J. Korean Math. Soc. **42** (2005), No. 2, pp. 255–268

PERIODIC SOLUTIONS IN NONLINEAR NEUTRAL DIFFERENCE EQUATIONS WITH FUNCTIONAL DELAY

MARIETTE R. MAROUN AND YOUSSEF N. RAFFOUL

ABSTRACT. We use Krasnoselskii's fixed point theorem to show that the nonlinear neutral difference equation with delay

$$x(t+1) = a(t)x(t) + c(t)\Delta x(t-g(t)) + q\big(t, x(t), x(t-g(t))\big)$$

has a periodic solution. To apply Krasnoselskii's fixed point theorem, one would need to construct two mappings; one is contraction and the other is compact. Also, by making use of the variation of parameters techniques we are able, using the contraction mapping principle, to show that the periodic solution is unique.

1. Introduction

Recently, there has been an increasing interest in the analysis of qualitative theory of solutions of difference equations. Motivated by the papers [2-5] and the references therein, we consider the nonlinear neutral difference equation

$$x(t+1) = a(t)x(t) + c(t)\Delta x(t-g(t)) + q\big(t,\, x(t),\, x(t-g(t))\big) \tag{1.1}$$

which arises in a (food-limited) population model.

$$q : \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

where $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers.
Our motivation comes from the neutral nonlinear difference equation

$$\Delta x(t) = b(t)x(t) + c(t)\Delta x(t-g(t)) + q\big(t,\, x(t),\, x(t-g(t))\big)$$

which can be easily put in the form of (1.1) with $a(t) = 1+b(t)$. Throughout this paper Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ for any sequence $\{x(n),\, n = 0,\, 1,\, 2\cdot,\, \cdot\}$. Also, we define

Received November 4, 2003.
2000 Mathematics Subject Classification: 39A10, 39A12.
Key words and phrases: Krasnoselski, contraction, nonlinear neutral difference equation, periodic solutions, unique solution.

the operator E by $Ex(n) = x(n+1)$. For more on the calculus of difference equations, we refer the reader to [1]. The purpose of this paper is to use Krasnoselskii's fixed point theorem to show the existence of a periodic solution for equation (1.1). To apply Krasnoselaskii's fixed point theorem we need to construct two mappings; one is contraction and the other is compact.

Using the variation of parameters techniques, enables us to show the uniqueness of the periodic solution by appealing to the contraction mapping principle.

2. Existence of periodic solutions

Let T be an integer such that $T \geq 1$. Define $P_T = \{\phi : C(R,R),\ \phi(t+T) = \phi(t)\}$ where $C(R,R)$ is the space of all real valued functions. Then P_T is a Banach space when it is endowed with the maximum norm

$$\|x(t)\| = \max_{t\in[0,T-1]} |x(t)|.$$

In this paper we assume the periodicity conditions

$$a(t+T) = a(t), \quad c(t+T) = c(t), \quad g(t+T) = g(t), \quad g(t) \geq g^* > 0 \tag{2.1}$$

for some constant g^*. Also, we assume that

$$\prod_{s=t-T}^{t-1} a(s) \neq 1. \tag{2.2}$$

Throughout this paper we assume that $a(t) \neq 0$ for all $t \in [0, T-1]$. It is interesting to note that equation (1.1) becomes of advanced type when $g(t) < 0$. Since we are searching for periodic solutions, it is natural to ask that $q(t,x,y)$ is periodic in t and Lipschitz continuous in x and y. That is

$$q(t+T,x,y) = q(t,x,y) \tag{2.3}$$

and

$$|q(t,x,y) - q(t,z,w)| \leq L\|x-z\| + K\|y-w\| \tag{2.4}$$

for some positive constants L and E.
Note that

$$\begin{aligned} |q(t,x,y)| - |q(t,0,0)| &\leq |q(t,x,y) - q(t,0,0)| \\ &\leq L\|x-0\| + K\|y-0\| \\ &= L\|x\| + K\|y\|. \end{aligned}$$

which gives us,

$$|q(t,x,y)| \leq L\|x\| + K\|y\| + |q(t,0,0)|. \tag{2.5}$$

LEMMA 2.1 *Suppose (2.1) and (2.2) hold. If $x(t) \in P_T$, then $x(t)$ is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) &= c(t-1)x(t-g(t)) \\ &+ \frac{1}{1-\prod_{s=t-T}^{t-1} a(s)} \sum_{r=t-T}^{t-1} \Big[x(r-g(r))\Big(a(r)c(r-1)-c(r)\Big) \\ &+ q(r,x(r),x(r-g(r)))\Big] \prod_{s=r+1}^{t-1} a(s) \end{aligned} \tag{2.6}$$

Proof. We consider two cases; $t \geq 1$ and $t \leq 0$. Let $x(t) \in P_T$ be a solution of (1.1). For $t \geq 1$ Equation (1.1) is equivalent to

$$\begin{aligned} &\Delta\Big[\prod_{s=0}^{t-1} a^{-1}(s)x(t)\Big] \\ &= \Big[c(t)\Delta x(t-g(t)) + q(t,x(t),x(t-r(t)))\Big] \prod_{s=0}^{t} a^{-1}(s). \end{aligned} \tag{2.7}$$

By summing (2.7) from $(t-T)$ to $(t-1)$ we obtain

$$\begin{aligned} &\sum_{r=t-T}^{t-1} \Delta\Big[\prod_{s=0}^{r-1} a^{-1}(s)x(r)\Big] \\ &= \sum_{r=t-T}^{t-1} [c(r)\Delta x(r-g(r)) + q(r,x(r),x(r-g(r)))] \prod_{x=0}^{r} a^{-1}(s). \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} &x(t)\prod_{s=0}^{t-1} a^{-1}(s) - x(t-T)\prod_{s=0}^{t-T-1} a^{-1}(s) \\ &= \sum_{r=t-T}^{t-1} [c(r)\Delta x(r-g(r)) + q(r,x(r),x(r-g(r)))] \prod_{s=0}^{r} a^{-1}(s). \end{aligned}$$

By dividing both sides of the above expression by

$$\prod_{s=0}^{t-1} a^{-1}(s)$$

and the fact that $x(t) = x(t-T)$, we get

$$x(t) = \frac{1}{1-\prod_{s=t-T}^{t-1} a(s)} \sum_{r=t-T}^{t-1} \Big[c(r)\Delta x(r-g(r)) + q(r, x(r), x(r-g(r)))\Big] \prod_{s=r+1}^{t-1} a(s). \tag{2.8}$$

Rewrite

$$\sum_{r=t-T}^{t-1} c(r)\Delta x(r-g(r)) \prod_{s=r+1}^{t-1} a(s) = \sum_{r=t-T}^{t-1} c(r) \prod_{s=r+1}^{t-1} a(s)\Delta x(r-g(r)).$$

By considering $Ey = c(r)\prod_{s=r+1}^{t-1} a(s)$ and $y = c(r-1)\prod_{s=r}^{t-1} a(s)$ we get $y = c(r-1)\prod_{s=r}^{t-1} a(s)$. Thus, by performing a summation by parts on the above equation using the summation by parts formula

$$\sum Ey\Delta z = yz - \sum z\Delta y$$

we have

$$\begin{aligned}
&\sum_{r=t-T}^{t-1} c(r)\Delta x(r-g(r)) \prod_{s=r+1}^{t-1} a(s) \\
&= c(t-1)\prod_{s=t}^{t-1} a(s)x(t-g(t)) \\
&- c(t-T-1) \prod_{s=t-T}^{t-1} a(s)x(t-T-g(t-T)) \\
&- \sum_{r=t-T}^{t-1} x(r-g(r))\Delta\Big((c(r-1)\prod_{s=r}^{t-1} a(s))\Big) \\
&= c(t-1)x(t-g(t)) - c(t-1) \prod_{s=t-T}^{t-1} a(s)x(t-g(t)) \\
&- \sum_{r=t-T}^{t-1} x(r-g(r))\Delta\Big((c(r-1)\prod_{s=r}^{t-1} a(s))\Big)
\end{aligned}$$

$$= c(t-1)x(t-g(t))\Big[1 - \prod_{s=t-T}^{t-1} a(s)\Big]$$
$$- \sum_{r=t-T}^{t-1} x(r-g(r))\Delta\Big((c(r-1)\prod_{s=r}^{t-1} a(s))\Big).$$

Thus,

$$\sum_{r=t-T}^{t-1} c(r)\Delta x(r-g(r)) \prod_{s=r+1}^{t-1} a(s) = c(t-1)x(t-g(t))\Big[1 - \prod_{s=t-T}^{t-1} a(s)\Big]$$
$$- \sum_{r=t-T}^{t-1} x(r-g(r))\Delta\Big((c(r-1)\prod_{s=r}^{t-1} a(s))\Big). \tag{2.9}$$

Finally, substituting (2.9) into (2.8) completes the proof.
Now for $t \leq 0$, Equation (1.1) is equivalent to

$$\Delta\Big[\prod_{s=t}^{0} a(s)x(t)\Big]$$
$$= \Big[c(t)\Delta x(t-g(t)) + q(t, x(t), x(t-r(t)))\Big] \prod_{s=t+1}^{0} a(s).$$

Summing the above expression from $(t-T)$ to $(t-1)$ we obtain (1.1) by a similar argument.
Using (2.6) we define the mapping $H : P_T \to P_T$ by

$$(H\varphi)(t)$$
$$= c(t-1)\varphi(t-g(t))$$
$$+ \Big(1 - \prod_{s=t-T}^{t-1} a(s)\Big)^{-1} \sum_{r=t-T}^{t-1} \Big[\varphi(r-g(r))[a(r)c(r-1) - c(r)] \tag{2.10}$$
$$+ q(r, \varphi(r), \varphi(r-g(r)))\Big] \prod_{s=r+1}^{t-1} a(s).$$

Next we state Krasnoselskii's fixed point theorem which enables us to prove the existence of a periodic solution. But first, we define what we mean by a mapping being compact.
Let $\mathcal{S}$ be a subset of a Banach space $\mathcal{B}$ and $f : \mathcal{S} \to \mathcal{B}$. If f is continuous

and $f(\mathcal{S})$ is contained in a compact subset of $\mathcal{B}$, then f is a compact mapping. □

THEOREM 2.2 (Krasnoselskii). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A and B maps $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) *$x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,*
(ii) *A is compact and continuous,*
(iii) *B is a contraction mapping.*
Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

We note that to apply the above theorem we need to construct two mappings; one is contraction and the other is compact. Therefore, we express equation (2.10) as

$$(H\varphi)(t) = (B\varphi)(t) + (A\varphi)(t) \tag{2.11}$$

where $A, B : P_T \to P_T$ are given by

$$(B\varphi)(t) = c(t-1)\varphi(t - g(t)) \tag{2.12}$$

and

(2.13)
$$(A\varphi)(t) = \Big(1 - \prod_{s=t-T}^{t-1} a(s)\Big)^{-1} \sum_{r=t-T}^{t-1} \Big[\varphi(r - g(r))[a(r)c(r-1) - c(r)] + q(r, \varphi(r), \varphi(r - g(r)))\Big] \prod_{s=r+1}^{t-1} a(s).$$

LEMMA 2.3. *Suppose (2.1)-(2.4) hold. If A is defined by (2.13), then $A : P_T \to P_T$ and is compact.*

Proof. First we want to show that $(A\varphi)(t+T) = (A\varphi)(t)$. Let $\varphi \in P_T$. Then using (2.13) we arrive at

$$\begin{aligned}&(A\varphi)(t+T)\\ &= \Big[1 - \prod_{s=t}^{t+T-1} a(s)\Big]^{-1} \sum_{r=t}^{t+T-1} \Big[\varphi(r - g(r))[a(r)c(r-1) - c(r)]\\ &+ q(r, \varphi(r), \varphi(r - g(r)))\Big] \prod_{s=r+1}^{t+T-1} a(s).\end{aligned}$$

Let $j = r - T$, then

$$
\begin{aligned}
&(A\varphi)(t+T) \\
&= \Big[1 - \prod_{s=t}^{t+T-1} a(s)\Big]^{-1} \sum_{j=t-T}^{t-1} \Big[\varphi(j+T-g(j+T))[a(j+T)c(j+T-1) \\
&\quad - c(j+T)] + q(j+T, \varphi(j+T), \varphi(j+T-g(j+T)))\Big] \prod_{s=j+T+1}^{t+T-1} a(s) \\
&= \Big[1 - \prod_{s=t}^{t+T-1} a(s)\Big]^{-1} \sum_{j=t-T}^{t-1} \Big[\varphi(j-g(j))[a(j)c(j-1) - c(j)] \\
&\quad + q(j, \varphi(j), \varphi(j-g(j)))\Big] \prod_{s=j+T+1}^{t+T-1} a(s).
\end{aligned}
$$

Now let $k = s - T$, then

$$
\begin{aligned}
&(A\varphi)(t+T) \\
&= \Big[1 - \prod_{k=t-T}^{t-1} a(k)\Big]^{-1} \sum_{j=t-T}^{t-1} \Big[\varphi(j-g(j))[a(j)c(j-1) - c(j)] \\
&\quad + q(j, \varphi(j), \varphi(j-g(j)))\Big] \prod_{k=j+1}^{t-1} a(s) = (A\varphi)(t).
\end{aligned}
$$

To see that A is continuous, we let $\varphi, \psi \in P_T$ with $\|\varphi\| \le C$ and $\|\psi\| \le C$. Let

$$
\begin{aligned}
&\eta = \max_{t\in[0,T-1]} \left| \frac{1}{(1 - \prod_{s=t-T}^{t-1} a(s)} \right|, \\
(2.14)\qquad &\beta = \max_{r\in[t-T,t]} |a(r)c(r-1) - c(r)|,\ \gamma = \max_{t\in[0,T-1]} \prod_{s=t-T}^{t-1} a(s).
\end{aligned}
$$

Given $\epsilon > 0$, take $\delta = \epsilon/M$ such that $\|\varphi - \psi\| < \delta$, where $M = T\gamma\eta[\beta + L + K]$. By making use of (2.4) into (2.13) we obtain

$$\begin{aligned}
&\Big\|\Big(A\varphi(t)\Big) - \Big(A\psi(t)\Big)\Big\| \\
= \ & \Big\|\frac{1}{1-\prod_{s=t-T}^{t-1} a(s)} \sum_{r=t-T}^{t-1} \Big[\Big(\varphi(r-g(r)) - \psi(r-g(r))\Big) \\
& \Big(c(r-1)a(r) - c(r)\Big) + \Big(q(r,\varphi(r),\varphi(r-g(r))) \\
& -q(r,\psi(r),\psi(r-g(r)))\Big)\Big] \prod_{s=r+1}^{t-1} a(s)\Big\| \\
\leq \ & \eta \sum_{r=t-T}^{t-1} \Big[\|\varphi-\psi\|\beta + L\|\varphi-\psi\| + K\|\varphi-\psi\|\Big]\gamma \\
\leq \ & \gamma\eta \sum_{r=t-T}^{t-1} (\beta+L+K)\|\varphi-\psi\| \\
= \ & \eta\gamma T(\beta+L+K)\|\varphi-\psi\| \\
= \ & M\|\varphi-\psi\| = M\delta < \epsilon
\end{aligned}$$

where L and K are given by (2.4). This proves A is continuous. Next, we show that A maps bounded subsets into compact sets. Let J be given, $S = \{\varphi \in P_T : \| \varphi \| \leq J\}$ and $Q = \{(A\varphi)(t) : \varphi \in S\}$, then S is a subset of R^T which is closed and bounded thus compact. As A is continuous in φ it maps compact sets into compact sets. Therefore $Q = A(S)$ is compact. □

LEMMA 2.4. *If B is given by (2.11) and*

$$\Big\|c(t-1)\Big\| \leq \zeta < 1, \tag{2.15}$$

then B is a contraction.

Proof. Let B be defined by (2.12). Then for $\varphi, \psi \in P_T$ we have

$$\begin{aligned}
\|B(\varphi) - B(\psi)\| &= \max_{t\in[0,T-1]} |B(\varphi) - B(\psi)| \\
&= \max_{t\in[0,T-1]} \Big|c(t-1)\varphi(t-g(t)) - c(t-1)\psi(t-g(t)\Big| \\
&= \max_{t\in[0,T-1]} \Big|c(t-1)\Big|\Big|\varphi(t-g(t)) - \psi(t-g(t))\Big| \\
&\leq \zeta\|\varphi-\psi\||.
\end{aligned}$$

Hence B defines a contraction mapping with contraction constant ζ. □

THEOREM 2.5. *Let $\alpha = ||q(t,0,0)||$. Let η, β and γ be given by (2.14). Suppose (2.1)-(2.4) and (2.15) hold. Suppose there is a positive constant G such that all solutions $x(t)$ of (1.1), $x(t) \in P_T$ satisfy $|x(t)| \le G$, the inequality*

$$\left\{\zeta + \eta\gamma T(\beta + L + K)\right\}G + \eta\gamma T\alpha \le G \tag{2.16}$$

holds. Then equation(1.1) has a $T-$periodic solution.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : ||\varphi|| \le G\}$. Then Lemma 2.3 implies $A : \mathbb{M} \to P_T$ and A is compact and continuous. Also, from Lemma 2.4, the mapping B is a contraction and it is clear that $B : \mathbb{M} \to P_T$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $||A\phi + B\psi|| \le G$. Let $\varphi, \psi \in \mathbb{M}$ with $||\varphi||, ||\psi|| \le G$. Then from (2.11)-(2.12) and the fact that $|q(t,x,y)| \le L||x|| + K||y|| + \alpha$, we have

$$\begin{aligned}
&\left\|\Big(A\varphi(t)\Big) + \Big(B\psi(t)\Big)\right\| \\
&= \left\|\frac{1}{1 - \prod_{s=t-T}^{t-1}} \sum_{r=t-T}^{t-1} \Big[\varphi(r - g(r))\Big(c(r-1)a(r) - c(r)\Big)\right. \\
&\left. + q(r, \varphi(r), \varphi(r - g(r)))\Big] \prod_{s=r+1}^{t-1} a(s) + c(t-1)\psi(t - g(t))\right\| \\
&\le \eta\gamma \sum_{r=t-T}^{t-1} \Big[L||\varphi|| + K||\varphi|| + \beta||\varphi|| + \alpha\Big] + \zeta||\psi|| \\
&\le \eta\gamma[(\beta + L + K)||\varphi|| + \alpha]T + \zeta||\psi|| \\
&\le \eta\gamma T(\beta + L + K)G + \eta\gamma T\alpha + G\zeta \\
&= \left\{\zeta + \eta\gamma T(\beta + L + K)\right\}G + \eta\gamma T\alpha \\
&\le G.
\end{aligned}$$

□

We see that all the conditions of Krasnoselskii's theorem are satisfied on the set $\mathbb{M}$. Thus there exists a fixed point z in $\mathbb{M}$ such that $z = Az + Bz$. By Lemma 2.1, this fixed point is a solution of (1.1). Hence (1.1) has a $T-$periodic solution.

REMARK: The constant G of Theorem 2.5 serves as a priori bound on all possible T-periodic solutions of equation (1.1)

THEOREM 2.6. *Suppose (2.1)-(2.4) and (2.15) hold. Let η, β and γ be given by (2.14). If*

$$\zeta + T\gamma\eta\Big(\beta + L + K\Big) \leq \nu < 1,$$

then equation (1.1) has a unique $T-$periodic solution.

Proof. Let the mapping H be given by (2.10). For $\varphi, \psi \in P_T$, in view of (2.10), we have

$$\begin{aligned}
&\Big\|\Big(H\varphi(t)\Big) - \Big(H\psi(t)\Big)\Big\| \\
&= \Big\|\Big(B\varphi(t)\Big) + \Big(A\varphi(t)\Big) - \Big(B\psi(t)\Big) - \Big(A\psi(t)\Big)\Big\| \\
&= \Big\|\Big(\Big(B\varphi(t)\Big) - \Big(B\psi(t)\Big)\Big) + \Big(\Big(A\varphi(t)\Big) - \Big(A\psi(t)\Big)\Big)\Big\| \\
&\leq \Big\|\Big(B\varphi(t)\Big) - \Big(B\psi(t)\Big)\Big\| + \Big\|\Big(A\varphi(t)\Big) - \Big(A\psi(t)\Big)\Big\| \\
&\leq \zeta\|\varphi - \psi\|| + \gamma\eta \sum_{r=t-T}^{t-1} \Big[L\|\varphi - \psi\| + K\|\varphi - \psi\| + \beta\|\varphi - \psi\|\Big] \\
&\leq \Big[\zeta + T\gamma\eta\Big(\beta + L + K\Big)\Big]\|\varphi - \psi\| \\
&< \nu\|\varphi - \psi\|
\end{aligned}$$

By the contraction mapping principle, (1.1) has a unique $T-$periodic solution. □

3. Example

Consider equation (1.1) along with conditions (2.1)-(2.4) and (2.13)-(2.14). Suppose that $a(t) \neq 1$ for all $t \in [0, T-1]$. Set

$$\rho = \min_{t\in[0,T-1]} |a(t) - 1|, \quad \delta = \max_{t\in[0,T-1]} k(t),$$

where $k(t) = c(t) - c(t-1)$.
Suppose $1 - ||c|| > 0$. If

$$\rho(1 - ||c||) > (1 - ||c||)(\delta + L + K) + T\rho(||a - 1|| + L + K)$$

holds, and G is defined by

$$G = \frac{\alpha(1 - ||c|| + T\rho}{\rho(1 - ||c||) - (1 - ||c||)(\delta + L + K) - T\rho(||a - 1|| + L + K)}$$

satisfies inequality (2.16), then (1.1) has a $T-$periodic solution.

Proof. We rewrite (1.1) as

$$(3.1)\quad \Delta x(t) = \Big(a(t) - 1\Big)x(t) + c(t)\Delta x(t - g(t)) + q\big(t, x(t), x(t - g(t)\big)$$

Let the mappings A and B be defined by (2.13) and (2.12), respectively. Let $x(t) \in P_T$. A summation of equation (1.1) from 0 to $T - 1$ gives

$$\sum_{s=0}^{T-1} \Delta x(s)$$
$$= \sum_{s=0}^{T-1} \Big[(a(s) - 1)x(s) + c(s)\Delta x(s - g(s)) + q\big(s, x(s), x(s - g(s))\big)\Big]$$
$$x(T) - x(0)$$
$$= \sum_{s=0}^{T-1} \Big[(a(s) - 1)x(s) + c(s)\Delta x(s - g(s)) + q\big(s, x(s), x(s - g(s))\big)\Big]$$

Since $x(t) \in P_T$, $x(T) = x(0)$. Therefore
(3.2)
$$0$$
$$= \sum_{s=0}^{T-1} \Big[(a(s) - 1)x(s) + c(s)\Delta x(s - g(s)) + q\big(s, x(s), x(s - g(s))\big)\Big].$$

Rewrite and then sum by parts, using the summation by parts formula

$$\sum Ey\Delta z = yz - \sum z\Delta y$$

with $Ey(s) = c(s)$ and $z = x(s - g(s))$. As a consequence, we have

$$\sum_{s=0}^{T-1} c(s)\Delta x(s - g(s))$$
$$= c(s - 1)x(s - g(s))\Big|_{s=0}^{T} - \sum_{s=0}^{T-1} x(s - g(s))\Delta c(s - 1)$$

$$= c(T-1)x(T-g(T)) - c(-1)x(0-g(0))$$
$$-\sum_{s=0}^{T-1} x(s-g(s))[c(s)-c(s-1)]$$
$$= -\sum_{s=0}^{T-1} x(s-g(s))[c(s)-c(s-1)].$$

As a result (3.2) becomes

$$\sum_{s=0}^{T-1}[a(s)-1]x(s)$$
$$= \sum_{s=0}^{T-1} x(s-g(s))[c(s)-c(s-1)] - q\big(s, x(s), x(s-g(s))\big).$$

Let $S = \sum_{s=0}^{T-1}\Big|a(s)-1\Big|\Big|x(s)\Big|$. We claim that there exists a $t^* \in [0, T-1]$such that

$$T\Big|a(t^*)-1\Big|\Big|x(t^*)\Big| \leq \sum_{s=0}^{T-1}\Big|a(s)-1\Big|\Big|x(s)\Big|.$$

Suppose such t^* does not exist. Then

$$T\Big|a(t^*)-1\Big|\Big|x(t^*)\Big| > S,$$

which implies that

$$T\Big|a(t^*)-1\Big|\Big|x(t^*)\Big| > S+\epsilon.$$

or

$$\sum_{t^*=0}^{T-1}\Big|a(t^*)-1\Big|\Big|x(t^*)\Big| > \sum_{t^*=0}^{T-1}\frac{S+\epsilon}{T}.$$

Hence, $S > S+\epsilon$, which is a contradiction. Therefore, such t^* exists.

From (3.3), it implies that there exists a $t^* \in (0, T-1)$ such that

$$T\Big|a(t^*)-1\Big|\Big|x(t^*)\Big| \leq \sum_{s=0}^{T-1}\Big|k(t)\Big|\Big|x(s-g(s))\Big| + \Big|q\big(s, x(s), x(s-g(s))\big)\Big|$$

By taking the maximum over $t \in [0, T-1]$, we obtain from the above inequality

$$\begin{aligned} T\rho||x(t^*)|| &\leq \sum_{s=0}^{T-1} \Big(\delta||x|| + L||x|| + E||x|| + \alpha\Big) \\ &= \sum_{s=0}^{T-1} \Big((\delta + L + E)||x|| + \alpha\Big) \\ &= T\Big((\delta + L + E)||x|| + \alpha\Big), \end{aligned}$$

which gives us

$$||x(t^*)|| \leq \frac{1}{\rho}(\delta + L + K)||x|| + \frac{\alpha}{\rho}. \tag{3.3}$$

Since for all $t \in [0, T-1]$

$$x(t) = x(t^*) + \sum_{s=t^*}^{t-1} \Delta x(s),$$

taking maximum over $t \in [0, T-1]$ and using

$$||x(t)|| \leq ||x(t^*)|| + \sum_{s=0}^{T-1} |\Delta x(s)|$$

yields

$$||x(t)|| \leq \frac{1}{\rho}(\delta + L + E)||x|| + \frac{\alpha}{\rho} + T||\Delta x||. \tag{3.4}$$

Taking the norm in (1.1) yields

$$||\Delta x(t)|| \leq ||a-1|| \, ||x|| + ||c|| \, ||\Delta x|| + K||x|| + L||x|| + \alpha.$$

or

$$\Big(1 - ||c||\Big)||\Delta x(t)|| \leq \Big(||a-1|| + E + L\Big)||x|| + \alpha.$$

Thus

$$||\Delta x(t)|| \leq \frac{\Big(||a-1|| + E + L\Big)||x|| + \alpha}{1 - ||c||}. \tag{3.5}$$

A substitution of (3.6) into (3.5) yields

$$||x(t)|| \leq \frac{1}{\rho}(\delta + L + K)||x|| + \frac{\alpha}{\rho} + T\frac{\Big(||a-1|| + K + L\Big)||x|| + \alpha}{1 - ||c||}.$$

Hence

$$||x(t)|| \leq \frac{\alpha(1 - ||c|| + T\rho)}{\rho(1 - ||c||) - (1 - ||c||)(\delta + L + E) - T\rho(||a - 1|| + L + E)} = G.$$

Thus, for all $x(t) \in P_T$ we have shown that

$$||x(t)|| \leq G.$$

Define $\mathbb{M} = \{\varphi \in P_T : ||\varphi|| \leq G\}$. Then by Theorem 2.5, equation (1.1) has a $T-$periodic solution. This completes the proof. □

References

[1] W. G. Kelly and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, 2001.

[2] Y. N. Raffoul, *Periodic Solutions for Scalar and Vector Nonlinear Difference Equations*, Panamer. Math. J. **9** (1999), no. 1, 97–111.

[3] ———, *Periodic Solutions in Nonlinear Differential Equations with Functional Delay*, Electron. J. of Differential Equations and Applications. 2003, no. 102, 1–7.

[4] ———, *Positive Periodic Solutions of Functional Discrete Systems and Population Models*, In Review.

[5] ———, *T-Periodic Solutions and a priori Bounds*, Math. Comput. Modeling **32** (2000), 643–652.

Mariette R. Maroun
Department of Mathematics
Baylor University
P.O. Box 97328, Waco, Texas 76798-7328, USA
E-mail: Mariette-Maroun@baylor.edu

Youssef N. Raffoul
Department of Mathematics
University of Dayton
Dayton, OH 45469-2316, USA
E-mail: youssef.raffoul@notes.udayton.edu