\star_{l-1})$. Assume that $\alpha < 2l$, $g > \max\{(\alpha-1)(\alpha-2), 2(\alpha-1)(\alpha-6)+11\}$ and suppose that $k < \alpha$. Then C has two different pencils g^1_α, g^1_k. Consider the morphism $\varphi = \Phi_{g^1_\alpha} \times \Phi_{g^1_k} : C \to \varphi(C) \subset \mathbb{P}^1 \times \mathbb{P}^1$. If it is birational, then Castelnuovo-Severi inequality tells us that $g \leq (\alpha-1)(k-1) \leq (\alpha-1)(\alpha-2)$. This contradicts our assumption, so φ is not birational, i. e. $s := \deg\varphi \geq 2$. Note that s divides both α and k, and $k \leq \alpha - s$. Let C' be the normalization of $\varphi(C)$ and denote its genus by g'. It has two different pencils $g'^1_{\alpha/s}$, $g'^1_{k/s}$ such that $g^1_\alpha = \varphi^*(g'^1_{\alpha/s})$, $g^1_k = \varphi^*(g'^1_{k/s})$. An easy calculation gives us that

$$1 \leq \frac{k}{s} \leq \frac{\alpha}{s} - 1 < \frac{2l}{s} - 1,$$

which shows that $s < l$. Then $k/s > 1$ (since $k \geq l$ by Lemma 2.10) and we then have $s < \frac{2}{3}l$ similarly as above. If $s \geq l/2$, then

$$2 \leq \frac{k}{s} \leq \frac{\alpha}{s} - 1 < 3,$$

which shows that $k/s = 2$ and $\alpha/s = 3$. Then C' is hyperelliptic and has a g_3^1 which is free from base points. Hence we obtain that $g' = 2$. Since C' has a $g_7'^5$, C has a g_{7s}^5. Then $5l \geq \frac{15}{2}s > 7s$. Then, from our hypothesis that C satisfies $(\star_{l-1})$, we obtain that $2g - 2 < (l-1) \cdot 7s \leq 7s(2s-1)$. On the other hand, from our assumption for g we have $2g - 2 > 2\alpha(\alpha - 3) = 18s(s-1)$. We then have $s = 2$, $k = 4$, $\alpha = 6$ and $l = 4$ by straightforward calculations. But then $2g - 2 < 42$, which contradicts that $48 \leq l^2 r \leq ld \leq 2g - 2$.

Hence we obtain $s < l/2$. Furthermore $k/s \geq 3$, since otherwise $k = (k/s) \cdot s < 2 \cdot (l/2) = l$. Take any linear system of degree $2g'+1$ on C'. Its dimension is $g'+1$ since it is non-special. Thus C' has a $g_{2g'+1}'^{g'+1}$. Then its pull-back by φ is a $g_{s(2g'+1)}^{g'+1}$ on C. Note that $2s \cdot (g'+1) > s(2g'+1)$. Now C satisfies $(\star_{2s-1})$ from our hypothesis, hence it follows that

$$2g - 1 \leq (2s-1) \cdot s(2g'+1).$$

By Remark 1.8, we also have

$$g' \leq \left(\frac{\alpha}{s} - 1\right)\left(\frac{k}{s} - 1\right).$$

Therefore

$$\begin{aligned}
g &\leq (2s-1) \cdot s\left(g' + \frac{1}{2}\right) + \frac{1}{2} \\
&\leq (2s-1) \cdot s\left\{\left(\frac{\alpha}{s} - 1\right)\left(\frac{k}{s} - 1\right) + \frac{1}{2}\right\} + \frac{1}{2} \\
&= \left(2 - \frac{1}{s}\right)\left\{(\alpha - s)(k - s) + \frac{1}{2}s^2\right\} + \frac{1}{2} \\
&= 2(\alpha - s)(k - s) + s^2 - (\alpha - s)\left(\frac{k}{s} - 1\right) - \frac{1}{2}s + \frac{1}{2} \\
&\leq 2(\alpha - s)(\alpha - 2s) + s^2 - 2(\alpha - s) - \frac{1}{2}s + \frac{1}{2} \quad \left(k \leq \alpha - s, \frac{k}{s} \geq 3\right)
\end{aligned}$$

It is easy to show that the last number attains its maximum as a function of s when $s = 2$ and the maximal value is $2(\alpha - 2)(\alpha - 5) + \frac{7}{2}$. Hence we obtain that $g \leq 2(\alpha-2)(\alpha-5)+3 = 2(\alpha-1)(\alpha-6)+11$, which is a contradiction.

If $r \geq 16$ then $\alpha < 2l$ by (1) and

$$g > \frac{1}{2}ld \geq \frac{1}{2}l^2 r \geq 8l^2 > 2\alpha^2.$$

Then our criterion is satisfied. □

3. The description of 3-Clifford curves

We obtained rough results for l-Clifford curves in the previous section. Here we concentrate on 3-Clifford curves and determine all of them. We may assume $(\star_2)$ because 2-Clifford curves are classified by Theorem 2.2.

THEOREM 3.1. *Assume that C satisfies $(\star_2)$. Let g^r_d be a complete linear system without base points on C satisfying $0 \leq d \leq \frac{2}{3}(g-1)$. If $4r > d$ then one of the following holds:*

(i-a) *$d = 4r-2$ and g^r_d embeds C in $\mathbf{P}^r$ as an extremal curve with $g = 6r-2$; or*

(i-b) *$d = 4r-1$ and g^r_d maps C birationally in $\mathbf{P}^r$ as a nearly extremal curve and $6r \leq g \leq 6r+2$ except for the case $r=2, g=15$ (C is smooth septic); or*

(ii) *C is a triple covering of another curve C' (of genus g') with $g \geq 18g'+6$. In this case $k = 3k'$ and $3r \leq d-(2k-9)$, where k (resp. k') is the gonality of C (resp. C').*

Proof. Let g^r_d be a complete linear system without base points on C with $d \leq \frac{2}{3}(g-1)$ and $4r > d$. Note that $d \geq 3r$. If $r=1$ then $d=3$ and $g \geq 6$ from our assumption, hence C is trigonal and (ii) holds. So assume that $r \geq 2$.

First consider the case that g^r_d is compounded. Let C' be the normalization of the image curve of $\Phi_{g^r_d}$ and let $\varphi : C \to C'$ be the lift of $\Phi_{g^r_d}$. All we have to show is that $n := \deg\varphi$ does not equal 2, since then we can apply Theorem 2.7. Suppose that $n=2$. Then d is even and C' has a $g'^r_{d/2}$, and its genus g' must satisfy $g \leq 6g'+2$ because otherwise C becomes a 2-Clifford curve by Remark 2.3. Since $d \leq \frac{2}{3}(g-1)$, we have $\frac{d}{2} \leq \frac{1}{3}(g-1) \leq 2g'+\frac{1}{3}$. Hence $\frac{d}{2} \leq 2g'$ and it follows from Clifford's Theorem that $2r \leq \frac{d}{2}$, i. e. $4r \leq d$. This contradicts our hypothesis.

Assume that the g^r_d is simple. First of all, it follows by Theorem 2.7 that the g^r_d maps C birationally in $\mathbf{P}^r$ as a nearly extremal curve. From our hypothesis and Castelnuovo's bound we have

$$(**) \qquad \frac{3}{2}d+1 \leq g \leq \pi_0(d,r) = \binom{m}{2}(r-1)+m\varepsilon,$$

where $m := [(d-1)/(r-1)]$ and $\varepsilon := d-1-m(r-1)$.

If $r=2$ then a direct calculation shows that

$$d=6 \quad \text{and} \quad g=\pi_0(6,2)=10, \quad \text{or}$$
$$d=7 \quad \text{and} \quad 12 \leq g \leq \pi_0(7,2)=15.$$

So assume $r \geq 3$. Then $m=3,4,$ or 5. But suppose that $m=3$, then $\pi_0(d,r)=3d-6r+3$ and $(**)$ implies that $d=4r-1$, contradicting to $m=3$. So $m=4$ or 5.

If $m=4$, then $\pi_0(d,r)=4d-10r+6$. By an easy calculation we obtain

$$d=4r-2 \quad \text{and} \quad g=\pi_0(4r-2,r)=6r-2, \quad \text{or}$$
$$d=4r-1 \quad \text{and} \quad 6r \leq g \leq \pi_0(4r-1,r)=6r+2.$$

If $m=5$, then $5 \leq \frac{d-1}{r-1} \leq \frac{4r-2}{r-1} = 4+\frac{2}{r-1}$, hence $r=3$ and $d=4r-1=11$. Then $(**)$ shows that $18 \leq g \leq \pi_0(11,3)=20$. So C is a curve of type (i-b). □

Remark 3.2. We would like to some remarks on the range of the degree in our hypothesis.
(1) Assume that C satisfies $(\star_2)$. If it admits a linear system g^r_d satisfying $d \leq g-2r-1$ and $4r > d$, then the conclusion of Theorem 3.1 holds. This is proved in a similar fashion.
(2) We can get a similar result for $d = \left[\frac{2}{3}(g-1)\right]+1$, but there exists a counterexample for $d=\left[\frac{2}{3}(g-1)\right]+2$ (see Example 2.8).

Let's proceed to the further step. From now on we consider 3-Clifford curves of type I. Let r be an integer not less than 3. Let C be a smooth 3-Clifford curve with a simple g^r_d satisfying $0 \leq d \leq \frac{2}{3}(g-1)$ and $4r > d$. We denote by C_0 the image of the morphism $\Phi_{g^r_d} : C \to \mathbf{P}^r$, by p_a its arithmetic genus. Then Theorem 1.5 tells us that C_0 lies on a surface S of degree $r-1$ in $\mathbf{P}^r$. Recall that S is a rational normal surface scroll or a Veronese surface.

In the former case, we will denote by H (resp. L, Δ) the linear equivalence class of its general hyperplane section (resp. of a line of the ruling, of the minimal section) and let α denote the degree of the map from C_0 to a general hyperplane section of S induced by the ruling on S.

If S is a cone over a smooth rational curve then consider the blowing-up of S at its vertex x_0, which gives us a smooth rational ruled surface $\tilde{S}$. Then $\Phi_{g_d^r}$ can be lifted to a morphism from C to $\tilde{S}$ because the fiber of $\Phi_{g_d^r}$ at x_0 is a divisor on C. Let $\tilde{C}_0$ be its image curve and we will denote by $\tilde{H}$ (resp. $\tilde{L}$) the class of the pull-back of a general hyperplane section (resp. of a line of the ruling) of S.

THEOREM 3.3. *Let the notation be as above. In particular, let C be a smooth 3-Clifford curve with a simple $g_d^r (r \geq 3)$ satisfying $0 \leq d \leq \frac{2}{3}(g-1)$ and $4r > d$. Assume that C satisfies $(\star_2)$. Let C_0 be the image curve of the morphism $\Phi_{g_d^r} : C \to \mathbf{P}^r$ and let S be a surface of degree $r-1$ in $\mathbf{P}^r$ containing C_0. Then one of the following holds:*
(1) If $d = 4r-2$ then C is extremal and $C \simeq C_0$ ($\simeq \tilde{C}_0$ if S is a cone) and there are three possibilities.
 (i) S is a smooth rational surface scroll and C_0 is a smooth and irreducible member of the linear system $|5H-(r-3)L|$ on S.
 (ii) S is a cone over a smooth rational curve and $\tilde{C}_0$ is a smooth and irreducible member of $|5\tilde{H}|$ on $\tilde{S}$. In this case $r=3$ and $d=10$.
 (iii) S is a Veronese surface and C_0 is the image of a smooth nonic under the Veronese embedding $\mathbf{P}^2 \to \mathbf{P}^5$. In this case $r=5$ and $d=18$.
(2) If $d=4r-1$ then there are two possibilities.
 (i) S is a smooth rational surface scroll and C_0 is a reduced and irreducible member of one of the following linear systems:

$$|4H+3L| \ ; \ g = p_a = 6r,$$
$$|5H-(r-4)L| \ ; \ 6r \leq g \leq p_a = 6r+2,$$
$$|6H-(2r-5)L| \ ; \ r = 3, 4, \text{ or } 5 \quad \text{and} \quad 6r \leq g \leq p_a = 5r+5,$$
$$|7H-3L| \ ; \ r=3 \quad \text{and} \quad g = p_a = 18.$$

 (ii) S is a cone over a smooth rational curve and $\tilde{C}_0$ is a reduced and irreducible member of one of the following linear systems:

$$|4\tilde{H}+3\tilde{L}| \ ; \ g = p_a = 6r,$$
$$|5\tilde{H}+\tilde{L}| \ ; \ r=3 \quad \text{and} \quad 18 \leq g \leq p_a = 20.$$

Proof. We restrict ourselves to the case that $d = 4r-1$ because otherwise the conclusion follows from Theorem 1.4. Then S cannot be a Veronese surface since the degree d of C_0 is odd, therefore S is a rational normal surface scroll. First suppose that S is smooth. Then $\mathrm{Pic}(S)$ is freely generated by H and L with $H^2 = r-1$, $H.L = 1$ and $L^2 = 0$.

Note that the canonical line bundle $K_S \sim -2H+(r-3)L$. Assume that C_0 belongs to the linear system $|\alpha H + \beta L|$. Then

$$d = 4r-1 = C_0.H = \alpha(r-1)+\beta, \quad C_0.L = \alpha.$$

By the adjunction formula

$$\begin{aligned} 2p_a - 2 &= C_0.(C_0+K_S) = C_0.\{(\alpha-2)H + (\beta+r-3)L\} \\ &= (\alpha-2)(4r-1) + \alpha\{4r-1-\alpha(r-1)+r-3\} \\ &= -(\alpha^2-9\alpha+8)r + \alpha^2 - 5\alpha + 2 \end{aligned}$$

Then it follows by $p_a \geq g \geq 6r$ that

$$12r - 2 \leq -(\alpha^2-9\alpha+8)r + \alpha^2 - 5\alpha + 2.$$

Simplifying it we obtain that

$$(\alpha-4)\{(\alpha-5)r - (\alpha-1)\} \leq 0,$$

which implies that $4 \leq \alpha \leq 7$. Then our conclusion follows from straightforward calculations.

If S is a cone then we will make a similar argument on $\tilde{S}$ instead of S. Assume that $\tilde{C}_0$ belongs to the linear system $|\alpha\tilde{H} + \beta\tilde{L}|$ of $\tilde{S}$. Note that $\tilde{H}$ does not intersect the minimal section $\tilde{\Delta}$ of $\tilde{S}$, which implies that

$$\begin{aligned} &\tilde{H} \sim \tilde{\Delta} + (r-1)\tilde{L}, \\ &\tilde{C}_0 \in |\alpha\tilde{\Delta} + \{\alpha(r-1)+\beta\}\tilde{L}|. \end{aligned}$$

Then it is necessary (and sufficient) that $\beta \geq 0$ for the last linear equivalence class to contain a reduced and irreducible member. Taking it into consideration, we can obtain our classification similarly as above. □

As a corollary of the theorem we can determine the gonality of 3-Clifford curves.

COROLLARY 3.4. *Let $r \geq 2$ be an integer and let the notation be as above if $r \geq 3$. Let C be a smooth 3-Clifford curve with a simple $g^r_d(r \geq 2)$ satisfying $0 \leq d \leq \frac{2}{3}(g-1)$ and $4r > d$. Assume that C satisfies $(\star_2)$. Then the gonality of C is determined as follows:*
(1) If $d = 4r-2$ then gon$(C) = 5$ *or* 8. *In the latter case, $r = 5$ and C is isomorphic to a smooth plane nonic.*
(2) If $d = 4r-1$, then gon$(C) = \alpha$ *if $r \geq 3$ except for the cases that $r = 3$ and $\alpha = 6, 7$. In the former case $k = 5$ and in the latter case $k = 4$. If $r = 2$ then*

$$\text{gon}(C) = \begin{cases} 6 & (g^2_7 \text{ is very ample}) \\ 5 & (\text{otherwise}) \end{cases}$$

In particular, $4 \leq \text{gon}(C) \leq 6$.

For simplifying the proof, we use a result by G. Martens for the gonality of smooth curves on a Hirzebruch surface. Let us denote by Σ_e the Hirzebruch surface with invariant e.

PROPOSITION 3.5 ([9]). *Let C be a smooth curve on Σ_e linearly equivalent to $a\Delta + bL$ $(a, b \in \mathbb{Z})$. Assume that $a \geq 2$, $b \geq ae$, and $a \neq b$ for $e = 1$, $b \geq a$ for $e = 0$. Then* $\text{gon}(X) = a$.

Proof. The first part of the conclusion is the direct consequence of Theorem 1.4. So we consider the second part.

Assume that $r \geq 3$. If $\alpha < 6$, then the conclusion directly follows from Theorem 2.11 (2).

Assume that $\alpha = 6$ (then S is smooth). If $r = 5$ then C_0 is smooth and we can apply Proposition 3.5. So we may assume $r = 3$ or 4. If $r = 4$ then S is isomorphic to the Hirzebruch surface Σ_1, which is the blow-up of $\mathbb{P}^2$ at one point and C is birational to a plane curve of degree 9 with a triple point. Then it is easy to show that $\text{gon}(C) = 6$ (see [10], Proposition 2). If $r = 3$ then S is isomorphic to $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $C_0 \sim 6\Delta + 5L$. Hence $\text{gon}(C) \leq 5$. If $\text{gon}(C) < 5$, then Castelnuovo-Severi inequality (Prop 1.7) gives a contradiction $18 \leq g \leq (5-1)(4-1) = 12$. Hence $\text{gon}(C) = 5$.

Assume that $\alpha = 7$. Then $r = 3$ and S is isomorphic to $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and $C \simeq C_0 \sim 7\Delta + 4L$. Hence $\text{gon}(C) = 4$.

Finally, assume that $r = 2$. If g^2_7 is very ample, then C is smooth septic ($\text{gon}(C) = 6$). Otherwise $12 \leq g \leq 14$ and the image of the g^2_7 is a singular plane septic with at most two nodes as its singularities. Then it is easy to show that $\text{gon}(C) = 5$ (for example, see [3], Theorem 2.1). □

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