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A MEAN CONDITION ON FORCING TERM FOR MULTIPLICITY OF PERIODIC SOLUTIONS FOR NONLINEAR DISSIPATIVE HYPERBOLIC EQUATIONS

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Abstract. A condition on forcing term insuring the multiplicity of Dirichlet-periodic solutions of nonlinear dissipative hyperbolic equations is discussed. The nonlinear term is assumed to have coercive growth.

1. Introuction

Let R be the set of all reals and $\Omega \subseteq R^n$, $n \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$ which is assumed to be of class C^2.

Let $Q = (0, 2\pi) \times \Omega$ and $L^2(Q)$ be the space of measurable and Lebesgue square integrable real-valued functions on Q with usual inner product $< \cdot, \cdot >$ and corresponding norm $\| \cdot \|_2$.

By $H_0^1(\Omega)$ we mean the completion of $C_0^1(\Omega)$ with respect to the norm $\| \cdot \|_1$ defined by

$$\|\varphi\|_1^2 = \int_\Omega \sum_{|\alpha| \leq 1} |D^\alpha \varphi(x)|^2 dx.$$

$H^2(\Omega)$ stands for the usual Sovolev space ; i.e., the completion of $C^2(\bar{\Omega})$ with respect to the norm $\| \cdot \|_2$ defined by

$$\|\varphi\|_2^2 = \int_\Omega \sum_{|\alpha| \leq 2} |D^\alpha \varphi(x)|^2 dx.$$

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We denote by $\lambda_1 < \lambda_2 \le \cdots$ the eigenvalues of the problem

$$-\Delta_x u = \lambda u, \quad u \in H_0^1(\Omega)$$

and ϕ_1 the positive normalized eigenfunction corresponding to λ_1.

The purpose of this work is to investigate a condition on forcing term insuring multiplicity of periodic solutions of the nonlinear dissipative hyperbolic equations of the form

$$(E) \qquad \frac{\partial^2 u}{\partial t^2} - \Delta_x u + \beta \frac{\partial u}{\partial t} - \lambda_1 u + g(u) = h(t,x) \text{ in } Q,$$

$$(B_1) \qquad u(t,x) = 0 \text{ on } (0, 2\pi) \times \partial\Omega,$$

$$(B_2) \qquad u(0,x) = u(2\pi, x) \text{ on } \Omega$$

where $\beta(\neq 0) \in R$, $g : R \to R$ is a continuous nonlinear function with coercive growth, and with a restriction on the left-hand side growth by the difference of λ_2 and λ_1 and $h \in L^2(Q)$ is 2π-periodic with respect to t.

Moreover, we assume that there exist constants $a_0 > 0$ and $b_0 \ge 0$ such that

$$(H_1) \qquad |g(u)| \le a_0|u| + b_0 \quad \text{for all } u \in R.$$

Dissipative hyperbolic equations of the form (E) are related to the mathematical interpretation of undistorted plane waves. A more general interpretation is given by the telegraph equations of the form in one space dimension

$$u_{tt} - c^2 \Delta_x u + (\alpha + \beta) u_t + \alpha\beta u = 0,$$

where c is the speed of light and α the capacity and β the inductive damping factor, satisfied by the voltage or the current u as a function of time t and the position x along a cable, here x measures the length of the cable from an initial point. For the existence of a generalized solution, Mawhin[18] give a solvabality of the doubly-periodic problem to the equation of the form

$$(1.1) \qquad u_{tt} - u_{xx} + \beta u_t + g(u) = h(t,x), \ (t,x) \in R^2,$$

where $\beta \neq 0$, $u = u(t,x)$ and g is a continuous function on R of at most linear growth. He obtain results for resonance and nonresonance situations related to the eigenvalues of the linear differential operator $u_{tt} - u_{xx} + \beta u_t$ for the doubly-periodic problem. The results do not depend upon the size of $|\beta|$ and non-vanishing of this number bing only assumed to insure the compactness properties required by a treatment using topological degree. The results are more in sprit of work on ordinary and elliptic partial differential equations. Mawhin[18], and Fucik and Mawhin[4] deal with the existence of a periodic solution in both variables to the equations of the form

$$(1.2)\qquad u_{tt} - u_{xx} + \beta u_t - \mu u^+ + \nu u^- + g(u) = h(t,x),\ (t,x) \in R^2,$$

where β, μ, $\nu \in R$, $\beta \neq 0$, $u = u(t,x)$ and g is a continuous bounded function. The authors decompose the set R^2 of all pairs (μ,ν) (see [21]) and obtain existence results of different types on different component. For the existence of a solution for the doubly-periodic problem for the equations of the form (1.1) whose nonlinearity grow exponentially, we refer Kim[11]. The author impose a condition on the mean of source term to apply continuation theorem(see [5]). In [8] and [17], the existence of a solution for the Dirichlet-periodic resonance problem for (1.1) are treated. The authors imposed on the source term the orthogonal condition and the Landesman-Lazer condition, respectively, with the first eigenfunction of the linear differential operator to prove the resonance problem. In [10], the existence of a solution for the Dirichlet-periodic problem for the equation of the form (1.1) is discussed when the rate of growth allowed in the nonlinear term g is any polynomial growth. The only condition on the nonlinear term ia a weak type of sign condition which is a strong tool in the proof of the result. Any restriction on the forcing term is not needed except that term is in L^2(cf. [9]). The author use Leray-Schauder theory to prove the result. Brezis and Nirenberg[3] apply their general results on the ranges of nonlinear operators to the Dirichlet-periodic problem for the equations of the form

$$(1.3)\qquad u_{tt} + Eu + \beta u_t + g(t,x,u) = h(t,x),\ (t,x) \in R \times \Omega,$$

where $\beta \neq 0$, $u = u(t,x)$ and

$$Ev(x) = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta v(x)).$$

The function g is supposed to be continuous with sublinear growth in u, satisfies a sign condition and Landesman-Lazer type conditions. The authors prove the existence of a solution with some additional conditions related to the kernel of the operator E and the regularity results are also stated. In [20], Nkashama and Willem treat the existence of a solution for the Dirichlet-periodic problem for (1.3) whose nonlinearity g is jumping over two conesecutive eigenvalues. They prove results by using coincidence degree. The readers can refer to the book[22] for a survey of boundary value problems for dissipative hyperbolic equations.

But few seems to be known about the multiple existence of generalized solutions of Dirichlet-periodic solutions of the form (E) with a mean condition on the forcing term from which is motivated the coercivity of the nonlinear term(cf. [1], [6], [7], [13], [15], [19]). In [12] and [14]. The multiple existence of doubly-periodic solutions of the equations of the form (1.1) and Dirichlet-periodic solutions of the equations of the form (E) are treated, respectively(cf. [16]). These results are related to Ambrosetti-prodi[2] who initiate so call Ambrosetti-Prodi type multiplicity in the study of the Dirichlet problem to elliptic equations. In our result, we discuss a mean condition on the forcing term insuring the multiple existnce of solutions. We impose the coercive growth condition on nonlinear term. We take advantage of the properties of the first eigenfunction ϕ_1 in the proofs of our results. Our proof of the main theorem is based on Mawhin's continuation theorem(see [5]).

2. Preliminary results

Let's define the linear operator

$$L : \mathrm{Dom}L \subseteq L^2(Q) \to L^2(Q)$$

by

$$\mathrm{Dom}L = \{u \in L^2((0,2\pi), H^2(\Omega) \cap H_0^1(\Omega)) \mid \frac{\partial u}{\partial t} \in L^2(Q), \frac{\partial^2 u}{\partial t^2} \in L^2(Q), \\ u(0,x) = u(2\pi, x), x \in \Omega\}$$

and

$$Lu = \frac{\partial^2 u}{\partial t^2} - \triangle_x u + \beta \frac{\partial u}{\partial t} - \lambda_1 u.$$

By Fourier series and Parseval inequality, we have easily

$$< Lu, \frac{\partial u}{\partial t} >= \beta \| \frac{\partial u}{\partial t} \|_{L^2}^2 \quad \text{for all} \quad u \in \mathrm{Dom}L.$$

Hence $\ker L = \ker(\triangle_x + \lambda_1 I) = \ker L^*$ since $\triangle_x + \lambda_1 I$ is self-adjoint and $\ker(\triangle_x + \lambda_1 I)$ is one space dimension generated by the eigenfunction ϕ_1. Therefore L is a closed, densely defined linear operator and $\mathrm{Im}(L) = [\ker L]^{\perp}$; i.e., $L^2(Q) = \ker L \bigoplus \mathrm{Im}L$.

Let us consider a continous projection $P_1 : L^2(Q) \to L^2(Q)$ such that $\mathrm{Im}P_1 = \mathrm{Ker}L$. Then $L^2(Q) = \ker L \bigoplus \ker P_1$. We consider another continuous projection $P_2 : L^2(Q) \to L^2(Q)$ defined by

$$(P_2 h)(t,x) = \frac{1}{2\pi} \iint_Q h(t,x)\phi_1(x)dtdx\phi_1(x).$$

Then we have $L^2(Q) = \mathrm{Im}P_2 \bigoplus \mathrm{Im}L$, $\ker P_2 = \mathrm{Im}L$, and $L^2(Q)/\mathrm{Im}L$ is isomorphic to $\mathrm{Im}P_2$.

Since $\dim[L^2(Q)/\mathrm{Im}L] = \dim[\mathrm{Im}P_2] = \dim[\ker L] = 1$, we have an isomorphism $J : \mathrm{Im}P_2 \to \ker L$.

By the closed graph theorm, the generalized right inverse of L defined by

$$K = [L|_{\mathrm{Dom}L\cap \mathrm{Im}L}]^{-1} : \mathrm{Im}L \to \mathrm{Im}L$$

is continuous.

If we equip the space $\mathrm{Dom}L$ with the norm

$$\|u\|^2_{\mathrm{Dom}L} = \iint_Q [u^2 + (\frac{\partial u}{\partial t})^2 + (\frac{\partial^2 u}{\partial t^2})^2 + \sum_{|\beta|\le 2} (D^\beta_x u)^2] dtdx.$$

Then there exist a constant $c > 0$ independent of $h \in \mathrm{Im}L$, $u = Kh$ such that

$$\|Kh\|_{\mathrm{Dom}L} \le c\|h\|_{L^2}.$$

Therefore $K : \mathrm{Im}L \to \mathrm{Im}L$ is continuous and by the compact imbedding of $\mathrm{Dom}L$ in $L^2(Q)$, we have that $K : \mathrm{Im}L \to \mathrm{Im}L$ is compact

LEMMA 2.1. *L is a closed, densely defined linear operator such that $\ker L = [\mathrm{Im}L]^\perp$ and such that the right inverse $K : \mathrm{Im}L \to \mathrm{Im}L$ is completely continuous.*

Proof. See [3] and [20]. □

3. Multiplicity results

Let us consider the following

$$(E_\mu) \qquad \frac{\partial^2 u}{\partial t^2} - \triangle_x u + \beta\frac{\partial u}{\partial t} - \lambda_1 u + \mu g(u) = \mu h(t,x) \ \text{ in } \ Q,$$

$$(B_1) \qquad u(t,x) = 0 \ \text{ on } \ (0,2\pi)\times\partial\Omega,$$

$$(B_2) \qquad u(0,x) = u(2\pi,x) \ \text{ on } \ \Omega,$$

where $\mu \in [0,1]$.

Let $L : \mathrm{Dom}L \subseteq L^2(Q) \to L^2(Q)$ be defined as before. If we define a sustitution operator $N : L^2(Q) \to L^2(Q)$ by

$$(Nu)(t,x) = g(u(t,x)) - h(t,x)$$

for $u \in L^2(Q)$ and $(t,x) \in Q$, then N maps $L^2(Q)$ continuously into itself and transforms bounded sets into bounded sets. Let G be any open bounded subset of $L^2(Q)$, then $P_2N : \bar{G} \to L^2(Q)$ is bounded and $K(I-P_2):\bar{G}\to L^2(Q)$ is compact and continuous. Thus N is L-compact on $\bar{G}$ (see [5]).

The coincidence degree $D_L(L+\mu N,G)$ is well defined and constant in μ if $Lu+\mu Nu\neq 0$ for $\mu\in[0,1]$ and $u\in \mathrm{Dom}L\cap\partial G$. It is easy to check that (u,μ) is a weak solution of $(E_\mu)(B_1)(B_2)$ if and only if $u\in\mathrm{Dom}L$ and

$$Lu+\mu Nu=0. \tag{3.1$_\mu$}$$

Here, we assume the following

$$\liminf_{|u|\to\infty} g(u)=+\infty, \tag{H_2}$$

$$m=\inf_{u\in R} g(u)>0. \tag{H_3}$$

Moreover, we assume that

$$\limsup_{u\to-\infty}|\frac{g(u)}{u}|<\lambda_2-\lambda_1. \tag{H_4}$$

Then it follows from (H_4) that there exists $a\in(0,\lambda_2-\lambda_1)$ and $b\geq 0$ satisfying

$$|g(u)|\leq a|u|+b \quad \text{for all } u\leq 0.$$

For $h\in L^2(Q)$, we write $\wedge h=\iint_Q h(t,x)\phi_1(x)dtdx$.

LEMMA 3.1. *Let $c>0$ be given. Then for every $\delta>0$ there exists a constant $K(\delta)>0$ such that, for every $\tilde{u}\in\mathrm{Im}L$, $\alpha\in R$, the following implication holds.*

$$\iint_Q g(\alpha\phi_1+\tilde{u})\phi_1 dtdx\leq c \;\Rightarrow\; |\alpha|\leq\delta||\tilde{u}||_{L^2}+K(\delta). \tag{3.2}$$

Proof. Let $\delta>0$ be arbitrarily given, and put

$$r:=\sqrt{2\pi+\frac{1}{\delta^2}}-\frac{1}{\delta}. \tag{3.3}$$

We have chosen ϕ_1 to satisfy the condition

$$\int_\Omega \phi_1^2(x)dx=1, \tag{3.4}$$

and find $\eta>0$ such that for every subset $M\subseteq Q$ we have the following

$$|M|<\eta \;\Rightarrow\; \iint_M \phi_1^2(x)dtdx<r^2, \tag{3.5}$$

where $|\cdot|$ denotes the Lebesgue measure.

Let M_ξ be the set $\{x \in \Omega \mid \phi_1(x) < \xi\}$. We fix $\xi > 0$ such that $|M_\xi| < \eta/(4\pi)$. Next, from (H_2) we can choose sufficiently large $\nu > 0$ such that

$$|u| > \nu \;\Rightarrow\; g(u) > \frac{2c}{\xi\eta}.$$

We define the sets $A = \{(t,x) \in Q \setminus ((0,2\pi) \times M_\xi) \mid |\alpha\phi_1 + \tilde{u}| \geq \nu\}$, $B = A \cup ((0,2\pi) \times M_\xi)$. Then we have

$$c \geq \iint_A g(\alpha\phi_1 + \tilde{u})\phi_1 dtdx \geq \frac{2c}{\eta}|A|,$$

and hence

$$|A| \leq \eta/2, \quad |B| \leq \eta. \tag{3.6}$$

On the other hand, we have

$$\begin{aligned} 0 &= 2\iint_Q \alpha\phi_1\tilde{u}dtdx \\ &\leq \iint_{Q\setminus B}(|\alpha\phi_1 + \tilde{u}|^2 - |\alpha\phi_1|^2)dtdx + 2\iint_B |\alpha\phi_1||\tilde{u}|dtdx \end{aligned} \tag{3.7}$$

and, from (3.4), it follows that

$$2\pi\alpha^2 = \iint_{Q\setminus B}|\alpha\phi_1|^2 dtdx + \iint_B |\alpha\phi_1|^2 dtdx.$$

Using (3.5), (3.6), (3.7) and the definition of the set B, we obtain that

$$\begin{aligned} 2\pi\alpha^2 &\leq \iint_{Q\setminus B}|\alpha\phi_1 + \tilde{u}|^2 dtdx + 2\iint_B |\alpha\phi_1||\tilde{u}|dtdx + \iint_B |\alpha\phi_1|^2 dtdx \\ &\leq |Q|\nu^2 + 2|\alpha|r||\tilde{u}||_{L^2} + \alpha^2 r^2. \end{aligned}$$

This yields that

$$\alpha^2 \leq \frac{2|\alpha|r}{2\pi - r^2}||\tilde{u}||_{L^2} + \frac{|Q|\nu^2}{2\pi - r^2}.$$

We conclude that

$$|\alpha| \leq \frac{2r}{2\pi - r^2}||\tilde{u}||_{L^2} + \nu\sqrt{\frac{|Q|}{2\pi - r^2}}. \tag{3.8}$$

Hence, (3.2) holds as a consequence of (3.3), (3.8), and the proof is complete. □

From Lemma 3.1, we have the following.

LEMMA 3.2. *Let $h \in L^2(Q)$ be given such that $\wedge h > 0$ and suppose hypotheses $(H_1) - (H_4)$ hold.*

Then there exist constsnts $\bar{\gamma} = \bar{\gamma}(h) > 0$, $\tilde{M} = \tilde{M}(h) > 0$ *such that, for every* $\mu \in [0,1]$, *every solution of problem* (3.1_μ) *of the form* $u = \alpha\phi + \tilde{u}$ *with* $\tilde{u} \in \mathrm{Im}L$ *and* $\alpha \in R$, *we have*

$$|\alpha| \le \bar{\gamma}, \ ||\tilde{u}||_{L^2} \le \tilde{M}.$$

Proof. Let us take inner product of (3.1_μ) with $\frac{\partial u}{\partial t}$, then from the periodicity of u in t, we have

$$||\frac{\partial u}{\partial t}||_{L^2} \le \frac{1}{|\beta|}||h||_{L^2}.$$

Again, take inner product of (3.1_μ) with u. This yields that

$$(\lambda_2 - \lambda_1)||\tilde{u}||^2_{L^2} - ||\frac{\partial u}{\partial t}||^2_{L^2} + \mu \iint_Q g(u)udtdx \le \mu \iint_Q |h||u|dtdx.$$

By hypotheses $(H_1) - (H_4)$, we have that there exists a constsnt $C_1 > 0$ such that $g(u)u \ge -a|u|^2 - C_1$ for every $u \in R$. There exists therefore a constant $C_2 > 0$ such that

$$(\lambda_2 - \lambda_1 - a)||\tilde{u}||^2_{L^2} \le C_2(1+\alpha^2) + ||h||_{L^2}||u||_{L^2}\ .$$

Hence we have

$$||\tilde{u}||^2_{L^2} \le C_3(1+\alpha^2) \tag{3.9}$$

for some constant $C_3 > 0$ independent of μ .

Taking inner product of (3.1_μ) with ϕ_1 and using the fact that $\mu > 0$, we obtain that

$$\iint_Q g(\alpha\phi_1 + \tilde{u})\phi_1 dtdx = \wedge h.$$

By Lemma 3,1 with $c = \wedge h$ and $\delta = 1/(2\sqrt{C_3})$, we find a constant $C_4 > 0$ such that

$$|\alpha| \le \frac{1}{2\sqrt{C_3}}||\tilde{u}||_{L^2} + C_4. \tag{3.10}$$

Combining (3.10) with (3.9), we obtain that $|\alpha| \le (1+|\alpha|)/2 + C_4$, hence $|\alpha| \le \ \bar{\gamma}$, $||\tilde{u}||_{L^2} \le \tilde{M}$ for some constant $\bar{\gamma} > 0$, $\tilde{M} > 0$, and the proof is complete. □

Remark. Let $h \in L^2(Q)$, then, by hypothesis (H_2) and Lemma 3.2, we have a number $\tilde{\gamma}(h) > \bar{\gamma}$ such that

$$\iint_Q g(\alpha\phi_1(x) + \tilde{u})\phi(x)dtdx > \iint_Q h\phi_1(x)dtdx \tag{3.11}$$

for $|\alpha| \ge \tilde{\gamma}$.

LEMMA 3.3. *Let $h \in L^2(Q)$ be given such that $\wedge h > 0$ and let hypotheses $(H_1) - (H_4)$ hold. Suppose $G(h) \subseteq L^2(Q)$ be a bounded open set such that the implication*

$$|\alpha| < \tilde{\gamma}_0,\ ||\tilde{u}||_{L^2} < \tilde{M}_0 \Rightarrow u = \alpha\phi + \tilde{u} \in G(h)$$

holds where $\tilde{\gamma}_0$ and $\tilde{M}_0$ are numbers such that $\tilde{\gamma}_0 > \tilde{\gamma}$ and $\tilde{M}_0 > \tilde{M}$.

Then, for any bounded open set G in $L^2(Q)$ such that $G(h) \subsetneq G$, we have $D_L(L+N, G) = 0$.

Proof. By Lemma 3.2, the equation

$$Lu + g(u) = \nu h \tag{3.12}$$

has no solution on ∂G for $\nu \in [0,1]$. Hence, we may apply the homotopy argument with respect to ν. Taking inner product of (3.12) with ϕ_1, we obtain, from hypothesis (H_3), that

$$\nu \wedge h = \iint_Q g(u)\phi_1 dtdx \geq 2\pi m \int_\Omega \phi_1 dtdx > 0.$$

Hence, (3.12) has no solution for $\nu = 0$. Therefore, we have $D_L(L + N, G) = D_L(L+g, G) = 0$, and the proof is complete. □

Now, let $\tilde{\gamma}$, $\tilde{M}$ be defined in Lemma 3.2 and Remark, and $h \in L^2(\Omega)$ with $\wedge h > 0$. Let $\alpha_0 \in [-\tilde{\gamma}, \tilde{\gamma}]$, $x_0 \in \bar{\Omega}$, $\tilde{u}_0 \in [-\tilde{M}, \tilde{M}]$ be such that

$$g(\alpha_0\phi_1(x_0) + \tilde{u}_0) = \min_{[x\in\bar{\Omega}, |\alpha|\leq\tilde{\gamma}, |\tilde{u}|\leq\tilde{M}]} g(\alpha\phi_1(x) + \tilde{u}).$$

THEOREM . *Let hypotheses $(H_1) - (H_4)$ hold and suppose that $h \in L^2(Q)$ and there exists a constant $C > 0$ be such that*

$$\iint_Q g(\alpha_0\phi_1(x) + \tilde{u}(t,x))\phi_1(x)dtdx < C \tag{3.13}$$

for all $\tilde{u} \in L^2$ with $\wedge\tilde{u} = 0$, $||\tilde{u}||_{L^2} \leq \tilde{M}_0$ and satisfying the conditions (B_1) and (B_2).

Then boundary value problem $(E), (B_1), (B_2)$ has at least two solutions if

$$C < \iint_Q h\phi_1(x)dtdx. \tag{3.14}$$

Proof. Let $\Delta G(h) \subsetneq G$ be an open bounded set such that the following implication

$$\alpha_0 < \alpha < \tilde{\gamma}_0,\ \|\tilde{u}\|_{L^2} < \tilde{M}_0 \Rightarrow u = \alpha\phi + \tilde{u} \in \Delta G(h)$$

holds and such that if $u \in \partial\Delta G(h)$, then necessary $u = \alpha_0\phi_1(x) + \tilde{u}$ or $u = \tilde{\gamma}_0\phi_1(x) + \tilde{u}$ with $\|\tilde{u}\|_{L^2} \leq \tilde{M}_0$. If $u = \alpha_0\phi_1(x) + \tilde{u}$ with $\|\tilde{u}\|_{L^2} \leq \tilde{M}_0$, then, by taking inner product (3.1_μ) with ϕ_1, we have

$$\iint_Q \mu g(\alpha_0\phi_1(x) + \tilde{u}(t,x))\phi_1(x)dtdx = \mu \iint_Q h\phi_1(x)dtdx$$

which, from (3.13) and (3.14), is impossible. If $u = \tilde{\gamma}_0\phi_1(x) + \tilde{u}$ with $\|\tilde{u}\|_{L^2} \leq \tilde{M}_0$, then, by (3.11), we have

$$\iint_Q g(\tilde{\gamma}_0\phi_1(x) + \tilde{u})\phi_1(x)dtdx > \iint_Q h\phi_1(x)dtdx$$

which is also impossible. Thus $D_L(L + N, \Delta G(h))$ is well defined and

$$D_L(L + N, \Delta G(h)) = D_B(JP_2N, \Delta G(h) \cap \ker L, 0),$$

where D_B is Brouwer degree and $P_2N : L^2(Q) \to \ker L$ is an operator defined by

$$(P_2Nu)(t,x) = \frac{1}{2\pi}[\iint_Q g(u(t,x))\phi_1(x)dtdx - \iint_Q h\phi_1(x)dtdx]\phi_1(x).$$

Now, let $T : \ker L \to R$ be defined by

$$T(\alpha\phi_1(x)) = \alpha.$$

Then

$$\begin{aligned} D_L(L + N, \Delta G(h) &= D_B(JP_2N, \Delta G(h) \cap \ker L, 0) \\ &= D_B(T(JP_2N)T^{-1}, T(\Delta G(h) \cap \ker L), 0). \end{aligned}$$

If we let $J : \mathrm{Im}P_2 \to \ker L$ be the identity map, then the operator $\Phi = 2\pi T(JP_2N)T^{-1}$ will be defined by

$$\Phi(\alpha) = \iint_Q g(\alpha\phi_1(x))\phi_1(x)dtdx - \iint_Q h\phi_1(x)dtdx.$$

Thus, we have, by (3.13) and (3.14),

$$\Phi(\alpha_0) = \iint_Q g(\alpha_0\phi_1(x))\phi_1(x)dtdx - \iint_Q h\phi_1(x)dtdx < 0,$$

and, by (3.11), we have

$$\Phi(\tilde{\gamma}_0) = \iint_Q g(\tilde{\gamma}_0\phi_1(x))\phi_1(x)dtdx - \iint_Q h\phi_1(x)dtdx > 0.$$

Hence, the coincidence degree exists and the corresponding value

$$|D_L(L+N,\Delta G(h))| = |D_B[JP_2N,\Delta G(h)\cap \mathrm{Ker}L,0]| = 1.$$

By Mawhin's continuation theorem, the problem $(E),(B_1),(B_2)$ has at least one solution in $\Delta G(h)$.

By the additivity of the coincidence degree, we have

$$0 = D_L(L+N,G) = D_L(L+N,\Delta G(h))) + D_L(L+N,G-\overline{\Delta G(h)})$$

and hence

$$|D_L(L+N,G-\overline{\Delta G(h)})| = 1.$$

Therefore, $(E),(B_1),(B_2)$ has another solution in $G-\overline{\Delta G(h)}$. This proves our assertion. □

REMARK. If

$$\frac{1}{|\Omega|}\iint_Q h(t,x)\phi_1(x)dtdx < \inf_{u\in R} g(u),$$

then the boundary value problem $(E)(B_1)(B_2)$ has no solution.

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