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DECAY CHARACTERISTICS OF THE HAT INTERPOLATION WAVELET COEFFICIENTS IN THE TWO-DIMENSIONAL MULTIRESOLUTION REPRESENTATION

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Abstract. The objective of this study is to analyze the decay characteristics of the hat interpolation wavelet coefficients of some smooth functions defined in a two-dimensional space. The motivation of this research is to establish some fundamental mathematical foundations needed in justifying the adaptive multiresolution analysis of the hat-interpolation wavelet-Galerkin method. Though the hat-interpolation wavelet-Galerkin method has been successful in some classes of problems, no complete error analysis has been given yet. As an effort towards this direction, we give estimates on the decaying ratios of the wavelet coefficients at children interpolation points to the wavelet coefficient at the parent interpolation point. We also give an estimate for the difference between non-adaptively and adaptively interpolated representations.

1. Introduction

Wavelet-based numerical methods have recently received much attention in both applied mathematics and engineering communities. Due to the multiscale characteristics of wavelet bases, numerical analysis can be carried out adaptively in a multiresolution setting. The idea of multiresolution analysis goes back to the analysis using the hierarchical finite element [35, 38, 39]. Inspired by their approach, sparse grids [4, 12, 24, 25, 37] and a natural hierarchical refinement approach [31]

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have been proposed. By using the hierarchical finite element method, we can avoid the cumbersome remeshing process inevitable when standard finite element based adaptive approaches [2, 3, 6, 8, 22] are used. The idea behind the hierarchical finite element method is closely related to the multiscale multiresolution characteristics of wavelets, but the adaptive multiresolution analysis appears more effective when the numerical analysis directly works with the wavelet bases.

Depending on the characteristics of wavelet bases, the wavelet methods can be classified as orthogonal wavelet methods [1, 7, 14, 16, 17, 18] and interpolation wavelet methods [13, 29, 30]. In the reference cited above, issues such as error analysis are discussed, but the discussions are mainly given for the orthogonal wavelet methods; studies on convergence and error analysis for the interpolation wavelet-Galerkin methods are rare. Nevertheless, some satisfactory numerical results by the wavelet-Galerkin methods are reported [13, 29, 30], so there is a need to provide the mathematical foundation to justify the success and to improve the method.

The major advantage of using the interpolation wavelets over the orthogonal wavelets is that the formers work directly with nodal variables and can handle easily general boundary conditions prescribed on curvilinear boundary curves. The interpolation wavelet was proposed by Donoho [20] in order to compute the wavelet coefficients directly from the sampled data at the interpolation points rather than from some integrations over an interval. See [19] for recent researches on the interpolation wavelets. When the interpolation wavelets are used within the Galerkin formulation, the so-called wavelet-Galerkin methods are developed [13, 29, 30]. In this approach, the field variables are expanded by the interpolation wavelets in multiscales and substituted into the Galerkin form. However, common multiresolution adaptive strategies directly work with the values of the wavelet coefficients over resolution levels. This means that the sup-norm measure is used for adaptive analysis, but the Sobolev-norm is used to calculate the wavelet coefficients. The use of different norms makes it very difficult to analyze the solution convergence or error estimates.

In this work, we aim at carrying out some fundamental error analysis that is needed for full error analysis for the interpolation wavelet-Galerkin method. Specifically, we carry out the pointwise error analysis for two-dimensional multiscale interpolations. This is equivalent to giving estimates on the decaying ratios of the wavelet coefficients at children interpolation points to the wavelet coefficients at the parent interpolation point. To investigate the effects of adaptive analysis on the approximation accuracy, we also estimate the difference between non-adaptively and adaptively interpolated representations. Though the complete error or convergence analysis of the interpolation wavelet-Galerkin method is not performed here, the present analysis will serve as a first step toward the complete analysis.

2. Hat-interpolation wavelet-Galerkin method

This work is motivated by the need to establish error analysis of an interpolation wavelet-Galerkin method. So, we begin with the brief presentation of the hat interpolation wavelet-Galerkin method, which will be compared with the standard bilinear element method. The hat-interpolation wavelet-Galerkin method can be applied to general partial differential equations having nonzero boundary conditions for which bilinear finite element methods can be used. To make the subsequent analysis simpler, however, we will explain the wavelet-Galerkin method applied to the second-order elliptic partial differential equation having zero boundary conditions.

Let $\Omega \subset \mathbf{R}^n$ be a Lipschitz domain and $f \in L^2(\Omega)$. Then the Dirichlet elliptic problem is formulated as follows:

$$-\operatorname{div}(a\nabla u) + b \cdot u + cu = f \qquad \text{in } \Omega \tag{2.1a}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{2.1b}$$

where a is a uniformly elliptic operator, b is a vector function whose elements are all $L^\infty(\Omega)$ functions and c is an $L^\infty(\Omega)$ function. Let us define the Sobolev space $H_0^1(\Omega)$ as follows:

$$H_0^1(\Omega) = \{f \in H^1(\Omega) | f(x) = 0 \text{ for all } x \in \partial\Omega\}$$

The weak formulation of (2.1) is:

$$\int_\Omega a\nabla u \cdot \nabla v + (b \cdot \nabla u)v + cuv = \int_\Omega fv \tag{2.2}$$

for $v \in H_0^1(\Omega)$. It is well known that (2.2) has a unique stable solution $u \in H_0^1(\Omega)$ at least if $b = 0$ and $c \geq 0$, which results from the Lax-Milgram lemma [23].

To solve (2.2) in a finite dimensional space, we use a sequence of finite dimensional subspaces such that

$$\{V_j\}_{j=1,\ldots,\infty} \subset H_0^1(\Omega), \overline{\bigcup_{j=1}^{\infty} V_j} = H_0^1(\Omega).$$

To introduce the multiresolution paradigm into the weak formulation, let us assume that

$$V_1 \subset V_2 \subset V_3 \subset V_4 \subset \cdots \subset H_0^1(\Omega).$$

Let the finite-dimensional approximation u_j of the solution u of Equation (2.2) the solution of the following equation:

$$\int_\Omega a\nabla u_j \cdot \nabla v + (b \cdot \nabla u_j)v + cu_j v = \int_\Omega fv \tag{2.3}$$

for $v \in V_j$. Solving (2.3) in V_j is called the Galerkin method or the wavelet-Galerkin method depending on the construction of the finite-dimensional function space V_j and its basis functions.

To introduce the hat interpolation wavelet, let us assume $\Omega = [0,1] \times [0,1] \subset \mathbf{R}^2$. Let the finite-dimensional subspace $\{V_j\}_{j=1,2,\ldots}$ be

$$\begin{aligned} V_j &= \{\phi(x,y) \in C_0(\Omega) | \phi(x,y) \text{ is bilinear in each } \Omega_{kl} \\ &\qquad \text{for } k,l = 1,\ldots,2^j\} \end{aligned}$$

where

$$\begin{aligned} C_0(\Omega) &= \{f | f \text{ is continuous on } \Omega \text{ and } f = 0 \text{ on } \partial\Omega\}, \\ \Omega_{kl} &= \left[\frac{k-1}{2^j}, \frac{k}{2^j}\right] \times \left[\frac{l-1}{2^j}, \frac{l}{2^j}\right]. \end{aligned}$$

Let B_j be a basis of V_j. The resulting algebraic system for (2.3) is then written as

$$K_j e_j = f_j \tag{2.4}$$

where the system matrix K_j and the loading vector f_j are defined as

$$\begin{aligned} K_j(l,k) &= \int_\Omega a\nabla\phi_k \nabla\phi_l + (b \cdot \nabla\phi_k)\phi_l + c\phi_k\phi_l, \\ f_j(k) &= \int_\Omega f\phi_k, \phi_k, \phi_l \in B_j, \qquad k,\, l = 1,\ldots,2^{2j}. \end{aligned} \tag{2.5}$$

The finite-dimensional approximation u_j is reconstructed as

$$u_j(x,y) = \sum_{k=1}^{2^{2j}} e_j(k)\phi_k(x,y)$$

and it is well known that u_j converges to u in the H_0^1 norm or equivalently energy norm, as j goes to infinity. If all the elements of B_j are so constructed as to enter the basis B_{j+1}, multiresolution analysis can be carried out. This will require a multiscale basis representation and we will present the hat interpolation wavelet based multiscale representation. Thus it is possible to implement multiresolution analysis in the present method.

Though elliptic problems in two or higher dimensions are our main concern, we will explain first the hat interpolation wavelet method in the one-dimensional case. From now on, the notation $\tilde{}$ will be used for the one-dimensional case.

Let the scaling function of $H_0^1([0,1])$ be $\tilde{\phi}$. The multiresolution analysis (MRA) of a Hilbert space $H_0^1([0,1])$ defined in a finite interval is based on the following properties:

(2.6a) $$\tilde{V}_1 \subset \cdots \subset H_0^1([0,1]),$$

(2.6b) $$\overline{\bigcup_{j=1}^{\infty} \tilde{V}_j} = H_0^1([0,1]),$$

(2.6c) $\{\tilde{\phi}(2^j \cdot - k) : k = 1, \ldots, 2^j - 1\}$ is a Riesz basis of $\tilde{V}_j, j \geq 1$.

For the multiresolution analysis, it is very useful to introduce the wavelet space $\widetilde{W}_{j+1}$ defined as the complement of $\tilde{V}_j$ in the space $\tilde{V}_{j+1}$:

$$\tilde{V}_{j+1} = \tilde{V}_j \bigoplus \widetilde{W}_{j+1}.$$

Let us define $\tilde{\phi}$ as follows:

(2.7) $$\tilde{\phi}(x) = \begin{cases} 1+x & \text{if } -1 \leq x \leq 0 \\ 1-x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise .} \end{cases}$$

Though $\tilde{\phi}$ in itself does not contained in $H_0^1([0,1])$, there is no problem in constructing the hat interpolation system. Thus we can call $\tilde{\phi}$ the scaling function. The interpolation wavelet $\tilde{\psi}(x)$ generating $\widetilde{W}_1$ becomes

(2.8) $$\tilde{\psi}(x) = \tilde{\phi}(2x-1) \in H_0^1([0,1]).$$

Let $\tilde{\phi}_{j,k}(x) = \tilde{\phi}(2^j x - k)$ and $\tilde{\psi}_{j,k}(x) = \tilde{\psi}(2^{j-1}x - k) = \tilde{\phi}(2^j x - (2k+1))$, where j is the scale index of the functions $\tilde{\phi}_{j,k}$ and $\tilde{\psi}_{j,k}$. Due to (2.7) and (2.8), the support size of the basis functions $\tilde{\phi}_{j,k}, \tilde{\psi}_{j,k}$ is $\frac{1}{2^{j-1}}$. Using $\tilde{\phi}_{j,k}(x)$, one can form the single-scale bases of $\tilde{V}_j$ and $\widetilde{W}_j$ as linear span of the set $\{\widetilde{\phi}_{j,k}\}_{k=1,\dots,2^j-1}$ and $\{\widetilde{\psi}_{j,k}\}_{k=0,\dots,2^{j-1}-1}$, respectively. Using the fact that $\widetilde{\psi} = \widetilde{\psi}_{1,0} = \widetilde{\phi}_{1,1}$, we define $\widetilde{W}_1 = \widetilde{V}_1$. Since the bases have different scales, the subspace $\widetilde{V}_j$ can be written as

$$\widetilde{V}_j = \widetilde{V}_{j_0} \bigoplus \left[\bigoplus_{m=j_0+1}^{j} \widetilde{W}_m \right] = \bigoplus_{m=1}^{j} \widetilde{W}_m. \tag{2.9}$$

The last expression in (2.9) is obtained by inserting $j_0 = 1$ in the first equation where $\widetilde{W}_1 = \widetilde{V}_1$ is used. It is worth remarking that interpolation wavelets including the hat interpolation wavelet can be generated by convolving any orthogonal wavelet with itself. The interpolation wavelets have the good property of recovering some class of signals by uniform sampling. Donoho has shown that the orthogonal projection of a function $f \in L^2(\mathbf{R})$ onto an approximation space $\widetilde{V}_j$ converges in the sense of the L^∞ norm as well as the L^2 norm (see [32]).

Returning to $\mathbf{R}^2$, the finite-dimensional space V_j can be constructed by the following tensor product:

$$V_j = \widetilde{V}_j \bigotimes \widetilde{V}_j \tag{2.10}$$

The space V_j may be represented in either the standard or nonstandard multiscale representation [33]. Because some two-dimensional wavelets in the standard representation have very long supports in the one direction and very short support in the other direction, two-dimensional localizations by these wavelets are difficult to achieve. Henceforth we will mainly consider the nonstandard multiscale representation in the subsequent discussions. However the results of Section 3 also apply to the case of standard multiscale representation.

The space V_j in (2.10) is decomposed into a multiscale form as

$$V_j = (\widetilde{V}_{j-1} \bigoplus \widetilde{W}_j) \bigotimes (\widetilde{V}_{j-1} \bigoplus \widetilde{W}_j) \tag{2.11a}$$

$$= V_{j-1} \bigoplus W_j^h \bigoplus W_j^v \bigoplus W_j^d \tag{2.11b}$$

$$= V_{j_0} \bigoplus_{m=j_0+1}^{j} \left(W_m^h \bigoplus W_m^v \bigoplus W_m^d \right) \tag{2.11c}$$

where

$$W_m^h = \widetilde{W}_m \bigotimes \widetilde{V}_{m-1}, \quad W_m^v = \widetilde{V}_{m-1} \bigotimes \widetilde{W}_m, \quad W_m^d = \widetilde{W}_m \bigotimes \widetilde{W}_m.$$

The subspaces W_m^h, W_m^v and W_m^d may be called , the horizontal, the vertical and the diagonal wavelet space, respectively. Since $\widetilde{V}_m$ and

$\widetilde{W}_m$ are the one-dimensional interpolation scaling and wavelet spaces, W_m^h, W_m^v and W_m^d on $\Omega = [0,1] \times [0,1]$ are represented as:

$$
\begin{aligned}
(2.12)\ V_m &= \operatorname{span}\{\phi_{m,k,l}(x,y) = \widetilde{\phi}_{m,k}(x)\widetilde{\phi}_{m,l}(y) | 1 \le k, l \le 2^m - 1\} \\
W_m^h &= \operatorname{span}\{\psi_{m,k,l}^h(x,y) = \widetilde{\psi}_{m,k}(x)\widetilde{\phi}_{m-1,l}(y) | \\
&\qquad 0 \le k \le 2^{m-1} - 1, 1 \le l \le 2^{m-1} - 1\} \\
W_m^v &= \operatorname{span}\{\psi_{m,k,l}^v(x,y) = \widetilde{\phi}_{m-1,k}(x)\widetilde{\psi}_{m,l}(y) | \\
&\qquad 1 \le k \le 2^{m-1} - 1, 0 \le l \le 2^{m-1} - 1\} \\
W_m^d &= \operatorname{span}\{\psi_{m,k,l}^d(x,y) = \widetilde{\psi}_{m,k}(x)\widetilde{\psi}_{m,l}(y) | 0 \le k, l \le 2^{m-1} - 1\}
\end{aligned}
$$

Turing to the linear system (2.3), the solution of (2.3) may be found by using any basis set B_j of the finite-dimensional subspace V_j of $H_0^1(\Omega)$. Let the jth resolution level basis set $B_j^{j_0}$ be composed of the functions having scales from j_0 up to j:

$$
\begin{aligned}
(2.13)\qquad & B_j^{j_0} \\
= \ & \{\phi_{j_0,k,l} | 1 \le k, l \le 2^{j_0} - 1\} \\
& \bigcup \Bigg[\bigcup_{m=j_0+1}^{j} \{\psi_{m,k_1,l_1}^h, \psi_{m,l_2,k_2}^v, \psi_{m,k_3,k_4}^d | \\
& 0 \le k_1, k_2, k_3, k_4 \le 2^{m-1} - 1, 1 \le l_1, l_2 \le 2^{m-1} - 1\}. \Bigg]
\end{aligned}
$$

In expressing the basis set $B_j^{j_0}$ in (2.11) the multiscale space decomposition (2.11c) and the subspace representation (2.12) are used. The subscripts k_1, k_2, k_3, and k_4 in (2.13) are indices for one-dimensional wavelets and the subscripts l_1 and l_2, the indices for one-dimensional scaling functions. If B_j^j are used to solve (2.3), the resulting method is the standard bilinear finite element method. However, the hat interpolation wavelet-Galerkin method uses the multiscale basis set $B_j^{j_0} (j_0 < j)$ encompassing scale j_0 to scale j. Furthermore some relations between the bilinear finite element method and the hat interpolation wavelet method hold, which can be stated as

$$
[K_j]_{WG} = T_j^t [K_j^{j_0}]_{FEM} T_j, \quad [f_j]_{WG} = T_j^t [f_j^{j_0}]_{FEM}
$$

where $[\cdot]_{FEM}$, or $[\cdot]_{WG}$ are the quantities defined in (2.4) with the basis set equal to B_j^j or $B_j^{j_0}$, respectively. The explicit expression of the transformation matrix T_j for two-dimensional cases is given in [29].

3. Decay analysis of wavelet coefficients in multiscale representations

In this section, we will give estimates on the decaying ratios of the wavelet coefficients at children interpolation points to the wavelet coefficients at the parent interpolation point when a multiscale representation of a function is used.

3.1. Multiscale approximation in one dimensional cases

Consider the problem of approximating a function $u \in H_0^1(\Omega)$ by the multiscale multiresolution approximation using the interpolation wavelet basis set B_J^1. Though Ω will be restricted to $\Omega = [0,1]$ in this section, the generalization of the analysis $\Omega = [a,b]$ is straightforward. Let us associate each wavelet function $\psi_{j,k}$ with the center point $x = \frac{2k+1}{2^j} \in \Omega$ of its support. For the notational convenience, $\psi_{j,k}$ will be denoted by ψ^x.

In the wavelet-based multiscale representation, there exist a unique pair of integers (j,k) assigned to represent the location of $x = x_{j,k} = \frac{2k+1}{2^j}$. Thus the notation (j,k) may be considered as an operator mapping from some point x in Ω to nonnegative integers, so it will be denoted as $(j(x), k(x))$. We also note that in the multiscale approximation using the basis set B_J^1, the wavelet basis function associated with $x = x_{j,k}$ will appear only in the resolution levels greater than j.

For subsequent analysis, we use the following terminology.

1. The scale of $x = x_{j,k}$: $j(x)$
2. The right child of x : $x^+ = x + (\frac{1}{2})^{j(x)+1}$
3. The left child of x : $x^- = x - (\frac{1}{2})^{j(x)+1}$
4. The children of x : $x^c = \{x^+, x^-\}$
5. The parent of x : x^p is $x - (\frac{1}{2})^{j(x)}$ or $x + (\frac{1}{2})^{j(x)}$ satisfying $x \in (x^p)^c$
6. The neighbor of x : x^n is $x - (\frac{1}{2})^{j(x)}$ or $x + (\frac{1}{2})^{j(x)}$ such that $x^n \neq x^p$

Table 1 illustrates how the families of the points at $j = 4$ are linked.

Suppose that $u \in H_0^1(\Omega)$ and that $P_{\widetilde{V}_J}$ is an interpolating projection from $H_0^1(\Omega)$ to $\widetilde{V}_J$ such that $[P_{\widetilde{V}_J}(u)](x) = u(x)$ for each $x = x_{j(x),k(x)}$

TABLE 1. The locations of the parents(x^p), the neighborhood(x^n),etc. for $x = x_{j,k}$ with $j = 4$. The subscripts stands for the binary representation of x.

x	x^p	x^n	$(x^p)^p$	$(x^p)^n$
$1/2^4 = 0.0001_{(2)}$	$0.001_{(2)}$	$0_{(2)}$	$0.01_{(2)}$	$0_{(2)}$
$3/2^4 = 0.0011_{(2)}$	$0.001_{(2)}$	$0.01_{(2)}$	$0.01_{(2)}$	$0_{(2)}$
$5/2^4 = 0.0101_{(2)}$	$0.011_{(2)}$	$0.01_{(2)}$	$0.01_{(2)}$	$0.1_{(2)}$
$7/2^4 = 0.0111_{(2)}$	$0.011_{(2)}$	$0.1_{(2)}$	$0.01_{(2)}$	$0.1_{(2)}$
$9/2^4 = 0.1001_{(2)}$	$0.101_{(2)}$	$0.1_{(2)}$	$0.11_{(2)}$	$0.1_{(2)}$
$11/2^4 = 0.1011_{(2)}$	$0.101_{(2)}$	$0.11_{(2)}$	$0.11_{(2)}$	$0.1_{(2)}$
$13/2^4 = 0.1101_{(2)}$	$0.111_{(2)}$	$0.11_{(2)}$	$0.11_{(2)}$	$1_{(2)}$
$15/2^4 = 0.1111_{(2)}$	$0.111_{(2)}$	$1_{(2)}$	$0.11_{(2)}$	$1_{(2)}$

with $j(x) \le J$. Then, the following holds [32]:

$$
\begin{aligned}
(3.1) \qquad P_{\widetilde{V}_J} u &= \sum_{k=1}^{2^J-1} u\left(\frac{k}{2^J}\right) \widetilde{\phi}_{J,k} \\
&= \sum_{k=1}^{2^{j_0}-1} u\left(\frac{k}{2^{j_0}}\right) \widetilde{\phi}_{j_0,k} + \sum_{j=j_0+1}^{J} \sum_{k=0}^{2^{j-1}-1} d_j[k] \widetilde{\psi}_{j,k}
\end{aligned}
$$

where

$$
\begin{aligned}
d_j[k] &= u\left(\frac{2k+1}{2^j}\right) - [P_{\widetilde{V}_{j-1}} u]\left(\frac{2k+1}{2^j}\right) \\
&= u\left(\frac{2k+1}{2^j}\right) - \frac{u(\frac{k}{2^{j-1}}) + u(\frac{k+1}{2^{j-1}})}{2}.
\end{aligned}
$$

Thus we can treat $d_j[k]$ as a function of x: $d(x) = d_{j(x)}[k(x)]$. Using the definition of x^p, x^n for $x = x_{j,k}$, $d(x)$ can be expressed as

$$
(3.2) \qquad d(x) = u(x) - \frac{u(x^p) + u(x^n)}{2}.
$$

Let us now define two sets $\widetilde{X}^u_{j_0}$ and $\widetilde{X}^{u,\epsilon}_{j_0}$ associated with some functions $u \in C^4(\Omega) \bigcap H^1_0(\Omega)$ such that:

$$
\begin{aligned}
\widetilde{X}^u_{j_0} &= \{x = x_{j,k} \in \Omega | j(x) \ge j_0, d(x) \ne 0\}, \\
\widetilde{X}^{u,\epsilon}_{j_0} &= \{x \in \widetilde{X}^u_{j_0} || u^{(2)}(x)| \ge \epsilon\}
\end{aligned}
$$

and introduce the following norms and seminorms:

$$\begin{aligned} \|f\|_0 &= \text{esssup}_{x\in\Omega} |f(x)|, \\ |f|_m &= \left\| f^{(m)} \right\|_0, \\ \|f\|_m &= \max_{m'=1,\dots,m} |f|_{m'}. \end{aligned}$$

THEOREM 3.1. *Let* $u \in C^4(\Omega) \cap H_0^1(\Omega)$ *and* $j_0 > \frac{1}{2}\log_2 \frac{\|u^{(4)}\|_0}{12\epsilon}$. *Then for* $x \in \widetilde{X}_{j_0}^{u,\epsilon}$, *the following estimates hold:*

$$\text{(3.3a)} \qquad \left|\frac{d(x^+)}{d(x)}\right| \le \frac{1}{4}\left(1 + \frac{\|u^{(3)}\|_0}{\epsilon} h + \frac{C_1}{\epsilon} h^2\right),$$

$$\text{(3.3b)} \qquad \left|\frac{d(x^-)}{d(x)}\right| \le \frac{1}{4}\left(1 + \frac{\|u^{(3)}\|_0}{\epsilon} h + \frac{C_2}{\epsilon} h^2\right),$$

$$\text{(3.3c)} \qquad \left|\frac{[d(x^+) + d(x^-)]/2}{d(x)}\right| \le \frac{1}{4}\left(1 + \frac{5}{12}\frac{\|u^{(4)}\|_0}{\epsilon} h^2 + \frac{C_3}{\epsilon} h^4\right),$$

where $h = x^+ - x = 2^{-j(x)-1}$, C_1, C_2 *and* C_3 *are constants depending on* $\|u^{(3)}\|_0$ *and* $\|u^{(4)}\|_0$.

The proof of Theorem 3.1 will be given at the end of this section.

Consider the Taylor series of a function u around a_0 depending on the regularity of u.

$$\text{(3.4)} \quad u(a) = u(a_0) + u^{(1)}(a_0)b + u^{(2)}(a_0)\frac{b^2}{2} + u^{(3)}(a_0)\frac{b^3}{6} + u^{(4)}(\xi)\frac{b^4}{24}$$

where $b = a - a_0$ and ξ is some number between a and a_0. For a point $x \in (0,1)$, one of the following conditions always hold:

$$\text{(3.5a)} \qquad x^n = (x^-)^n < x^- < x < x^+ < (x^+)^n = x^p$$

$$\text{(3.5b)} \qquad x^p = (x^-)^n < x^- < x < x^+ < (x^+)^n = x^n.$$

Let us assume that (3.5a) holds. Then we get $h = x^p - x^+ = x^+ - x = x - x^- = x^- - x^n$ and the following Lemma 3.2 holds.

LEMMA 3.2. *Suppose that* $u \in C^4(\Omega) \bigcap H_0^1(\Omega)$ *and choose* $x \in \Omega$ *such that* $d(x) \neq 0$. *Then there exist 4 numbers* $\xi_1 \in (x, x+h), \xi_2 \in (x-h, x), \xi_3 \in (x, x+2h), \xi_4 \in (x-2h, x)$ *with* $h = 2^{-j(x)-1}$ *for which* *the following equations hold:*

$$\frac{d(x^+)}{d(x)} = \frac{1}{4}\frac{u^{(2)}(x)+hu^{(3)}(x)+h^2\left(-\frac{u^{(4)}(\xi_1)}{12}+\frac{2u^{(4)}(\xi_3)}{3}\right)}{u^{(2)}(x)+h^2\left(\frac{u^{(4)}(\xi_3)}{6}+\frac{u^{(4)}(\xi_4)}{6}\right)}, \tag{3.6}$$

$$\frac{d(x^-)}{d(x)} = \frac{1}{4}\frac{u^{(2)}(x)-hu^{(3)}(x)+h^2\left(-\frac{u^{(4)}(\xi_2)}{12}+\frac{2u^{(4)}(\xi_4)}{3}\right)}{u^{(2)}(x)+h^2\left(\frac{u^{(4)}(\xi_3)}{6}+\frac{u^{(4)}(\xi_4)}{6}\right)},$$

$$\frac{[d(x^+)+d(x^-)]/2}{d(x)} = \frac{1}{4}\frac{u^{(2)}(x)+h^2\left(-\frac{u^{(4)}(\xi_1)}{24}-\frac{u^{(4)}(\xi_2)}{24}+\frac{u^{(4)}(\xi_3)}{3}+\frac{u^{(4)}(\xi_4)}{3}\right)}{u^{(2)}(x)+h^2\left(\frac{u^{(4)}(\xi_3)}{6}+\frac{u^{(4)}(\xi_4)}{6}\right)}.$$

Proof. Assume that the condition (3.5a) hold, without loss of generality. Then inserting $a_0 = x$ and $a = x^+, x^-, x^p, x^n$ into (3.4) yields

$$\begin{aligned}
u(x^+) &= u(x)+hu^{(1)}(x)+\frac{h^2}{2}u^{(2)}(x)+\frac{h^3}{6}u^{(3)}(x) \\
&\quad +\frac{h^4}{24}u^{(4)}(\xi_1) \\
u(x^-) &= u(x)-hu^{(1)}(x)+\frac{h^2}{2}u^{(2)}(x)-\frac{h^3}{6}u^{(3)}(x) \\
&\quad +\frac{h^4}{24}u^{(4)}(\xi_2) \\
u(x^p) &= u(x)+2hu^{(1)}(x)+2h^2u^{(2)}(x)+\frac{4h^3}{3}u^{(3)}(x) \\
&\quad +\frac{2h^4}{3}u^{(4)}(\xi_3) \\
u(x^n) &= u(x)-2hu^{(1)}(x)+2h^2u^{(2)}(x)-\frac{4h^3}{3}u^{(3)}(x) \\
&\quad +\frac{2h^4}{3}u^{(4)}(\xi_4)
\end{aligned} \tag{3.7}$$

where ξ_1, ξ_2, ξ_3, and ξ_4 are some constants in the intervals of $(x, x+h), (x-h, x), (x, x+2h), (x-2h, x)$, respectively. Using $(x^+)^p = (x^-)^p = x$, (3.2), (3.5a), and (3.7), we obtain

$$\text{(3.8a)}\quad d(x^+) = -\frac{h^2}{2}u^{(2)}(x) - \frac{h^3}{2}u^{(3)}(x) + h^4\left(\frac{u^{(4)}(\xi_1)}{24} - \frac{u^{(4)}(\xi_3)}{3}\right)$$

$$\text{(3.8b)}\quad d(x^-) = -\frac{h^2}{2}u^{(2)}(x) + \frac{h^3}{2}u^{(3)}(x) + h^4\left(\frac{u^{(4)}(\xi_2)}{24} - \frac{u^{(4)}(\xi_4)}{3}\right)$$

$$\text{(3.8c)}\quad d(x) = -2h^2u^{(2)}(x) + h^4\left(-\frac{u^{(4)}(\xi_3)}{3} - \frac{u^{(4)}(\xi_4)}{3}\right).$$

It is now straightforward to obtain (3.6) from the expressions (3.8). □

LEMMA 3.3. *Let $u \in C^4(\Omega) \cap H_0^1(\Omega)$ and*

$$\text{(3.9)}\quad \frac{\left\|u^{(4)}\right\|_0}{12 \cdot 4^{j(x)}\left|u^{(2)}(x)\right|} < 1.$$

Then the following estimates hold:

$$\text{(3.10)}\quad \begin{aligned} \frac{d(x^+)}{d(x)} &= \frac{1}{4}\left(1 + \frac{u^{(3)}(x)}{u^{(2)}(x)}h + \frac{C_1(x)}{u^{(2)}(x)}h^2\right), \\ \frac{d(x^-)}{d(x)} &= \frac{1}{4}\left(1 - \frac{u^{(3)}(x)}{u^{(2)}(x)}h + \frac{C_2(x)}{u^{(2)}(x)}h^2\right), \\ \left|\frac{[d(x^+) + d(x^-)]/2}{d(x)}\right| &\leq \frac{1}{4}\left(1 + \frac{5}{12}\frac{\left\|u^{(4)}\right\|_0}{\left|u^{(2)}(x)\right|}h^2 + \frac{C_3(x)}{u^{(2)}(x)}h^4\right), \end{aligned}$$

where $h = 2^{-j(x)-1}$, $C_1(x), C_2(x)$ and $C_3(x)$ are constants depending on $u^{(3)}(x)$ and $\left\|u^{(4)}\right\|_0$.

Proof. We will begin with the following equation:

$$\text{(3.11)}\quad \frac{1}{1+s} = 1 - s + \frac{s^2}{1+s}.$$

If $|s| < \delta < 1$, (3.11) means $\frac{1}{1+s} = 1 - s + O(\frac{s^2}{1-\delta})$. To apply (3.11) to (3.6), we must check the following:

$$\text{(3.12)}\quad \left|\frac{h^2}{u^{(2)}(x)}\left(\frac{u^{(4)}(\xi_3)}{6} + \frac{u^{(4)}(\xi_4)}{6}\right)\right| \leq \frac{\left\|u^{(4)}\right\|_0}{12 \cdot 4^{j(x)}\left|u^{(2)}(x)\right|} < 1$$

which is derived from (3.9). Thus taking $\delta = \frac{\|u^{(4)}\|_0}{12\cdot 4^{j(x)}|u^{(2)}(x)|}$, dividing the numerator and the denominator in (3.6) by $u^{(2)}(x)$, and using the estimates (3.11) and (3.12), we get the lemma. □

The proof of Theorem 3.1. Equation (3.9) is satisfied for all $x \in \widetilde{X}_{j_0}^{u,\epsilon}$, since

$$\frac{\|u^{(4)}\|_0}{12\cdot 4^{j(x)}\left|u^{(2)}(x)\right|} \leq \frac{\|u^{(4)}\|_0}{12\cdot 4^{j_0}\epsilon} < 1.$$

Thus we get the inequalities (3.10) with the constants $C_1(x), C_2(x)$ and $C_3(x)$ are bounded by constants C_1, C_2 and C_3 for all $x \in \widetilde{X}_{j_0}^{u,\epsilon}$ depending only on $\|u^{(2)}\|_0, \|u^{(3)}\|_0$ and $\|u^{(4)}\|_0$. Further using $\frac{|u^{(3)}(x)|}{|u^{(2)}(x)|}$ and $\frac{\|u^{(4)}\|_0}{|u^{(2)}(x)|}$ are bounded by $\frac{\|u^{(3)}\|_0}{\epsilon}$ and $\frac{\|u^{(4)}\|_0}{\epsilon}$, the theorem is proved. □

3.2. Multiscale approximation in two-dimensional cases

In this subsection, let $\Omega = [0,1]\times[0,1]$ and $x = x_{j(x),k(x)} = \frac{2k(x)+1}{2^{j(x)}}$, $y = y_{j(y),l(y)} = \frac{2l(y)+1}{2^{j(y)}} \in [0,1]$. The definition of the operator $l(y)$ mapping from $[0,1]$ to nonnegative integers is the same as that of $k(x)$. The definitions of children and parents for two-dimensional cases are somewhat different from those for one-dimensional cases. The scale $j(x)$ is not necessarily the same as $j(y)$ due to the multiscale characteristics of the basis set $B_J^{j_0}$. Therefore, some care must be taken to give consistent definitions of children and parents in two-dimensional cases. We will use the following terminology for two-dimensional cases.

1. The x-scale of (x,y) : $j(x)$
2. The y-scale of (x,y) : $j(y)$
3. The scale $j(x,y)$ of (x,y) : $j(x,y) = \max(j(x), j(y))$
4. The mesh size h at scale $j(x,y)$: $h = 2^{-j(x,y)-1}$
5. The point (x,y) is the horizontal point $(x,y)_h$ if $j(x,y) = j(x) > j(y)$
   (See Figure 1(a))
   - The children of $(x,y)_h$: $(x,y)_h^c = \{(x-h, y-2h), (x+h, y-2h), (x-h, y), (x+h, y), (x-h, y+2h), (x+h, y+2h)\} \cap \Omega$
6. The point (x,y) is the vertical point $(x,y)_v$ if $j(x,y) = j(y) > j(x)$
   (See Figure 1(b))
   - The children of $(x,y)_v$: $(x,y)_v^c = \{(x-2h, y-h), (x-2h, y+h), (x, y-h), (x, y+h), (x+2h, y-h), (x+2h, y+h)\} \cap \Omega$

7. The point (x,y) is the diagonal point $(x,y)_d$ if $j(x,y) = j(x) = j(y)$ (See Figure 1(c))
   - The diagonal children of $(x,y)_d$: $(x,y)_d^c = \{(x-h, y-h), (x-h, y+h), (x+h, y-h), (x+h, y+h)\} \cap \Omega$
8. The parent of (x,y) : $(x,y)^p$ for which $(x,y) \in ((x,y)^p)^c$.

Note that the children of horizontal, vertical, and diagonal points are also horizontal, vertical, and diagonal, respectively. The number of parent of diagonal point is just one, but that of horizontal or vertical point can be two depending the position of the point. The wavelet basis function $\psi^{(x,y)}$ related to the point (x,y) is defined as

$$\psi^{(x,y)} = \begin{cases} \psi^h_{j(x,y),k(x),2^{j(x,y)-1-j(y)}(2l(y)+1)} & \text{for horizontal } (x,y)_h, \\ \psi^v_{j(x,y),2^{j(x,y)-1-j(x)}(2k(x)+1),l(y)} & \text{for vertical } (x,y)_v, \\ \psi^d_{j(x,y),k(x),l(y)} & \text{for diagonal } (x,y)_d, \end{cases}$$

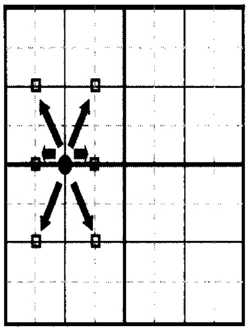
or simply as

$$\psi^{(x,y)} = \widetilde{\phi}_{jx(x,y),2^{jx(x,y)-j(x)}(2k(x)+1)}\widetilde{\phi}_{jy(x,y),2^{jy(x,y)-j(y)}(2l(y)+1)},$$

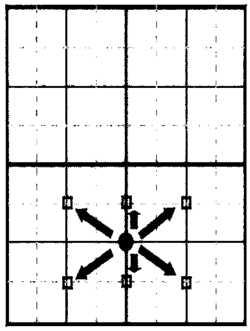
where $jx(x,y) = \max(j(x), j(x,y)-1)$ and $jy(x,y) = \max(j(y), j(x,y)-1)$.

Suppose that $u \in H^1_0(\Omega)$ and P_{V_J} is the interpolating projection from $H^1_0(\Omega)$ to V_J such that $[P_{V_J}(u)](x,y) = u(x,y)$ if $j(x,y) \le J$. Then the following holds:

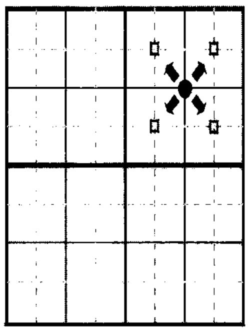
$$\begin{aligned} & P_{V_J}u \\ (3.13) \quad = & \sum_{k,l=1}^{2^J-1} u\left(\frac{k}{2^J}, \frac{l}{2^J}\right)\phi_{J,k,l} \\ = & \sum_{k,l=1}^{2^{j_0}-1} u\left(\frac{k}{2^{j_0}}, \frac{l}{2^{j_0}}\right)\phi_{j_0,k,l} \\ & + \sum_{j=j_0+1}^{J}\left[\sum_{k=0}^{2^{j-1}-1}\sum_{l=1}^{2^{j-1}-1} d_j^h[k,l]\psi^h_{j,k,l}\right. \\ & \left. + \sum_{k=1}^{2^{j-1}-1}\sum_{l=0}^{2^{j-1}-1} d_j^v[k,l]\psi^v_{j,k,l} + \sum_{k=0}^{2^{j-1}-1}\sum_{l=0}^{2^{j-1}-1} d_j^d[k,l]\psi^d_{j,k,l}\right] \end{aligned}$$



(a) For horizontal point $(x, y)_h$



(b) For vertical point $(x, y)_v$



(c) For diagonal point $(x, y)_d$

FIGURE 1. The children points (□) of the horizontal, vertical and diagonal points (•).

where

(3.14a)
$$\begin{aligned} & d_j^h[k,l] \\ = \ & u\left(\frac{2k+1}{2^j}, \frac{l}{2^{j-1}}\right) - [P_{V_{j-1}}u]\left(\frac{2k+1}{2^j}, \frac{l}{2^{j-1}}\right) \\ = \ & u\left(\frac{2k+1}{2^j}, \frac{l}{2^{j-1}}\right) \\ & -\frac{1}{2}\left[u\left(\frac{k}{2^{j-1}}, \frac{l}{2^{j-1}}\right) + u\left(\frac{k+1}{2^{j-1}}, \frac{l}{2^{j-1}}\right)\right] \end{aligned}$$

(3.15a) $d_j^v[k,l]$

$$= u\left(\frac{k}{2^{j-1}},\frac{2l+1}{2^j}\right)-[P_{V_{j-1}}u]\left(\frac{k}{2^{j-1}},\frac{2l+1}{2^j}\right)$$
$$= u\left(\frac{k}{2^{j-1}},\frac{2l+1}{2^j}\right)$$
$$-\frac{1}{2}\left[u\left(\frac{k}{2^{j-1}},\frac{l}{2^{j-1}}\right)+u\left(\frac{k}{2^{j-1}},\frac{l+1}{2^{j-1}}\right)\right]$$

(3.15b) $d_j^d[k,l]$

$$= u\left(\frac{2k+1}{2^j},\frac{2l+1}{2^j}\right)-[P_{V_{j-1}}u]\left(\frac{2k+1}{2^j},\frac{2l+1}{2^j}\right)$$
$$= u\left(\frac{2k+1}{2^j},\frac{2l+1}{2^j}\right)-\frac{1}{2}\left[u\left(\frac{2k+1}{2^j},\frac{l}{2^{j-1}}\right)\right.$$
$$+u\left(\frac{2k+1}{2^j},\frac{l+1}{2^{j-1}}\right)+u\left(\frac{k}{2^{j-1}},\frac{2l+1}{2^j}\right)$$
$$\left.+u\left(\frac{k+1}{2^{j-1}},\frac{2l+1}{2^j}\right)\right]+\frac{1}{4}\left[u\left(\frac{k}{2^{j-1}},\frac{l}{2^{j-1}}\right)\right.$$
$$\left.+u\left(\frac{k+1}{2^{j-1}},\frac{l}{2^{j-1}}\right)+u\left(\frac{k+1}{2^{j-1}},\frac{l+1}{2^{j-1}}\right)+u\left(\frac{k}{2^{j-1}},\frac{l+1}{2^{j-1}}\right)\right].$$

As in the one-dimensional case, $d(x,y)$ represents $d_j^\alpha[k,l], (\alpha=h,d,v)$ at (x,y) such that $(x,y)\in\left\{\left(\frac{2k+1}{2^j},\frac{l}{2^{j-1}}\right),\left(\frac{k}{2^{j-1}},\frac{2l+1}{2^j}\right),\left(\frac{2k+1}{2^j},\frac{2l+1}{2^j}\right)\right\}$:

$$d(x,y)=\begin{cases} d^h_{j(x,y)}[k(x),2^{j(x,y)-j(y)-1}(2l(y)+1)] \\ \qquad \text{if } (x,y) \text{ is a horizontal point} \\ d^v_{j(x,y)}[2^{j(x,y)-j(x)-1}(2k(x)+1),l(y)] \\ \qquad \text{if } (x,y) \text{ is a vertical point} \\ d^d_{j(x,y)}[k(x),l(y)] \\ \qquad \text{if } (x,y) \text{ is a diagonal point.} \end{cases}$$

For two-dimensional cases, we define $X^u_{j_0}$ and $X^{u,\epsilon}_{j_0}$ associated with some functions $u\in C^4(\Omega)\bigcap H^1_0(\Omega)$ as

$$X^u_{j_0} = \{(x,y)\in\Omega | j(x,y)\geq j_0, d(x,y)\neq 0\},$$
$$X^{u,\epsilon}_{j_0} = \{(x,y)\in X^u_{j_0} | \left|\frac{\partial^2 u(x,y)}{\partial x^2}\right|,\left|\frac{\partial^2 u(x,y)}{\partial y^2}\right|,\left|\frac{\partial^4 u(x,y)}{\partial x^2\partial y^2}\right|\geq\epsilon\},$$

and $\|f\|_0, |f|_m, \|f\|_m$ as

$$\begin{aligned}\|f\|_0 &= \operatorname{esssup}_{(x,y)\in\Omega} |f(x,y)|,\\ |f|_m &= \max_{n=0,\dots,m} \left\| \frac{\partial^m f}{\partial x^n \partial y^{m-n}} \right\|_0,\\ \|f\|_m &= \max_{n=1,\dots,m} |f|_n.\end{aligned}$$

THEOREM 3.4. *Let $u \in C^6(\Omega) \cap H_0^1(\Omega)$ and*

(3.16)
$$j_0 > \max\left(\frac{1}{2}\log_2\left(\frac{\left\|\frac{\partial^4 u}{\partial x^4}\right\|_0}{12\epsilon}\right), \frac{1}{2}\log_2\left(\frac{\left\|\frac{\partial^4 u}{\partial y^4}\right\|_0}{12\epsilon}\right), \frac{1}{2}\log_2\left(\frac{11\,|u|_6}{30\epsilon}\right)\right).$$

For $(x,y) \in X_{j_0}^{u,\epsilon}$ the following estimates hold:

(3.17a) $$\left|\frac{d(\widetilde{x},\widetilde{y})}{d(x,y)}\right| \le \frac{1}{4}\left(1 + \frac{\left\|\frac{\partial^3 u}{\partial x^3}\right\|_0 + 2\left\|\frac{\partial^3 u}{\partial x^2 \partial y}\right\|_0}{\epsilon} h + \frac{C_4}{\epsilon} h^2\right)$$

if $(\widetilde{x},\widetilde{y}) \in (x,y)_h^c$,

(3.17b) $$\left|\frac{d(\widetilde{x},\widetilde{y})}{d(x,y)}\right| \le \frac{1}{4}\left(1 + \frac{\left\|\frac{\partial^3 u}{\partial y^3}\right\|_0 + 2\left\|\frac{\partial^3 u}{\partial x \partial y^2}\right\|_0}{\epsilon} h + \frac{C_5}{\epsilon} h^2\right)$$

if $(\widetilde{x},\widetilde{y}) \in (x,y)_v^c$,

(3.17c) $$\left|\frac{d(\widetilde{x},\widetilde{y})}{d(x,y)}\right| \le \frac{1}{16}\left(1 + \frac{\left\|\frac{\partial^5 u}{\partial x^3 \partial y^2}\right\|_0 + \left\|\frac{\partial^5 u}{\partial x^2 \partial y^3}\right\|_0}{\epsilon} h + \frac{C_6}{\epsilon} h^2\right)$$

if $(\widetilde{x},\widetilde{y}) \in (x,y)_d^c$,

where C_4 and C_5 are constants depending on $|u|_3, |u|_4$, and C_6 is a constant depending on $|u|_5, |u|_6$.

Proof. The proof of Theorem 3.4 is basically the same as the proof of Theorem 3.1 for one dimensional cases. We can prove (3.17a) and (3.17b) by Lemma 3.5 and (3.11). Equation (3.17c) is obtained by using Lemma 3.6 and Equation (3.11). □

LEMMA 3.5. *Suppose that (x,y) is a horizontal or vertical point and $u \in C^6(\Omega) \cap H_0^1(\Omega)$. Then there are functions $M_i, (i = 1,\dots,6)$ such* *that*

$$\text{(3.18a)} \quad \frac{d(x\pm h,y)}{d((x,y)_h)} = \frac{1}{4}\frac{\frac{\partial^2 u(x,y)}{\partial x^2} \pm h\frac{\partial^3 u(x,y)}{\partial x^3} + M_1(x,y)}{\frac{\partial^2 u(x,y)}{\partial x^2} + M_2(x,y)}$$

$$\text{(3.18b)} \quad \frac{d(x\pm h,y+2h)}{d((x,y)_h)} = \frac{1}{4}\frac{\frac{\partial^2 u(x,y)}{\partial x^2} + h(\pm\frac{\partial^3 u(x,y)}{\partial x^3} + 2\frac{\partial^3 u(x,y)}{\partial x^2 \partial y}) + M_3(x,y)}{\frac{\partial^2 u(x,y)}{\partial x^2} + M_2(x,y)}$$

$$\text{(3.18c)} \quad \frac{d(x\pm h,y-2h)}{d((x,y)_h)} = \frac{1}{4}\frac{\frac{\partial^2 u(x,y)}{\partial x^2} + h(\pm\frac{\partial^3 u(x,y)}{\partial x^3} - 2\frac{\partial^3 u(x,y)}{\partial x^2 \partial y}) + M_3(x,y)}{\frac{\partial^2 u(x,y)}{\partial x^2} + M_2(x,y)}$$

$$\text{(3.18d)} \quad \frac{d(x,y\pm h)}{d((x,y)_v)} = \frac{1}{4}\frac{\frac{\partial^2 u(x,y)}{\partial y^2} \pm h\frac{\partial^3 u(x,y)}{\partial y^3} + M_4(x,y)}{\frac{\partial^2 u(x,y)}{\partial y^2} + M_5(x,y)}$$

$$\text{(3.18e)} \quad \frac{d(x+2h,y\pm h)}{d((x,y)_v)} = \frac{1}{4}\frac{\frac{\partial^2 u(x,y)}{\partial y^2} + h(\pm\frac{\partial^3 u(x,y)}{\partial y^3} + 2\frac{\partial^3 u(x,y)}{\partial x \partial y^2}) + M_6(x,y)}{\frac{\partial^2 u(x,y)}{\partial y^2} + M_5(x,y)}$$

$$\text{(3.18f)} \quad \frac{d(x-2h,y\pm h)}{d((x,y)_v)} = \frac{1}{4}\frac{\frac{\partial^2 u(x,y)}{\partial y^2} + h(\pm\frac{\partial^3 u(x,y)}{\partial y^3} - 2\frac{\partial^3 u(x,y)}{\partial x \partial y^2}) + M_6(x,y)}{\frac{\partial^2 u(x,y)}{\partial y^2} + M_5(x,y)}$$

and

$$\text{(3.19a)} \quad |M_1(x,y)| \le \frac{3h^2}{4}\left\|\frac{\partial^4 u}{\partial x^4}\right\|_0,$$

$$\text{(3.20a)} \quad |M_2(x,y)| \le \frac{h^2}{3}\left\|\frac{\partial^4 u}{\partial x^4}\right\|_0,$$

$$\text{(3.21a)} \quad |M_3(x,y)| \le h^2\left(\frac{3}{4}\left\|\frac{\partial^4 u}{\partial x^4}\right\|_0 + 2\left\|\frac{\partial^4 u}{\partial x^3 \partial y}\right\|_0\right),$$

$$\text{(3.22a)} \quad |M_4(x,y)| \le \frac{3h^2}{4}\left\|\frac{\partial^4 u}{\partial y^4}\right\|_0,$$

$$\text{(3.23a)} \quad |M_5(x,y)| \le \frac{h^2}{3}\left\|\frac{\partial^4 u}{\partial y^4}\right\|_0,$$

$$\text{(3.24a)} \quad |M_6(x,y)| \le h^2\left(\frac{3}{4}\left\|\frac{\partial^4 u}{\partial y^4}\right\|_0 + 2\left\|\frac{\partial^4 u}{\partial x \partial y^3}\right\|_0\right).$$

Proof. Equations (3.18a) and (3.18d) directly come from (3.6). By using (3.8), we obtain

$$\frac{d(x \pm h, y+2h)}{d((x,y)_h)} = \frac{1}{4}\frac{\frac{\partial^2 u(x,y+2h)}{\partial x^2} \pm \frac{\partial^3 u(x,y+2h)}{\partial x^3}h + M_1(x,y)}{\frac{\partial^2 u(x,y)}{\partial x^2} + M_2(x,y)} \tag{3.25}$$

where M_1 and M_2 are the functions satisfying (3.19a) and (3.20a), respectively. Consider the following Taylor expansions at (x,y) in y

$$\begin{aligned} \frac{\partial^2 u(x,y+2h)}{\partial x^2} &= \frac{\partial^2 u(x,y)}{\partial x^2} + 2h\frac{\partial^3 u(x,\xi_1)}{\partial x^2 \partial y} \\ \frac{\partial^3 u(x,y+2h)}{\partial x^3} &= \frac{\partial^3 u(x,y)}{\partial x^3} + 2h\frac{\partial^4 u(x,\xi_2)}{\partial x^3 \partial y} \end{aligned} \tag{3.26}$$

where ξ_1 and ξ_2 are constants in $(y, y+2h)$. Inserting (3.26) into (3.25), we obtain (3.18b). The proof of (3.18c),(3.18e), and (3.18f) is similar to that of (3.18b). □

LEMMA 3.6. *Suppose that (x,y) is a diagonal point and $u \in C^6(\Omega) \cap H_0^1(\Omega)$. Then the following equations hold.*

$$\begin{aligned} &\frac{d(x \pm h, y \pm h)}{d((x,y)_d)} \\ &= \left[\frac{h^4}{4}\frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} \pm \frac{h^5}{4}\left(\frac{\partial^5 u(x,y)}{\partial x^2 \partial y^3} + \frac{\partial^5 u(x,y)}{\partial x^3 \partial y^2}\right)\right. \\ &\quad \left. + M_7(x,y)\right] \cdot \left[4h^4 \frac{\partial^4 u(x,y)}{\partial x^2 y^2} + M_8(x,y)\right]^{-1}, \end{aligned} \tag{3.27}$$

where $M_7(x,y)$ and $M_8(x,y)$ satisfy

$$|M_7(x,y)| \le \frac{1121h^6}{720}|u|_6, \tag{3.28a}$$

$$|M_8(x,y)| \le \frac{88h^6}{15}|u|_6. \tag{3.28b}$$

Proof. For some points $(z,y),(x,w),(z,w)$ around (x,y) we can write the following Taylor expansions:

$$u(z,y) = \sum_{i=0}^{5} \frac{\partial^i u(x,y)}{\partial x^i}\frac{(z-x)^i}{i!} + \frac{\partial^6 u(\xi,y)}{\partial x^6}\frac{(z-x)^6}{6!}$$

$$\begin{aligned} u(x,w) &= \sum_{j=0}^{5} \frac{\partial^j u(x,y)}{\partial y^j}\frac{(w-y)^j}{j!} + \frac{\partial^6 u(x,\eta)}{\partial y^6}\frac{(w-y)^6}{6!} \\ u(z,w) &= \sum_{k=0}^{5} \frac{D^k_{z-x,w-y}(x,y)}{k!} + \frac{D^6_{z-x,w-y}(\widetilde{\xi},\widetilde{\eta})}{6!} \end{aligned}$$

where ξ and $\widetilde{\xi}$ are some numbers between x and z, η and $\widetilde{\eta}$ are some numbers between y and w, and

$$D^k_{\xi',\eta'}(x,y) = \sum_{j=0}^{k} {}_kC_j \frac{\partial^k u}{\partial x^j \partial y^{k-j}}(x,y)(\xi')^j(\eta')^{k-j}.$$

To evaluate $d(x,y)$, the following difference-checking property, which is another expression of (3.15b), is used:

$$\begin{aligned} (3.29)\quad & d(x,y) \\ = \; & u(x,y) \\ & -\frac{1}{2}[u(x-2h,y)+u(x+2h,y)+u(x,y-2h)+u(x,y+2h)] \\ & +\frac{1}{4}[u(x-2h,y-2h)+u(x+2h,y-2h) \\ & +u(x+2h,y+2h)+u(x-2h,y+2h)]. \end{aligned}$$

Consider the Taylor series expansions of u around (x,y) to obtain the following results.

$$\begin{aligned} (3.30\text{a})\quad & \frac{1}{2}\big(u(x-2h,y)+u(x+2h,y)+u(x,y-2h)+u(x,y+2h)\big) \\ = \; & 2u(x) \\ & +2h^2\left(\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}\right) \\ & +\frac{2h^4}{3}\left(\frac{\partial^4 u(x,y)}{\partial x^4}+\frac{\partial^4 u(x,y)}{\partial y^4}\right) \\ & +\frac{2h^6}{45}\left(\frac{\partial^6 u(\xi_1,y)}{\partial x^6}+\frac{\partial^6 u(\xi_2,y)}{\partial x^6}+\frac{\partial^6 u(x,\eta_1)}{\partial y^6}+\frac{\partial^6 u(x,\eta_2)}{\partial y^6}\right) \end{aligned}$$

$$\begin{aligned}
(3.31a)\qquad & \frac{1}{4}\left(u(x-2h,y-2h)+u(x+2h,y-2h)+\right.\\
& \left. u(x+2h,y+2h)+u(x-2h,y+2h)\right)\\
= \;& u(x,y)+2h^2\left(\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}\right)\\
& +\frac{2h^4}{3}\left(\frac{\partial^4 u(x,y)}{\partial x^4}+6\frac{\partial^4 u(x,y)}{\partial x^2 y^2}+\frac{\partial^4 u(x,y)}{\partial y^4}\right)\\
& +\frac{1}{45}\Big(D^6_{-h,-h}(\xi_3,\eta_3)+D^6_{h,-h}(\xi_4,\eta_4)+D^6_{h,h}(\xi_5,\eta_5)\\
& +D^6_{-h,h}(\xi_6,\eta_6)\Big)
\end{aligned}$$

where $\xi_i, \eta_j, i, j = 1, \ldots, 6$ are some numbers such that $\xi_1, \xi_3, \xi_6 \in (x - 2h, x), \xi_2, \xi_4, \xi_5 \in (x, x + 2h), \eta_1, \eta_3, \eta_4 \in (y - 2h, y)$ and $\eta_2, \eta_5, \eta_6 \in (y, y + 2h)$. Substituting (3.30) into (3.29) yields

$$(3.32a)\qquad d(x,y)=4h^4\frac{\partial^4 u}{\partial x^2 y^2}(x,y)+M_8(x,y)$$

$$\begin{aligned}
M_8(x,y) \;=\; & -\frac{2h^6}{45}\Big(\frac{\partial^6 u(\xi_1,y)}{\partial x^6}-\frac{\partial^6 u(\xi_2,y)}{\partial x^6}+\frac{\partial^6 u(x,\eta_1)}{\partial y^6}\\
(3.33a)\qquad & +\frac{\partial^6 u(x,\eta_2)}{\partial y^6}\Big)+\frac{1}{45}\Big(D^6_{-h,-h}(\xi_3,\eta_3)+D^6_{h,-h}(\xi_4,\eta_4)\\
& +D^6_{h,h}(\xi_5,\eta_5)+D^6_{-h,h}(\xi_6,\eta_6)\Big).
\end{aligned}$$

Equation (3.28b) is derived by using (3.33a) and the fact that ${}_6C_0 + \cdots + {}_6C_6 = 2^6$. The value of $d(x + h, y + h)$ can be obtained from the following analysis.

$$\begin{aligned}
(3.34a)\;\; & d(x+h,y+h)\\
= \;& u(x+h,y+h)-[u(x,y+h)+u(x+2h,y+h)+u(x+h,y)\\
& +u(x+h,y+2h)]/2+[u(x,y)+u(x+2h,y)\\
& +u(x+2h,y+2h)+u(x,y+2h)]/4\\
= \;& \frac{h^4}{4}\frac{\partial^4 u(x,y)}{\partial x^2\partial y^2}+\frac{h^5}{4}\Big(\frac{\partial^5 u(x,y)}{\partial x^2\partial y^3}\\
& +\frac{\partial^5 u(x,y)}{\partial x^3\partial y^2}\Big)+M_7(x,y)
\end{aligned}$$

(3.35a)
$$
\begin{aligned}
& M_7(x,y) \\
= \; & -\frac{1}{1440}\left(h^6\frac{\partial^6 u(\xi_8,y)}{\partial x^6}+h^6\frac{\partial^6 u(x,\eta_8)}{\partial y^6}\right. \\
& \left. +D^6_{2h,h}(\xi_9,\eta_9)+D^6_{h,2h}(\xi_{10},\eta_{10})\right) \\
& +\frac{1}{45}\left(h^6\frac{\partial^6 u(\xi_{11},y)}{\partial x^6}+h^6\frac{\partial^6 u(x,\eta_{11})}{\partial y^6}+D^6_{h,h}(\xi_{12},\eta_{12})\right)
\end{aligned}
$$

with $\xi_i \in (x-2h, x+2h), \eta_j \in (y-2h, y+2h), i,j = 7,\dots,12$. In writing Equation (3.34), we used the following expansions:

(3.36a)
$$
\begin{aligned}
& u(x+h,y+h) \\
= \; & \sum_{k=0}^{5}\frac{D^k_{h,h}(x,y)}{k!}+\frac{D^6_{h,h}(\xi_7,\eta_7)}{6!}
\end{aligned}
$$

(3.36b)
$$
\begin{aligned}
& [u(x,y+h)+u(x+2h,y+h)+u(x+h,y) \\
& +u(x+h,y+2h)]/2 \\
= \; & 2u(x,y)+\sum_{i=1}^{5}(2^{i-1}+1)\frac{h^i}{i!}\frac{\partial^i u(x,y)}{\partial x^i} \\
& +\sum_{j=1}^{5}(2^{j-1}+1)\frac{h^j}{j!}\frac{\partial^j u(x,y)}{\partial y^j} \\
& +\sum_{i+j=1,i,j\ge 1}^{5}(2^{i-1}+2^{j-1})\frac{{}_{i+j}C_i h^i h^j}{(i+j)!}\frac{\partial^{i+j}u(x,y)}{\partial x^i\partial y^j} \\
& +\frac{1}{2\cdot 6!}\left(h^6\frac{\partial^6 u(\xi_8,y)}{\partial x^6}+h^6\frac{\partial^6 u(x,\eta_8)}{\partial y^6}\right. \\
& \left. +D^6_{2h,h}(\xi_9,\eta_9)+D^6_{h,2h}(\xi_{10},\eta_{10})\right)
\end{aligned}
$$

(3.36c)
$$
\begin{aligned}
& [u(x,y)+u(x+2h,y)+u(x+2h,y+2h)+u(x,y+2h)]/4 \\
= \; & u(x,y)+\sum_{i=1}^{5}2^{i-1}\frac{h^i}{i!}\frac{\partial^i u(x,y)}{\partial x^i}+\sum_{j=1}^{5}2^{j-1}\frac{h^j}{j!}\frac{\partial^j u(x,y)}{\partial y^j} \\
& +\sum_{i+j=1,i,j\ge 1}^{5}2^{i+j-2}\frac{{}_{i+j}C_i h^i h^j}{(i+j)!}\frac{\partial^{i+j}u(x,y)}{\partial x^i\partial y^j} \\
& +\frac{16}{6!}\left(h^6\frac{\partial^6 u(\xi_{11},y)}{\partial x^6}+h^6\frac{\partial^6 u(x,\eta_{11})}{\partial y^6}+D^6_{h,h}(\xi_{12},\eta_{12})\right)
\end{aligned}
$$

Using the procedure to evaluate $d(x+h, y+h)$, we can also evaluate $d(x+h, y-h), d(x-h, y+h)$, and $d(x-h, y-h)$. We can prove (3.28a) by using (3.35a) and the fact that ${}_6C_0 + \cdots + {}_6C_6 = 2^6$. Finally, (3.27) is obtained by dividing (3.34a) by (3.32a). □

4. Error analysis for adaptive multiscale interpolation

In this section, we will first state a typical interpolation wavelet-based adaptive scheme. Then, we will show that the pointwise error at the interpolation node (x, y) between the adaptive solution and the non-adaptive solution in multiscale representation at resolution J depends on $h^2 = 2^{-2j(x,y)-2}$ at each interpolation point and the adaptive parameters. If there is no danger of confusion, $(x, y) \in C$ will be used instead of $\psi^{(x,y)} \in C$ for a basis set C in this section.

Let's consider the following adaptive scheme working from an initial resolution j_0 to the maximum resolution J. Assume that η is a small positive number and r is a number in $(0, 1)$.

Adaptive Scheme (j_0, J, η, r)

1. Choose an initial resolution level $j_0 > 0$ and let $j = j_0 + 1$. Let basis set C_j be $B_j^{j_0}$ and set the initial value of a parameter e_{max} to be η.
2. Use (3.13) to interpolate a given function by $d(x, y)$ such that $(x, y) \in C_j$. Or substitue (3.13) into (2.3) to compute $d(x, y)$ such that $(x, y) \in C_j$ using the wavelet-Galerkin analysis.
3. Add $\psi^{(\widetilde{x},\widetilde{y})}$ to C_j where $(\widetilde{x}, \widetilde{y}) \in (x, y)^c$ for (x, y) such that $(x, y) \in C_j$ and $|d(x, y)| \geq e_{max}$.
4. Set $e_{max} = r\eta$
5. If $j = J$, stop. Otherwise, set $j = j + 1$ and go to step 2.

Let us divide the (nonadaptive) interpolation $P_{V_J}u$ in (3.13) as the sum of $P_{C_J}u$ and Q_{C_J}.

(4.1)
$$\begin{aligned} & P_{V_J}u \\ = & \sum_{k,l=1}^{2^{j_0}-1} u\left(\frac{k}{2^{j_0}}, \frac{l}{2^{j_0}}\right)\phi_{j_0,k,l} + \sum_{j_0+1 \leq j(x,y) \leq J} d(x, y)\psi^{(x,y)} \\ = & P_{C_J}u + Q_{C_J}u \end{aligned}$$

where

$$
\begin{aligned}
P_{C_J}u &= \sum_{k,l=1}^{2^{j_0}-1} u\left(\frac{k}{2^{j_0}}, \frac{l}{2^{j_0}}\right)\phi_{j_0,k,l} + \sum_{(x,y)\in C_j, j=j_0+1,\dots,J} d(x,y)\psi^{(x,y)} \\
Q_{C_J}u &= \sum_{(x,y)\notin C_j, j=j_0+1,\dots,J} d(x,y)\psi^{(x,y)}
\end{aligned}
$$

If we use (3.13) to interpolate a function u by Step 2 in Adaptive Scheme (j_0, J, η, r), the resulting approximation must be $P_{C_J}u$ in (4.1). However, if we solve (2.3) using the representation (3.13) to find $d(x,y)$ such that $(x,y) \in C_j$, the resulting approximate solution may not be $P_{C_J}u$, in general. Thus let the nonadaptive solution of (2.3) in the finite-dimensional space V_J be u_J and the adaptive solution of (2.3) through Adaptive Scheme (j_0, J, η, r) be $\widetilde{u}_J$. In the finite dimensional space V_J, u_J and $P_{V_J}u$ differ by $O(2^{-2J})$ at interpolation points. Similarly, one can show that $\widetilde{u}_J$ and $P_{C_J}u$ differ by $O(2^{-2j_0})$ at interpolation points. Thus $|P_{V_J}u - P_{C_J}u|$ will be estimated in this section, instead of $|u_J - \widetilde{u}_J|$.

Before stating Theorem 4.1, we note the following result:

$$P_{V_J}u(x_0,y_0) = P_{C_J}u(x_0,y_0) \text{ for } (x_0,y_0) \in C_J.$$

Note that if $j(x_0,y_0) \le j_0+1$, $(x_0,y_0) \in C_J$. Equivalently, $(x_0,y_0) \notin C_J$ implies $j(x_0,y_0) \ge j_0+2$.

THEOREM 4.1. *Assume that j_0 satisfies (3.16). Also assume that $u \in C^6(\Omega) \cap H_0^1(\Omega), (x_0,y_0) \in X_{j_0}^{u,\epsilon}$ for some $\epsilon > 0$, and C_J is chosen through the Adaptive Scheme (j_0, J, η, r). Then the following estimate holds for $(x_0,y_0) \in V_J \setminus C_J$.*

$$
\begin{aligned}
&|P_{V_J}u(x_0,y_0) - P_{C_J}u(x_0,y_0)| \\
\le\ & 3(J-j_0-1)\max\left((2\epsilon + 2^{j_0+3}\|u\|_5)\max\left(\frac{6\epsilon}{\left\|\frac{\partial^4 u}{\partial x^4}\right\|_0}, \frac{6\epsilon}{\left\|\frac{\partial^4 u}{\partial y^4}\right\|_0}, \frac{4\epsilon^2}{|u|_6^2}\right),\right. \\
&\left.\frac{1}{4}r^{J-(j_0+1)}\eta\left[1 + \frac{C_7}{\epsilon}\left(\frac{1}{2}\right)^{j_0+2} + \frac{C_8}{\epsilon}\left(\frac{1}{2}\right)^{j_0+3}\right]\right)
\end{aligned}
$$

where $C_7 = \max\left(\left\|\frac{\partial^3 u}{\partial x^3}\right\|_0 + 2\left\|\frac{\partial^3 u}{\partial x^2 \partial y}\right\|_0, \left\|\frac{\partial^3 u}{\partial y^3}\right\|_0 + 2\left\|\frac{\partial^3 u}{\partial x \partial y^2}\right\|_0, \left\|\frac{\partial^5 u}{\partial x^3 \partial y^2}\right\|_0 + \left\|\frac{\partial^5 u}{\partial x^2 \partial y^3}\right\|_0\right)$ *and C_8 is a constant depending on $\|u\|_6$ and J.*

Proof. It is trivial to obtain the following identity using Equation (4.1)

$$\begin{aligned} |P_{V_J}u(x_0,y_0) - P_{C_J}u(x_0,y_0)| &= |Q_{C_J}u(x_0,y_0)| \\ &= \left| \sum_{(x,y)\notin C_J} d(x,y)\psi^{(x,y)}(x_0,y_0) \right| \\ &\le \sum_{(x,y)\notin C_J} |d(x,y)|\,\psi^{(x,y)}(x_0,y_0). \end{aligned} \tag{4.2}$$

At each resolution level $j = j_0 + 2, \ldots, J$, there are at most 3 points such that $\psi^{(x,y)}(x_0, y_0) \neq 0$ and $j(x,y) = j$. Thus the number of $(x,y)'s$ satisfying the conditions $(x,y) \notin C_J$ and $\psi^{(x,y)}(x_0,y_0) \neq 0$ is at most $3(J - j_0 - 1)$. To estimate the bound of $|d(x,y)|$ for $(x,y) \notin C_J$, let us define $(x,y)^{p^2} := ((x,y)^p)^p$ and $(x,y)^{p^k} := ((x,y)^{p^{k-1}})^p, k \geq 3$. Then $(x,y) \notin C_J$ implies that there is a positive integer $k \leq j(x,y) - (j_0 + 1)$ such that $(x,y)^{p^k} \in C_J$ and $(x,y)^{p^{k'}} \notin C_J, 0 \leq k' \leq k-1$, since $V_{j_0+1} \subset C_J$. There exist two cases for (x,y) such that

- Case I: $(x,y)^{p^{k'}} \in X_{j_0}^{u,\epsilon}, 0 \leq k' \leq k$
- Case II: There is an integer $\widetilde{k} \leq k$ such that $(x,y)^{p^{\widetilde{k}}} \notin X_{j_0}^{u,\epsilon}$.

The bound of $|d(x,y)|$ is estimated in Lemmas 4.2 and 4.3 for cases I and II, respectively. By using $j(x,y) - k \geq j_0 + 2$ and $(\frac{1}{4})^k, (\frac{1}{16})^k \leq \frac{1}{4}$ and inserting the bound in Lemmas 4.2 and 4.3 into (4.2) for $3(J - j_0 - 1)$ nonvanishing terms, we prove the theorem. □

LEMMA 4.2. *Let $u \in C^6(\Omega) \cap H_0^1(\Omega)$ and C_J be determined through Adaptive scheme (j_0, J, η, r) for j_0 satisfying (3.16). For (x,y) such that $j_0 + 1 \leq j(x,y) \leq J$, assume that there is a positive integer $k \leq j(x,y) - j_0 - 1$ such that $(x,y)^{p^k} \in X_{j_0}^{u,\epsilon} \cap C_J$ and $(x,y)^{p^{k'}} \in X_{j_0}^{u,\epsilon} \setminus C_J, 0 \leq k' \leq k-1$. Then $d(x,y)$ satisfies*

$$\begin{aligned} |d(x,y)| \leq\ & C_9^k r^{J-j_0-1}\eta \\ & \left[1 + \frac{C_7}{\epsilon}\left(\frac{1}{2}\right)^{j(x,y)+1-k} + \frac{C_8}{\epsilon}\left(\frac{1}{2}\right)^{j(x,y)-k} \right], \end{aligned}$$

where C_7 and C_8 are given in Theorem 4.1 and $C_9 = 1/4$ for horizontal or vertical points and $C_9 = 1/16$ for diagonal points.

Proof. Since $(x,y)^{p^k} \in C_J$ and $(x,y)^{p^{(k-1)}} \notin C_J$, $\left| d((x,y)^{p^k}) \right| \leq r^{J-j_0-1}\eta$. Since $(x,y)^{p^{k'}} \in X_{j_0}^{u,\epsilon}, 0 \leq k' \leq k$, we obtain, by Theorem

3.4 recursively,

$$
\begin{aligned}
|d(x,y)| &\le C_9 |d((x,y)^p)| \left[1 + \frac{C_7}{\epsilon} \left(\frac{1}{2}\right)^{j(x,y)} \frac{C_{10}}{\epsilon} \left(\frac{1}{2}\right)^{2j(x,y)} \right] \\
&\le \cdots\cdots \qquad (4.3) \\
&\le C_9^k |d((x,y)^{p^k})| \prod_{k'=1,\ldots,k} \left[1 + \frac{C_7}{\epsilon} \left(\frac{1}{2}\right)^{j(x,y)+1-k'} \right. \\
&\quad \left. + \frac{C_{10}}{\epsilon} \left(\frac{1}{2}\right)^{2j(x,y)+2-k'} \right] \\
&\le C_9^k r^{J-j_0-1} \eta \left[1 + \frac{C_7}{\epsilon} \left(\frac{1}{2}\right)^{j(x,y)+1-k} + \frac{C_8}{\epsilon} \left(\frac{1}{2}\right)^{j(x,y)-k} \right]
\end{aligned}
$$

where $C_{10} = \max(C_4, C_5, C_6)$ and C_8 is a constant depending C_7, C_{10}, $j(x,y)$, and k. Since $C_7, C_{10}, j(x,y)$, and k are constants depending on $\|u\|_6$ and J, also C_8 depends only on $\|u\|_6$ and J. □

LEMMA 4.3. *Let $u \in C^6(\Omega) \cap H_0^1(\Omega)$ and C_J be determined through Adaptive scheme (j_0, J, η, r) for j_0 satisfying (3.16). For (x,y) such that $j_0 + 1 \le j(x,y) \le J$, assume that there is a nonnegative integer $\widetilde{k} \le j(x,y) - j_0 - 1$ such that $(x,y)^{p^{\widetilde{k}}} \notin X_{j_0}^{u,\epsilon} \cup C_J$. Then the following estimate holds*

$$
|d(x,y)| \le (2\epsilon + 2^{j_0+3} \|u\|_5) \max \left(\frac{6\epsilon}{\left\| \frac{\partial^4 u}{\partial x^4} \right\|_0}, \frac{6\epsilon}{\left\| \frac{\partial^4 u}{\partial y^4} \right\|_0}, \frac{4\epsilon^2}{|u|_6^2} \right).
$$

Proof. From Equations (3.8c), (3.32), and (3.28b), we can conclude that there are bounded functions $M^h(x,y), M^v(x,y)$ and $M^d(x,y)$ defined on horizontal, vertical, and diagonal points, respectively such that

$$
\begin{aligned}
d(x,y) &= -2h^2 \frac{\partial^2 u}{\partial x^2}(x,y) + M^h(x,y) \text{ if } (x,y) \text{ is horizontal} \\
d(x,y) &= -2h^2 \frac{\partial^2 u}{\partial y^2}(x,y) + M^v(x,y) \text{ if } (x,y) \text{ is vertical} \\
d(x,y) &= 4h^4 \frac{\partial^4 u}{\partial x^2 \partial y^2}(x,y) + M^d(x,y) \text{ if } (x,y) \text{ is diagonal}
\end{aligned}
$$

where

$$
\begin{aligned}
|M^h(x,y)| &\leq \frac{2h^4}{3}\left\|\frac{\partial^4 u}{\partial x^4}\right\|_0, \\
|M^v(x,y)| &\leq \frac{2h^4}{3}\left\|\frac{\partial^4 u}{\partial y^4}\right\|_0, \\
|M^d(x,y)| &\leq \frac{88h^6}{15}|u|_6.
\end{aligned}
$$

Let $(x,y)-(x,y)^{p^{\tilde{k}}}=(h_x,h_y)$, then $|h_x|,|h_y|\leq 2^{-j(x,y)-1+\tilde{k}}\leq 2^{j_0+2}$. By using the fact that $\left|\frac{\partial^2 u((x,y)^{p^{\tilde{k}}})}{\partial x^2}\right|, \left|\frac{\partial^2 u((x,y)^{p^{\tilde{k}}})}{\partial y^2}\right|, \left|\frac{\partial^4 u((x,y)^{p^{\tilde{k}}})}{\partial x^2\partial y^2}\right| \leq \epsilon$ due to $(x,y)^{p^{\tilde{k}}}\notin X_{j_0}^{u,\epsilon}$, and using Taylor series around $(x,y)^{p^{\tilde{k}}}$, we obtain

$$
\begin{aligned}
\left|\frac{\partial^2 u(x,y)}{\partial x^2}\right| &\leq \epsilon+2^{j_0+2}\left[\left\|\frac{\partial^3 u}{\partial x^3}\right\|_0+\left\|\frac{\partial^3 u}{\partial x^2\partial y}\right\|_0\right] \\
\left|\frac{\partial^2 u(x,y)}{\partial y^2}\right| &\leq \epsilon+2^{j_0+2}\left[\left\|\frac{\partial^3 u}{\partial x\partial y^2}\right\|_0+\left\|\frac{\partial^3 u}{\partial y^3}\right\|_0\right] \\
\left|\frac{\partial^4 u(x,y)}{\partial x^2\partial y^2}\right| &\leq \epsilon+2^{j_0+2}\left[\left\|\frac{\partial^5 u}{\partial x^3\partial y^2}\right\|_0+\left\|\frac{\partial^5 u}{\partial x^2\partial y^3}\right\|_0\right].
\end{aligned} \tag{4.4}
$$

From (4.4), we can conclude that $\left|\frac{\partial^2 u(x,y)}{\partial x^2}\right|, \left|\frac{\partial^2 u(x,y)}{\partial y^2}\right|, \left|\frac{\partial^4 u(x,y)}{\partial x^2\partial y^2}\right| \leq \epsilon + 2^{j_0+3}\|u\|_5$. Therefore we obtain

$$
\begin{aligned}
|d(x,y)| &\leq 2(\epsilon+2^{j_0+3}\|u\|_5)h^2+\frac{2h^4}{3}\left\|\frac{\partial^4 u}{\partial x^4}\right\|_0 \leq \frac{6\epsilon(2\epsilon+2^{j_0+3}\|u\|_5)}{\left\|\frac{\partial^4 u}{\partial x^4}\right\|_0} \\
&\quad \text{if } (x,y) \text{ is horizontal} \\
|d(x,y)| &\leq 2(\epsilon+2^{j_0+3}\|u\|_5)h^2+\frac{2h^4}{3}\left\|\frac{\partial^4 u}{\partial y^4}\right\|_0 \leq \frac{6\epsilon(2\epsilon+2^{j_0+3}\|u\|_5)}{\left\|\frac{\partial^4 u}{\partial y^4}\right\|_0} \\
&\quad \text{if } (x,y) \text{ is vertical} \\
|d(x,y)| &\leq 4(\epsilon+2^{j_0+3}\|u\|_5)h^4+\frac{88h^6}{15}|u|_6 \leq \frac{4\epsilon^2(2\epsilon+2^{j_0+3}\|u\|_5)}{|u|_6^2} \\
&\quad \text{if } (x,y) \text{ is diagonal}
\end{aligned}
$$

This completes the proof of the lemma. □

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