J. Korean Math. Soc. **42** (2005), No. 2, pp. 291–304

CAUCHY COMPLETION OF CSÁSZÁR FRAMES

SE HWA CHUNG

ABSTRACT. We introduce a concept of Cauchy complete Császár frames and construct Cauchy completions of Császár frames using strict extensions of frames and show that the Cauchy completion gives rise to a coreflection in the categories CsFrm and UCsFrm.

1. Introduction and preliminaries

In order to study topological, proximal and uniform structures in a single setting, Császár [7] has introduced the concept of syntopogenous structures on a set. He defined the concept of complete syntopogenous spaces and constructed the completions of syntopogenous spaces using the Cauchy filters. For the concept of the Cauchy filters, he introduced the join-irreducible set in the spaces. Moreover, he also showed that proximity structures occupy an intermediate position between topologies and uniformities to the effect that the topology deduced from a uniformity is really constructed in two steps: first deduced from a uniformity a proximity structure, then from this topology in question. In [6], the author proved the following:

(1) The category of topological spaces and continuous maps and the category of perfect topogenous spaces and continuous maps are isomorphic.

(2) The category of proximity spaces and proximal maps and the category of symmetrical topogenous spaces and continuous maps are isomorphic.

Received November 12, 2003.

2000 Mathematics Subject Classification: 06D99, 18A40, 54A15, 54D35.

Key words and phrases: frame, frame homomorphism, *Császár* frame, Cauchy complete Császár frame, Cauchy completion, uniform Császár frame, Cauchy homomorphism, uniform homomorphism, regular Cauchy filter.

This work was supported by Kroea Research Foundation Grant (KRF-2002-015-CP0032).

(3) The category of uniform spaces and uniformly continuous maps and the category of symmetrical biperfect syntopogenous spaces and continuous maps are isomorphic.

The study of structured frames started with Isbell [10], where uniformities on frames were introduced as exact translation into frame terms of Tukey's approach([14]) to uniform spaces via covers. Nearness frames, which is uniform frames without the star-refinement condition, was introduced recently by Banaschwiski and Pultr [5], and they introduced the concept of Cauchy complete nearness frames and constructed the completions of nearness frames which furnish coreflections in the category of strong nearness frames and uniform homomorphisms(see [3-5]). For the Cauchy complete nearness frames, they introduced the concept of regular Cauchy filters via covers. Hong and Kim [9] showed more recently that every nearness frame has the Cauchy completions, which furnish coreflections in the category of nearness frames and Cauchy homomorphisms.

The purpose of this paper is twofold: One of them is to introduce the frame counterpart of the syntopogenous space. The other is to define the Cauchy complete Császár frames and then construct the Cauchy completions of Császár frames, that is, the frame counterpart of the Császár completion of syntopogenous spaces([7]). In addition, we present the following two results:

(1) The category CCCsFrm of Cauchy complete Császár frames is coreflective in the category CsFrm of Császár frames and Cauchy homomorphisms.

(2) The category CCUCsFrm of Cauchy complete uniform Császár frames is coreflective in the category UCsFrm of uniform Császár frames and uniform homomorphisms.

Now we briefly recall some of the basic concepts of frame theory and introduce notation and terminology that is consistent throughout this paper. For the general background of frame theory, we refer to [11] and for the category theory, we refer to [1].

DEFINITION 1.0.1. A *frame* is a complete lattice L in which

$$x \wedge \bigvee_I x_i = \bigvee_I (x \wedge x_i)$$

for any $x \in L$ and family $\{x_i\}_{i \in I}$ contained L.

In a frame L, e (0, resp.) denotes the top (bottom, resp.) of L.

DEFINITION 1.0.2. A subset A of a frame L is said to be a *cover* of L if $\bigvee A = e$.

DEFINITION 1.0.3. A frame L is said to be *regular* if

$$a = \bigvee\{x : x^* \vee a = e\}$$

for all $a \in L$, where x^* is the psedocomplement of x in L, given by $x^* = \bigvee\{y \in L : x \wedge y = 0\}$.

DEFINITION 1.0.4. (1) A *frame homomorphism* is a map $h\colon M \longrightarrow L$ between frames preserving finite meets (including the top) and arbitrary joins (including the bottom).

(2) A frame homomorphism $h\colon M \longrightarrow L$ is said to be *dense* if $h(x) = 0$ implies $x = 0$.

Because of join preservation, each frame homomorphism $h : M \to L$ has a right Galois adjoint h_*, that is, a monotone map $h_*\colon L \to M$ such that

$$h(x) \leq y \text{ if and only if } x \leq h_*(y).$$

PROPOSITION 1.0.5. *Every dense frame homomorphism between regular frames is a monomorphism.*

PROPOSITION 1.0.6. *Let $h\colon M \longrightarrow L$ be a frame homomorphism.*
(1) $x \leq h_*(h(x))$ for all $x \in L$.
(2) If h is onto then $hh_* = id$ and $h(x) \wedge y^* = 0$ implies $y^* \leq h(x^*)$.
(3) If h is dense onto then $h_*(x^*) = (h_*(x))^*$ for all $x \in L$.

DEFINITION 1.0.7. [3] A *strict extension* of L is an onto dense frame homomorphism $h\colon M \longrightarrow L$ satisfying the following

$$M = \{\bigvee A : A \subseteq h_*(L)\}.$$

2. Császár frames

In this section, we introduce the concept of Császár orders on frames and that of the Császár frames and then state some properties of the Császár frames in the form applicable to the situations arising later.

2.1. Császár orders

In this subsection, we introduce the concept of Császár orders on frames and properties of the Császár orders.

DEFINITION 2.1.1. A *Császár order* $\lhd$ on a frame L is a binary relation on L satisfying the following properties:
(CO_1) $0 \lhd 0$ and $e \lhd e$,

(CO_2) $x \lhd y$ implies $x \leq y$
(CO_3) $x \leq a \lhd b \leq y$ implies $x \lhd y$.

REMARK 2.1.2. Let L be a frame and $\mathcal{S} = \{\lhd_i : i \in I\}$ a family of Császár orders on L. Then $\lhd_{\mathcal{S}}$ is a Császár order on L, where $\lhd_{\mathcal{S}} = \cup\{\lhd_i : i \in I\}$.

DEFINITION 2.1.3. Let L be a frame and $\mathcal{S}$ a family of Császár orders on L. Then $\mathcal{S}$ is said to be *admissible* if for any $a \in L$,

$$a = \bigvee\{x \in L : x \lhd_{\mathcal{S}} a\}$$

DEFINITION 2.1.4. A *symmetric Császár order* $\lhd$ on a frame L is a Császár order satisfying the following property:
(ST) $a \lhd b$ implies $b^* \lhd a^*$.

The following two propositions are immediate from Proposition 0.0.6.

PROPOSITION 2.1.5. *Let $h\colon M \to L$ be an onto frame homomorphism and $\lhd$ a Császár order on L. Define a binary relation $h_*(\lhd)$ on M as follows: $xh_*(\lhd)y$ if and only if there are points a and b in L such that*

$$h(x) \leq a \lhd b \text{ and } h_*(b) \leq y.$$

Then (1) $h_*(\lhd)$ is a Császár order on M,
(2) if h is dense and $\lhd$ is a symmetric Császár order on L then $h_*(\lhd)$ is a symmetric Császár order on M.
(3) if $\lhd_1$ and $\lhd_2$ are Császár orders on L and $\lhd_1 \subseteq \lhd_2$ then $h_*(\lhd_1) \subseteq h_*(\lhd_2)$.
(4) $x \lhd y$ if and only if $h_*(x)h_*(\lhd)h_*(y)$.

PROPOSITION 2.1.6. *Let $h\colon M \to L$ be a frame homomorphism and $\lhd$ a Császár order on M. Define a binary relation $h(\lhd)$ on L as follows: $xh(\lhd)y$ if and only if there are points a and b in M such that*

$$x \leq h(a), a \lhd b \text{ and } h(b) \leq y.$$

Then (1) $h(\lhd)$ is a Császár order on L.
(2) If h is onto and $\lhd$ is a symmetric Császár order on L then $h(\lhd)$ is a symmetric Császár order on L.
(3) If $\lhd_1$ and $\lhd_2$ are Császár orders on M and $\lhd_1 \subseteq \lhd_2$ then $h(\lhd_1) \subseteq h(\lhd_2)$.
(4) If $x \lhd y$ then $h(x)h(\lhd)h(y)$.
(5) If h is onto then $\lhd = h(h_*(\lhd))$, where $\lhd$ is a Császár order on L.

REMARK 2.1.7. For any onto frame homomorphism $h\colon M \to L$ and any family $\mathcal{S}$ of Császár orders on L, $h_*(\lhd_{\mathcal{S}}) = \cup\{h_*(\lhd)\colon \lhd \in \mathcal{S}\}$.

DEFINITION 2.1.8. Let L be a frame and $\lhd$ a Császár order on a frame L. An element $a \in L$ is said to be *$\lhd$-small* if $x \lhd y$ implies $a \leq x^*$ or $a \leq y$.

Denote by $B(\lhd)$ the set of $\lhd$-small elements and $B(\lhd)$ is called the *$\lhd$-small set.*

REMARK 2.1.9. Let L be a frame and $\lhd, \lhd_\circ$ Császár orders on L. Then one has the following:

(1) $B(\lhd)$ is a down set, that is, if $x \in B(\lhd)$ and $y \leq x$, then $y \in B(\lhd)$.

(2) $\lhd \subseteq \lhd_\circ$ implies $B(\lhd_\circ) \subseteq B(\lhd)$.

Let L be a frame and $A, B \subseteq L$. Then A is said to *refine* B if for any $a \in A$ there exists $b \in B$ with $a \leq b$. In this case, we write $A \leq B$.

PROPOSITION 2.1.10. *Let $h\colon M \to L$ be a frame homomorphism and $\lhd$ a Császár order on L. Then*

(1) if h is dense, then $h_*(B(\lhd)) \subseteq B(h_*(\lhd))$,

(2) if h is onto, then $B(h_*(\lhd)) \leq h_*(B(\lhd))$,

(3) if h is onto dense, then $h(B(h_*(\lhd))) = B(\lhd)$.

Proof. (1) Take any $a \in B(\lhd)$ and suppose $xh_*(\lhd)y$. Then there exist $u, v \in L$ such that $h(x) \leq u \lhd v$ and $h_*(v) \leq y$. Suppose $h_*(a) \wedge x \neq 0$. Then $a \wedge h(x) \neq 0$, for h is dense and $h(h_*(a)) \leq a$. Hence $a \wedge u \neq 0$, for $h(x) \leq u$. Since $a \in B(\lhd)$, $a \leq v$ and $h_*(a) \leq y$. Therefore $h_*(a) \in B(h_*(\lhd))$. In all $h_*(B(\lhd)) \subseteq B(h_*(\lhd))$. This completes the proof.

(2) Take any $a \in B(h_*(\lhd))$ and suppose $x \lhd y$. Then $h_*(x)h_*(\lhd)h_*(y)$. Suppose $h(a) \wedge x \neq 0$. Then $a \wedge h_*(x) \neq 0$ for h is onto. Since $a \in B(h_*(\lhd))$, $a \leq h_*(y)$ and hence $h(a) \leq y$. Therefore $h(a) \in B(\lhd)$ and then $a \leq h_*(h(a)) \in h_*(B(\lhd))$. In all $B(h_*(\lhd))) \leq h_*(B(\lhd))$. This completes the proof.

(3) It is immediate from (2) and (3). □

2.2. The category of Császár frames

In this subsection, we introduce two categories: The category CsFrm of Császár frames and the category CCCsFrm of Cauchy complete Császár frames.

Now we recall syntopogenous spaces introduced by Császár. In a syntopogenous space $(X, \mathcal{S})$, the syntopogenous structure S is determined by topogenous orders on $P(X)$. Such an order is a sublattice of $P(X) \times P(X)$. The condition is required to deduce a topology on X from the syntopogenous structure $\mathcal{S}$(see [7]). But frame itself is a (pointless) topology and the right adjoint of a frame homomorphism does not preserve joins. This motivates the following definition.

DEFINITION 2.2.1. Let L be a frame and $\mathcal{L}$ a family of Császár orders on L. Then $\mathcal{L}$ is said to be a *Császár structure* on L if it satisfies the following:

(1) $\mathcal{L}$ is up-directed (relative to set inclusion)

(2) $\lhd_{\mathcal{L}}$ is a meet-sublattice of $L \times L$, that is, $x \lhd_{\mathcal{L}} y, z$ imply $x \lhd_{\mathcal{L}} y \wedge z$.

(3) $\mathcal{L}$ is admissible.

In this case, the pair $(L, \mathcal{L})$ is called a *Császár frame.*

REMARK 2.2.2. (1) It is easy to see that for any frame L, $\{\leq\}$ is a Császár structure on the frame L.

(2) Every nearness structure $\mathcal{N}$ on a frame L admits Császár structure $\mathcal{L}_{\mathcal{N}}$ on L:

$$\mathcal{L}_{\mathcal{N}} = \{\lhd_A : A \in \mathcal{N}\},$$

where $x \lhd_A y$ if and only if $\bigvee\{a \in A : x \wedge a \neq 0\} \leq y$.

(3) If L is a regular frame, then $(L, \{\prec\})$ is a Császár structure where $x \prec y$ if and only if $x^* \vee y = e$.

(4) If L is a completely regular frame then $(L, \{\prec^2\})$ is a Császár structure where $x \prec^2 y$ if and only if $x \prec z \prec y$ for some $z \in L$.

DEFINITION 2.2.3. Császár frame $(L, \mathcal{L})$ is said to be *regular* if $\lhd_{\mathcal{L}} \subseteq \prec$.

In what follows, we assume that every Császár frame is regular.

REMARK 2.2.4. (1) It is clear that if $(L, \mathcal{L})$ is a Császár frame then L is a regular frame.

(2) It is immediate from the above definition that $\{\prec\}$ is the largest regular Császár structure on the frame.

Cauchy filters play an important role in syntopogenous spaces inasmuch as such notions as convergence and completeness can be characterized in terms of Cauchy filters. We now define their counterparts, namely regular Cauchy filters, in a Császár frame using $\lhd$-small sets.

DEFINITION 2.2.5. Let $(L, \mathcal{L})$ be a Császár frame and F a filter on L. Then F is said to be :

(1) a *Cauchy filter* if for any $\lhd \in \mathcal{L}$, $F \cap B(\lhd) \neq \emptyset$.

(2) a *regular filter* if for any $a \in F$, there is $b \in F$ with $b \lhd_{\mathcal{L}} a$.

(3) a *completely prime filter* if $\bigvee a_i \in F (i \in I)$ implies $a_i \in F$ for some $i \in I$.

Using regular Cauchy filters, we now define a Cauchy homomorphism between Császár frames.

DEFINITION 2.2.6. Let $(L, \mathcal{L})$ and $(M, \mathcal{M})$ be Császár frames. A frame homomorphism $h\colon M \to L$ is said to be a *Cauchy homomorphism* if for any regular Cauchy filter F on $(L, \mathcal{L})$, there exists a regular Cauchy filter G on $(M, \mathcal{M})$ with $G \subseteq h^{-1}(F)$.

It is easy to see that every identity map is a Cauchy homomorphism and the composition of two Cauchy homomorphisms is also a Cauchy homomorphism.

CsFrm denotes the category of Császár frames and Cauchy homomorphisms.

We now recall that a filter F on a frame L is said to be convergent if for any cover A of L, $F \cap A \neq \emptyset$([8]) and that a regular filter on a frame is convergent if and only if it is completely prime filter on the frame, and every regular filter on a Császár frame $(L, \mathcal{L})$ is a regular filter on L.

DEFINITION 2.2.7. A Császár frame is said to be *Cauchy complete* if every regular Cauchy filter on the Császár frame is convergent.

CCCsFrm denotes the category of Cauchy complete Császár frames and Cauchy homomorphisms.

REMARK 2.2.8. The category CCCsFrm is a full subcategory of the category CsFrm.

REMARK 2.2.9. Note that a Császár frame is Cauchy complete if and only if every regular Cauchy filter on the Császár frame is a completely prime filter; therefore a Császár frame is Cauchy complete if and only if regular Cauchy filters on the Császár frame are precisely Cauchy completely prime filters.

3. Cauchy completion of Császár frames

In this section we construct the Cauchy completion of Császár frames and show that the Cauchy completion gives rise to a coreflection in the categories CsFrm. We introduce the concept of uniform Császár frames and show that the Cauchy completion gives rise to a coreflection in the categories UCsFrm.

3.1. Cauchy completion of Császár frames

Using strict extension of frames, we construct the Cauchy completion of Császár frames and then show that the Cauchy completion gives rise to a coreflection in the category CsFrm of Császár frames and Cauchy homomorphisms. .

Banaschewski and Hong [3] introduce strict extensions of frames. The following summary is extracted from the paper:

For a set X of filters on a frame L, let $P(X)$ denote the power set lattice of X and $L \times P(X)$ the product frame L and $P(X)$. Then $S_X L = \{(x, \sum_x) : x \in L\}$ is a subframe of $L \times P(X)$, where $\sum_x = \{F \in X : x \in F\}$. Let $s\colon s_X L \longrightarrow L$ be the restriction of the first projection and the right adjoint s^* of s is given by $s^*(x) = (x, \sum_x)(x \in L)$. Since s^* preserves meets, $s^*(L)$ is closed under finite meets and hence the subframe cL of $s_X L$ generated by $s^*(L)$ is given by $\{(\bigvee A, \sum_A) : A \subseteq L\}$.

The restriction $c_L\colon cL \longrightarrow L$ of s is called the strict extension of L associated with X. In what follows, X denotes the set of regular Cauchy filters on a frame L.

LEMMA 3.1.1. *Let $h\colon M \to L$ be an onto frame homomorphism and $\lhd$ a meet-sublattice of $L \times L$. Then $h_*(\lhd)$ is a meet-sublattice of $M \times M$.*

Proof. Suppose $ah_*(\lhd)b, c$. Then there exist $x_1, x_2, y_1, y_2 \in L$ such that

$$h(a) \le x_1 \lhd y_1, h_*(y_1) \le b$$

and

$$h(a) \le x_2 \lhd y_2, h_*(y_2) \le c.$$

By the assumption,

$$h(a) \le x_1 \wedge x_2 \lhd y_1 \wedge y_2, h_*(y_1 \wedge y_2) \le b \wedge c.$$

Hence $ah_*(\lhd)b \wedge c$. This completes the proof. □

For a Császár frame $(L, \mathcal{L})$, let $\mathcal{L}^* = \{c_{L_*}(\lhd) : \lhd \in \mathcal{L}\}$. Using this notion and Proposition 1.1.5, we have the following:

PROPOSITION 3.1.2. *For any Császár frame $(L, \mathcal{L})$, $(cL, \mathcal{L}^*)$ is a Császár frame.*

Proof. By Proposition 1.1.5, each of $\mathcal{L}^*$ is a Császár order on cL and $\mathcal{L}^*$ is directed. Firstly, we show that $\mathcal{L}^*$ is admissible, that is, $(\bigvee A, \sum_A) = \bigvee\{(\bigvee B, \sum_B) : (\bigvee B, \sum_B) \lhd_{\mathcal{L}^*} (\bigvee A, \sum_A)\}$ for each $A \subseteq$

L. Since $(\bigvee A, \sum_A) = \bigvee\{c_{L_*}(a) : a \in A\}$ and $b \lhd_{\mathcal{L}} a$ implies $c_{L_*}(b) \lhd_{\mathcal{L}^*} c_{L_*}(a)$, it is enough to show that $c_{L_*}(a) = \bigvee\{c_{L_*}(b) : b \lhd_{\mathcal{L}} a\}$ for each $a \in L$. Clearly, $\cup\{\sum_b : b \lhd_{\mathcal{L}} a\} \subseteq \sum_a$. Take any $F \in \sum_a$. Then there exists $b \in F$ with $b \lhd_{\mathcal{L}} a$ for F is regular; hence $F \in \cup\{\sum_b : b \lhd_{\mathcal{L}} a\}$. By the above lemma, $\lhd_{\mathcal{L}^*}$ is a meet-sublattice of $cL \times cL$. Now we show that $(L, \mathcal{L})$ is regular. Suppose $(a, \sum_A) \lhd_{\mathcal{L}^*} (b, \sum_B)$. Then there exist $x, y \in L$ such that $a \leq x \lhd_{\mathcal{L}} y$ and $c_{L_*}(y) \leq (b, \sum_B)$. Since $(L, \mathcal{L})$ is regular, $a \lhd_{\mathcal{L}} y$ implies $a^* \vee y = e$. Since F is a Cauchy filter L, $a^* \in F$ or $y \in F$ and hence $c_{L_*}(a^*) \vee c_{L_*}(y) = (e, X)$. Thus $(a, \sum_A)^* \vee (b, \sum_B) = (e, X)$ for $c_{L_*}(a^*) \leq (a, \sum_A)^*$. This completes the proof. $\square$

Proposition 3.1.3. *Let $(L, \mathcal{L})$ be a Császár frame. Then one has the following:*

(1) $c_L\colon (cL, \mathcal{L}^*) \to (L, \mathcal{L})$ is a Cauchy homomorphism.

(2) for any regular Cauchy filter G on $(cL, \mathcal{L}^*)$, $c_L(G)$ is also a regular Cauchy filter on $(L, \mathcal{L})$.

Proof. (1) Take any regular Cauchy filter F on $(L, \mathcal{L})$ and let $G = \{a \in cL : b \leq a \text{ for some } b \in c_{L_*}(F)\}$. Since c_L is onto and c_{L_*} preserves finite meets, G is a filter on $(cL, \mathcal{L}^*)$ and $G \subseteq {c_L}^{-1}(F)$, for c_L is onto. It remains to show that G is a regular Cauchy filter on $(cL, \mathcal{L}^*)$. We firstly show that G is a Cauchy filter on $(cL, \mathcal{L}^*)$. Take any $\lhd \in \mathcal{L}^*$. Then there exists $\lhd_L \in$ L with $\lhd = c_{L_*}(\lhd_L)$. Since F is a Cauchy filter on $(L, \mathcal{L})$, $B(\lhd_L) \cap F \neq \emptyset$ and hence $c_{L_*}(B(\lhd_L)) \cap c_{L_*}(F) \neq \emptyset$. By Proposition 1.1.10, $B(c_{L_*}(\lhd_L)) \cap c_{L_*}(F) \neq \emptyset$. Since $\lhd = c_{L_*}(\lhd_L)$, $B(\lhd) \cap c_{L_*}(F) \neq \emptyset$. Thus G is a cauchy filter on $(cL, \mathcal{L}^*)$. Now we show that G is a regular filter on $(cL, \mathcal{L}^*)$. Take any $a \in G$. Then there exists $b \in F$ with $c_{L_*}(b) \leq a$. Since F is a regular filter on $(L, \mathcal{L})$, there exists $d \in F$ with $d \lhd_L b$. By Proposition 1.1.5, $c_{L_*}(d) c_{L_*}(\lhd_{\mathcal{L}}) c_{L_*}(b)$ and hence $c_{L_*}(d) \lhd_{\mathcal{L}} a$. Thus G is a regular filter on $(cL, \mathcal{L}^*)$. This completes the proof.

(2) We first note that since cL is dense, $c_L(G)$ is a filter on L. It remains to show that $c_L(G)$ is a regular Cauchy filter on $(L, \mathcal{L})$. We firstly show that $c_L(G)$ is a Cauchy filter on $(L, \mathcal{L})$. Take any $\lhd \in \mathcal{L}$. Since G is a Cauchy filter on $(cL, \mathcal{L}^*)$, $G \cap B(c_{L_*}(\lhd)) \neq \emptyset$. Since c_L is onto dense, $c_L(G) \cap B(\lhd) \neq \emptyset$. Thus $c_L(G)$ is a Cauchy filter on $(L, \mathcal{L})$. Now we show that $c_L(G)$ is a regular Cauchy filter on $(L, \mathcal{L})$. Take any $a \in G$. Then there exists $b \in G$ and $c_{L_*}(\lhd) \in \mathcal{L}^*$ with $b c_{L_*}(\lhd) a$ for G is regular. Since c_L is onto, $c_L(b) \lhd_{\mathcal{L}^*} c_L(a)$ and $c_L(b) \in c_L(G)$. Thus $c_L(G)$ is a regular filter on $(L, \mathcal{L})$. This completes the proof. $\square$

Using the above, we construct the Cauchy completion of a Császár frame.

THEOREM 3.1.4. *The Császár frame $(cL, \mathcal{L}^*)$ is Cauchy complete.*

Proof. Take any regular Cauchy filter F on $(cL, \mathcal{L}^*)$ and take any basic cover S of cL, i.e., $S = \{(a, \sum_a) : a \in A\}$ for some $A \subseteq L$. By the above proposition, $c_L(F)$ is a regular Cauchy filter on $(L, \mathcal{L})$. Since $\bigvee S = (\bigvee A, \sum_A) = (e, X)$, $\sum_A = X$ and hence $c_L(F) \in \sum_A$. Pick $a \in c_L(F) \cap A \neq \emptyset$. Then there exists $b \in F$ with $b \leq c_{L_*}(a)$; hence $c_{L_*}(a) = (a, \sum_a) \in F \cap S$. Thus F is convergent. This completes the proof. $\square$

DEFINITION 3.1.5. For any Császár frame $(L, \mathcal{L})$, $c_L\colon (cL, \mathcal{L}^*) \to (L, \mathcal{L})$ or simply $(cL, \mathcal{L}^*)$ is called the *Cauchy completion* of $(L, \mathcal{L})$.

REMARK 3.1.6. It is worthwhile to note that $c_L\colon (cL, \mathcal{L}^*) \to (L, \mathcal{L})$ is an isomorphism if and only if every filter in X is Cauchy completely prime filter on L. Using this note, we know that a Császár frame $(L, \mathcal{L})$ is Cauchy complete if and only if $cL = \{(a, \sum_a) : a \in L\}$.

Using the above, we now have the main theorem in this subsection:

THEOREM 3.1.7. *The category CCCsFrm is coreflective in the category CsFrm.*

Proof. Take any Császár frame $(L, \mathcal{L})$. By Theorem 2.1.4, $(cL, \mathcal{L}^*)$ is a Cauchy complete Császár frame and by Proposition 2.1.3, the Cauchy completion $c_L\colon (cL, \mathcal{L}^*) \to (L, \mathcal{L})$ is a Cauchy homomorphism. Take any Cauchy complete Császár frame $(M, \mathcal{M})$ and Cauchy homomorphism $h\colon (M, \mathcal{M}) \to (L, \mathcal{L})$. We define $\bar{h}\colon cM \to cL$ by $\bar{h}(a, \sum_A) = (h(a), \cup\{\sum_{h(b)} : b \lhd_{\mathcal{M}} a\})$ for all $A \subseteq M$. Then $\bar{h}$ is a map with $h \circ c_M = c_L \circ \bar{h}$. Take any regular Cauchy filter H on $(cL, \mathcal{L}^*)$. By Proposition 2.1.3, $c_L(H)$ is a regular Cauchy filter on $(L, \mathcal{L})$. Since h is a Cauchy homomorphism, there exists a regular Cauchy filter G on $(M, \mathcal{M})$ with $G \subseteq h^{-1}(c_L(H))$. Now we show that $c_{M_*}(G) \subseteq \bar{h}^{-1}(H)$. Take any $c_{M_*}(g) \in c_{M_*}(G)$ then $g \in G$. Since G is regular, there exists $u \in G$ with $u \lhd_{\mathcal{M}} g$. Then there exists $(a, \sum_A) \in H$ with $h(u) = c_L(a, \sum_A) = a$, for $G \subseteq h^{-1}(c_L(H)$. Then $c_{L_*}(a) = c_{L_*}(h(u)) \leq \bar{h}(c_{L_*}g))$. Indeed, take any $F \in \sum_{h(u)}$. Since $u \lhd_{\mathcal{M}} g$ and $h(u) \in F$, by the definition of $\bar{h}$, $F \in \cup\{\sum_{h(u)} : u \lhd_{\mathcal{M}} g\}$. Thus $c_{L_*}(h(u)) \leq \bar{h}(c_{L_*}(g))$. Since $(a, \sum_A) \in H$ and H is a upset, $\bar{h}(c_{L_*}(g)) \in H$ and then $c_{M_*}(G) \subseteq \bar{h}^{-1}(H)$. Let $k = \bar{h} \circ c_{M_*}$. Clearly k preserves finite meets, for $\lhd_{\mathcal{M}}$ is a meet-sublattice of $M \times M$. In order to show that k preserves arbitrary joins, it is enough to show that for any subset $A \subseteq M$, $\cup\{\sum_{h(x)} : x \lhd_{\mathcal{M}} \bigvee A\} = \cup\{\sum_{h(x)} : x \lhd_{\mathcal{M}} a$ for some $a \in A\}$. Clearly $\cup\{\sum_{h(x)} : x \lhd_{\mathcal{M}} a$ for some $a \in A\} \subseteq \cup\{\sum_{h(x)} : x \lhd_{\mathcal{M}} \bigvee A\}$. In order to show that the reverse inclusion, take any $F \in \cup\{\sum_{h(x)} : x \lhd_{\mathcal{M}} \bigvee A\}$. Then F is a regular Cauchy filter on $(L, \mathcal{L})$ and $h(x) \in F$ for some $x \lhd_{\mathcal{M}} \bigvee A$. Since h is a Cauchy homomorphism, there exists a regular Cauchy filter G on $(M, \mathcal{M})$ with $G \subseteq h^{-1}(F)$. Since $(M, \mathcal{M})$ is a Cauchy complete Császár frame, G is a completely prime filter and $\{x^*\} \cup A$ is a cover of M; hence $G \cap A \neq \emptyset$, for $h(x) \in F$. Pick $a \in G \cap A$. Since G is regular,there exists $y \in G$ with $y \lhd_{\mathcal{M}} a$. Hence $F \in \cup\{\sum_{h(x)} : x \lhd_{\mathcal{M}} a$ for some $a \in A\}$ for $h(y) \in F$ and $a \in A$. Thus k is a frame homomorphism. In all k is a Cauchy homomorphism. Since c_L is a monomorphism, by Proposition 0.0.5 and Remark 1.2.3, such a k is unique. Thus $c_L \colon (cL, \mathcal{L}^*) \to (L, \mathcal{L})$ is the CCCsFrm-coreflection for $(L, \mathcal{L})$ in CsFrm. This completes the proof. □

3.2. Cauchy completion of uniform Császár frames

In this subsection, we introduce the concept of uniform Császár frames and show that the Cauchy completion gives rise to a coreflection in the categories UCsFrm.

DEFINITION 3.2.1. A Császár frame $(L, \mathcal{L})$ is said to be a *uniform Császár frame* if each of $\mathcal{L}$ is symmetric and for any $\lhd \in \mathcal{L}$, there exists $\lhd_0 \in \mathcal{L}$ with $\lhd \subseteq \lhd_0^2$, i.e., $a \lhd b$ implies $a \lhd_0 d \lhd_0 b$ for some $d \in L$.

The frame in the above definition is regarded as the topology deduced from a uniformity (= symmetrical biperfect syntopogenous structure) on a set.

The following is immediate from Propositions 1.1.5 and 2.1.2.

THEOREM 3.2.2. *For any uniform Császár frame* $(L, \mathcal{L})$, $(cL, \mathcal{L}^*)$ *is a Cauchy complete uniform Császár frame.*

DEFINITION 3.2.3. Let $(L, \mathcal{L})$ and $(M, \mathcal{M})$ be uniform Császár frames. A frame homomorphism $h \colon M \to L$ is said to be a *uniform homomorphism* if for any $\lhd_M \in \mathcal{M}$ there exists $\lhd_L \in \mathcal{L}$ with $B(\lhd_L) \leq h(B(\lhd_M))$.

It is easy to see that every identity frame homomorphism is a uniform homomorphism and the composition of two uniform homomorphisms is a uniform homomorphism.

UCsFrm denotes the category of uniform Császár frames and uniform homomorphisms.

CCUCsFrm denotes the category of Cauchy complete uniform Császár frames and uniform homomorphisms.

REMARK 3.2.4. The category CCUCsFrm is a full subcategory of the category UCsFrm.

Let $(L, \mathcal{L})$ be a Császár frame and $A, B \subseteq L$. Then A is said to $\lhd$-*refine* B if for any $a \in A$ there exists $b \in B$ with $a \lhd b$. In this case, we write $A \lhd B$.

LEMMA 3.2.5. *Let $(L, \mathcal{L})$ be a uniform Császár frame. Then for any $\lhd \in \mathcal{L}$, there exists $\lhd_\circ \in \mathcal{L}$ with $B(\lhd_\circ) \lhd_\circ B(\lhd)$.*

Proof. Take any $\lhd \in \mathcal{L}$ and let $\lhd_1, \lhd_\circ \in \mathcal{L}$ such that $\lhd \subseteq \lhd_1^2$ and $\lhd_1 \subseteq \lhd_\circ^2$.Take any $x \in B(\lhd_\circ)$. Suppose $z \in L$ with $x \lhd_\circ z$ and let $x^\circ = \bigwedge\{y \in L : x \lhd_\circ z \lhd_\circ y\}$. Then $x \lhd_\circ x^\circ$, for $\lhd$ is transitive. Now we show that $x^\circ \in B(\lhd)$. Suppose $a \lhd b$ and $x^\circ \wedge a \neq 0$. Since $\lhd_1 \subseteq \lhd_\circ^2$, there exist $a_1, a_2, a_3 \in L$ with $a \lhd_\circ a_1 \lhd_\circ a_2 \lhd_\circ a_3 \lhd_\circ b$. Then $x \wedge a_2 \neq 0$, else $x \leq a_2^* \lhd_\circ a_1^* \lhd_\circ a^*$, for $\lhd_\circ$ is symmetric and hence $x^\circ \leq a^*$, which is contradiction to the fact that $x^\circ \wedge a \neq 0$. Since $x \in B(\lhd_\circ)$, $x \leq a_3 \lhd_\circ b$ and so $x^\circ \leq b$. Thus $x^\circ \in B(\lhd)$ and so $B(\lhd_\circ) \lhd_\circ B(\lhd)$. This completes the proof. □

For a filter F on a uniform Császár frame $(L, \mathcal{L})$, F° denotes the filter $\{x \in L : a \lhd_{\mathcal{L}} x \text{ for some } a \in F\}$. Using the above lemma, we have the following:

PROPOSITION 3.2.6. *Let $(L, \mathcal{L})$ be a uniform Császár frame and F a Cauchy filter on $(L, \mathcal{L})$. Then F° is a regular Cauchy filter on $(L, \mathcal{L})$.*

Proof. Clearly F° is a regular filter. In order to show that F° is a Cauchy filter, take any $\lhd \in \mathcal{L}$ and let $\lhd^\circ \in \mathcal{L}$ with $B(\lhd_\circ) \lhd_\circ B(\lhd)$. Since F is a Cauchy filter on $(L, \mathcal{L})$, $F \cap B(\lhd^\circ) \neq \emptyset$. Pick $x \in F \cap B(\lhd^\circ)$. Since $x \in B(\lhd^\circ)$, there exists $y \in B(\lhd)$ with $x \lhd^\circ y$; hence $y \in F$ for $x \in F$. Thus F° is a Cauchy filter. This completes the proof. □

LEMMA 3.2.7. *Every uniform homomorphism $h\colon (M, \mathcal{M}) \to (L, \mathcal{L})$ between uniform frames is a Cauchy homomorphism.*

Proof. Take any regular Cauchy filter F on $(L, \mathcal{L})$ and $\lhd_M \in \mathcal{M}$. Since h is uniform, there exists $\lhd_L \in \mathcal{L}$ with $B(\lhd_L) \leq h(B(\lhd_M))$. Since F is a Cauchy filter, $F \cap B(\lhd_L) \neq \emptyset$ and hence $F \cap h(B(\lhd_M)) \neq \emptyset$. Thus $h^{-1}(F) \cap B(\lhd_M) \neq \emptyset$ and so $h^{-1}(F)$ is a Cauchy filter on $(M, \mathcal{M})$. Since $(M, \mathcal{M})$ is uniform, by Proposition 2.2.5, there exists a regular Cauchy filter G on $(M, \mathcal{M})$ with $G \subseteq h^{-1}(F)$. □

Using the above, we now have the main theorem in this subsection:

THEOREM 3.2.8. *The category CCUCsFrm is coreflective in the category UCsFrm.*

Proof. Take any uniform Császár frame $(L, \mathcal{L})$ and let $c_L \colon (cL, \mathcal{L}^*) \to (L, \mathrm{L})$ be the Cauchy completion of $(L, \mathcal{L})$. By Theorem 2.2.3, $(cL, \mathcal{L}^*)$ is a Cauchy complete uniform Császár frame and by Proposition 1.1.10, c_L is uniform, for c_L is onto dense. Take any Cauchy complete uniform Császár frame $(M, \mathcal{M})$ and any uniform homomorphism $h \colon (M, \mathcal{M}) \to (L, \mathcal{L})$. Since h is a uniform homomorphism, by the above lemma, h is a Cauchy homomorphism and hence by Theorem 2.1.7, there is a unique Cauchy homomorphism $\bar{h} \colon cM \to cL$ with $c_L \circ \bar{h} = h$. Now we show that $\bar{h}$ is uniform. Take any $c^*_M(\lhd) \in c^*_M(\mathcal{M})$ and then $\lhd \in \mathcal{M}$. Since $(M, \mathcal{M})$ is uniform, there exists $\lhd_\circ \in \mathcal{M}$ with $B(\lhd_\circ) \lhd_\circ B(\lhd)$. Since h is uniform, there exists $\lhd_L \in \mathcal{L}$ with $B(\lhd_L) \leq h(B(\lhd_\circ))$. Now we show that $c_{L_*}(B(\lhd_L)) \leq \bar{h}(c^*_M(B(\lhd)))$. Take any $c^*_L(u) \in c^*_L(B(\lhd_L))$ and then $u \in B(\lhd_L)$. Then there exists $v \in B(\lhd_\circ)$ with $u \leq h(v)$. Since $v \in B(\lhd_\circ)$, there exists $x \in B(\lhd)$ with $v \lhd_{\mathcal{M}} x$. Take any $F \in \sum_u$. Since $v \lhd_{\mathcal{M}} x$ and $u \leq h(v)$, $F \in \cup\{\sum_{h(z)} : z \lhd_{\mathcal{M}} x\}$ and hence $c^*_L(u) \leq \bar{h}(c^*_M(x))$ and hence $c^*_L(B(\lhd_L)) \leq \bar{h}(c^*_M(B(\lhd)))$. Since c_M and c_L are onto dense, by Proposition 1.1.10, $B(c^*_L(\lhd_L)) \leq \bar{h}(B(c^*_M(\lhd)))$. Thus $c_L \colon (cL, \mathcal{L}^*) \to (L, \mathcal{L})$ is the CCUCsFrm-coreflection for $(L, \mathcal{L})$ in UCsFrm. This completes the proof. □

ACKNOWLEDGEMENT. The author would like to express deep gratitude to Prof. S. S. Hong for his valuable seminar lectures at Hanyang Woman College and also to the referee for many kind suggestions and comments.

References

[1] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories*, Wiley Interscience, New York, 1990.

[2] B. Banschewski, *Compactifications of frames*, Math. Nachr. **149** (1990), 105–115.

[3] B. Banschewski and S. S. Hong, *Filters and strict extensions of frames*, Kyungpook Math. J. **39** (1999) no. 1, 215–230.

[4] B. Banschewski and A. Pultr, *Cauchy points of uniform and nearness frames*, Quaestions Math. **19** (1996), 101–127.

[5] ———, *Samuel compactification and completion of uniform frames*, Math. Proc. Cambridge Phil. Soc. **108** (1990), 63–78.

[6] S. H. Chung, *Categories of syntopogenous spaces*, Ph.D. Thesis, Sogang University, 1988.

[7] Á. Császár, *Foundation of General Topology*, MacMillan Company, New York, 1963.
[8] S. S. Hong, *Convergence in frames*, Kyungpook Math. J. **35** (1995), 85–91.
[9] S. S. Hong and Y. K. Kim, *Cauchy completions of nearness frames*, Applied Categorical Structures, **3** (1995), 371–377.
[10] J. R. Isbell, *Atomless parts of space*, Math. Scand. **31** (1972), 5–32.
[11] P. T. Johnstone, *Stone Spaces*, Cambridge Stud. Adv. Math. J. **3** (1982).
[12] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge University Press, Cambridge, 1970.
[13] J. Picado, *Structured frames by Weil Entourages*, Applied Categorical Structures **8** (2000), 351–366.
[14] J. W. Tukey, *Convergence and Uniformity in Topology*, Ann. Math. Stud. 2, Princeton University Press, 1940.

Department of Mathematics and Information
Kyungwon University
Sungnam 461-701, Korea
E-mail: sehwa@kyungwon.ac.kr