Fibers and Polymers 2005, Vol.6, No.1, 55-65

Extensible Elastica Solutions on the Large Deflection of Fiber Cantilever with Circular Wavy Crimp (I) –Derivation of Models and Their Solutions–

Jae Ho Jung* and Tae Jin Kang

School of Materials Science and Technology, Department of Engineering, Seoul National University, Seoul 151-742, Korea

(Received February 9, 2004; Revised December 3, 2004; Accepted December 10, 2004)

Abstract: Extensible elastica solutions of two-dimensional deflection of crimped fiber cantilever of circular wavy crimp were obtained for one end clamped boundary under concentrated, inclined and dead tip load Fiber was also regarded as a linear elastic material. Crimp was described as a combination of semicircular arcs smoothly connected with each other having constant curvature of all the same magnitude and alternative sign. Also the inclined load direction was taken into account. The solutions were expressed as the recursive forms of integrals in two different cases, which can also be transformed to elliptic integrals respectively. Comparing the data with inextensible ones was carried out. Consequently in the solution, the normal strain of neutral axis is expressed in terms of cross-sectional area, second moment of area and normalized load parameter. Examples of the circular cross-sectioned fiber are presented. As a result, the differences of normalized load between inextensible and extensible elastica solutions when the radius ratio becomes 0.1 were maximum $\Lambda = 0.1$.

Keywords: Extensible elastica, Dead tip load, Crimp, Recursive form, Elliptic integrals

Introduction

In our previous paper [1], we derived the elliptic integral solutions on the large deflection of fiber cantilever with circular wavy crimp under dead tip load, and estimated it. The possibility of extension of neutral axis during deflection can also be considered. In applied mathematical area, several researchers dealt with the extensibility of straight beam or column. Magnusson *et al.* [2,4] dealt with the post-buckling problem of simply hinged column withy two end considering extensibility. But there is few report about extensible elastica solutions of two-dimensional crimped beam with inclined loading. This result can be explained as follow. First, the concept of crimpis neither familiar nor easy to treat to the researchers in solid and fiber mechanics. As is emphasized in our previous paper, the modeling of plainly circular crimp is more complex than that of helical one. Unlike the case of helical one, one should deal with a series of governing equations at each segment of crimp. Second, although the extensibility is relatively easy to consider in establishing governing equation, it is very difficult to obtain its exact solution (See reference 2 and check its mathematical complexity). Thus the extensible elastica solution of crimped cantilever has never been attempted. In this work, we are to get it by applying new mathematical techniques summarized to three key concepts.

Mathematical Modeling

The description of fiber characteristic is almost the same as our previous paper [1]. A fiber is considered as a linear elastic and extensible beam with negligible shear effect. Segment numbering is the same as our previous paper. To solve the extensible elastica solution, the geometrical scheme must be reformulated as shown in Figure 1.

Geometric Relationships and Equilibrium Equations

The geometrical relationships and the equilibrium equations are as follow.

$$d\gamma = (1+\varsigma)d\Gamma,\ \frac{dx}{d\Gamma} = (1+\varsigma)\cos\theta,\ \frac{dy}{d\Gamma} = (1+\varsigma)\sin\theta \quad (1)$$

$$H = P_x = P\sin\alpha,\ V = P_y = P\cos\alpha \quad (2)$$

$$M(\Gamma) = M_1 - P_y x - P_x y = P\{\cos\alpha(X-x) + \sin\alpha(Y-y)\} \quad (3)$$

$$\frac{M(\gamma)}{EI} = \frac{d\theta}{d\Gamma} - \left(\frac{d\theta}{d\Gamma}\right)_0 \quad (4)$$

Where, the notation of *H*, *V*, *N* and P characters denotes the

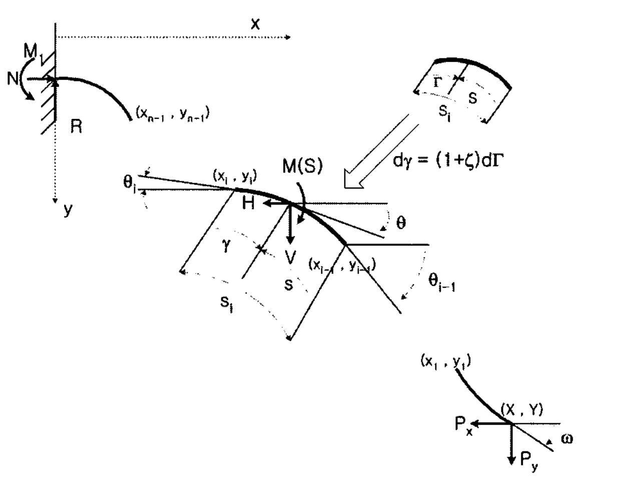

Figure 1. Free body diagram of extensible beam element under load.

*Corresponding author: pobee0@hanmail.net

load, M the bending moment. Also, $d\gamma$ is the extended arc length of neutral line, $d\Gamma$ the initial length. Meanwhile, the tensile strain (ς) for the extension of neutral axis must be expressed as follow.

$$EA\cdot\varsigma=(-H\cos\theta+V\sin\theta)=-P(\sin\alpha\cos\theta-\cos\alpha\sin\theta)$$
$$\therefore\ \varsigma=-\mu\sin(\alpha-\theta),\quad P/(EA)=\mu \tag{5}$$

Governing Equations

Substituting the equations (2) and (3) into (4) gives the following governing equations.

$$\frac{d\theta}{d\Gamma}=\pm\frac{1}{\rho}+\frac{P}{EI}\{\cos\alpha(X-x)+\sin\alpha(Y-y)\} \tag{6}$$

$$\frac{dx}{dS}=-(1+\varsigma)\cos\theta,\quad \frac{dy}{dS}=-(1+\varsigma)\sin\theta \tag{7a,b}$$

Using the relationship $S=S_i-\Gamma$ (See Figure 1) and the equation (5) and differentiating the equation (6) with respect to S gives:

$$\frac{d\theta}{dS}=\pm\frac{1}{\rho}-\frac{P}{EI}\{\cos\alpha(X-x)+\sin\alpha(Y-y)\} \tag{8}$$

$$\frac{d^2\theta}{dS^2}=-\frac{P}{EI}\{1-\mu\sin(\alpha-\theta)\}\cos(\alpha-\theta) \tag{9}$$

Where, the sign of the $1/\rho$ in the right side of the equation (8) is plus for the segment of odd order, minus for the segment of even order.

Mathematical Solutions

Integrating the equation (9) gives

$$\frac{1}{2}\left(\frac{d\theta}{dS}\right)^2=\frac{P}{EI}\left\{\sin(\alpha-\theta)-\frac{1}{2}\mu\sin^2(\alpha-\theta)\right\}+D \tag{10}$$

Where, D is the integral constant we should evaluate. Normalizing the equation (10) and determining the integral constant gives the following equation (11) and (12).

$$\left(\frac{d\theta}{d\varepsilon_i}\right)^2=2\lambda_i\left\{\sin(\alpha-\theta)-\frac{1}{2}\mu\sin^2(\alpha-\theta)-D_i\right\} \tag{11a}$$

$$\frac{S}{S_i}=\varepsilon_i,\quad \frac{PS_i^2}{EI}=\lambda_i,\quad \frac{PR^2}{EI}=\Lambda \tag{11b}$$

$$D_i=\sin(\alpha-\theta_{i-1})-\frac{1}{2}\mu\sin^2(\alpha-\theta_{i-1})$$
$$-\frac{1}{2}\left[\sqrt{2\left\{\sin(\alpha-\theta_{i-1})-\frac{1}{2}\mu\sin^2(\alpha-\theta_{i-1})-D_{i-1}\right\}}-\frac{2}{\sqrt{\Lambda}}\right]^2\quad(i\geq 2)$$
$$D_1=\sin(\alpha-\omega)-\frac{1}{2}\mu\sin^2(\alpha-\omega)-\frac{1}{2\Lambda} \tag{12}$$

These equations cannot be directly transformed to elliptic integrals. It appears to be the reason why many former researchers did not attempt to deal with the crimped and extensible elastica. Here we introduce new constant C_i that satisfies:

$$D_i=C_i-\frac{1}{2}\mu C_i^2,\quad C_i=\frac{1}{\mu}\{1-\sqrt{1-2\mu D_i}\} \tag{13}$$

The above constant is the first key concept of the modeling. Then the equation (11) can be rewritten as:

$$\left(\frac{d\theta}{d\varepsilon_i}\right)^2=\lambda_i(1-C_i)\left[1-\frac{2}{1-C_i}\sin^2\left\{\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)\right\}\right.$$
$$\left.-\mu C_i-\mu\left[1-2\sin^2\left\{\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)\right\}\right]\right] \tag{14}$$

Note that if the value of C_i is greater than unity, the equation (14) has no solution because the right side of the equation (14) has minus value. The proof is the following.

If, $C_i>1$, the term $1-\frac{2}{1-C_i}\sin^2\left\{\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)\right\}$ in the above equation is always greater than zero since the value of $\frac{2}{1-C_i}$ is minus. Using the equation (13), we can find the following inequality.

$$2-\mu\left[C_i+1-2\sin^2\left\{\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)\right\}\right]\geq 2-\mu[C_i+1]$$
$$=1+\sqrt{1-2\mu D_i}-\mu>0\ (\mu=1)$$

Thus the right side of the equation (14) has minus value. Meanwhile, one can easily confirm the value of the right side of the equation (14) should be zero so that the value of C_i may be equal to unity. Therefore, we discard the case of $C_i>1$ in our model. Similar to the previous paper, the further transformation of the equation (14) must be performed according to the range of the value of C_i.

Case 1: $\frac{2}{1-C_i}\geq 1$ or $-1\leq C_i<1$

Introducing and transforming variables into the equation (14) as below give the equation (15).

$$\sqrt{\frac{1-C_i}{2}}=k_i,\quad \sin\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)=k_i\sin\phi,$$

$$d\theta=\frac{2k_i\cos\phi}{\cos\left\{\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)\right\}}d\phi$$

$$\left(\frac{d\phi}{d\varepsilon_i}\right)^2=\lambda_i\cos^2\left\{\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)\right\}(1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi)$$
$$=\lambda_i(1-k_i^2\sin^2\phi)(1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi) \tag{15}$$

The above equation can be rewritten as:

$$d\varepsilon_i = \frac{(\pm 1)d\phi}{\sqrt{\lambda_i(1-k_i^2\sin^2\phi)}\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}} \tag{16}$$

Here we meet an important problem. The question whether the sign should be chosen as plus or minus in the equation (16) will arise. Although we did not discuss it in our previous paper, choosing the sign in the equation (16) is originated from choosing the start point of the arc length as loaded free or clamped end for the case of straight beam. If the point is chosen as clamped end, the sign becomes plus. If the point is chosen as loaded free end, the sign becomes minus. However, it is both possible to choose the sign in the equation (16) either plus or minus in our cases. For the case of n elements of crimped segments, there exist total 2^n choices. This is the second key concept of the modeling. It enables the solutions to have relatively much more equilibrium configurations of different shapes. As the simplest choice, we put all the signs as minus. Other choices will be dealt with in part II. Integrating the equation (16) from ϕ_{i-1} to ϕ gives:

$$\int_0^1 d\varepsilon_i = \int_{\phi_i}^{\phi_{i-1}} \frac{d\phi}{\sqrt{\lambda_i(1-k_i^2\sin^2\phi)}\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}} \tag{17}$$

The equation (17) can also be transformed to elliptic integral. This is inspired from the transformation of Weierstrass type elliptic integral to Legendre-Jacobian type one [3]. Detailed procedure is demonstrated in Appendix A. As a result, the equation (17) is transformed to the following elliptic integral.

$$\sqrt{\lambda_i} = \frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\{F(p_i,\gamma_{i-1})-F(p_i,\gamma_i)\} \tag{18}$$

$$a_i = 1-\mu+\mu k_i^2,\ b_i = \mu k_i^2,\ q_i^2 = \frac{b_i}{a_i+b_i},\ p_i = q_i^2+k_i^2-q_i^2k_i^2$$

$$\tan\phi = \sqrt{1-q_i^2}\tan\gamma,\ \tan\phi_i = \sqrt{1-q_i^2}\tan\gamma_i,$$

$$\tan\phi_{i-1} = \sqrt{1-q_i^2}\tan\gamma_{i-1}$$

Using the equations (7a,b), (11b) and the following equation, we also express the infinitesimal elements of x and y in terms of the variable ϕ, ε_i.

$$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = \begin{bmatrix}\sin\alpha & \cos\alpha\\ -\cos\alpha & \sin\alpha\end{bmatrix}\begin{bmatrix}1-2k_i^2\sin^2\phi\\ \pm 2k_i\sin\phi\sqrt{1-k_i^2\sin^2\phi}\end{bmatrix} \tag{19a}$$

Where, the sign of the right side of the equation (19) becomes plus as the slope angle θ satisfies $\cos\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)\geq 0$, minus as $\cos\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)\leq 0$. It results from the following relationship.

$$\cos\left\{\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)\right\} = \pm\sqrt{1-k_i^2\sin\phi} \tag{19b}$$

Also there exists a method of fixing the sign in the equation (19). The value of θ must satisfy the following relationship so that the sign of the equation (19b) may be plus.

$$4n\pi-\frac{3\pi}{2}+\alpha\leq\theta\leq 4n\pi+\frac{\pi}{2}+\alpha \tag{19c}$$

Where, n is an appropriate integer. If we choose the value of n appropriately so that $\theta = 0$ may be within the above range with given value of α and define all the value of θ and ω within the above range, we can fix the sign in the equation (19) as plus. For example, if we choose the value of n as zero and define all the values of θ and ω within the following range with the given range of $0<\alpha<\pi/2$, we can fix the sign as plus. This is the third key concept of the modeling.

$$-\frac{3\pi}{2}+\alpha\leq\theta\ \&\ \omega\leq\frac{\pi}{2}+\alpha \tag{19d}$$

Note that the interval $\left[-\frac{3\pi}{2}+\alpha,\ \frac{\pi}{2}+\alpha\right]$ covers all possible slope angles, which is equivalent to the interval $[0, 2\pi]$. Also the sign in the equation (16) are different from that in the equation (19).

In summary, the choice of the sign in the equation (16) and (19c) is described as the following.

1. The sign in the equation (16) can be chosen as either plus or minus. Therefore, there may exist multiple equilibrium configurations according to the sign. In our present work, we choose the sign as all minus. Other choices will be dealt with in part II.

2. The sign in the equation (19b) can be chosen as plus if we confine all existent angles of slope within the range shown in the equation (19d).

Applying this concept, we obtain the following equation (20) by integrating the equation (7) with the aid of the equation (5) and (19) as:

$$\frac{L\sqrt{\lambda_i}}{S_i}(\bar{x}_{i-1}-\bar{x}_i) = \int_{\phi_i}^{\phi_{i-1}}\{1+\mu-2\mu(1-k_i^2\sin^2\phi)\}$$
$$\times[\sin\alpha(1-2k_i^2\sin^2\phi)+\cos\alpha\cdot 2k_i\sin\phi\cdot\sqrt{1-k_i^2\sin^2\phi}]$$
$$\times\frac{d\phi}{\sqrt{(1-k_i^2\sin^2\phi)}\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}} \tag{20a}$$

$$\frac{L\sqrt{\lambda_i}}{S_i}(\bar{y}_{i-1}-\bar{y}_i) = \int_{\phi_i}^{\phi_{i-1}}\{1+\mu-2\mu(1-k_i^2\sin^2\phi)\}$$
$$\times[-\cos\alpha(1-2k_i^2\sin^2\phi)+\sin\alpha\cdot 2k_i\sin\phi\cdot\sqrt{1-k_i^2\sin^2\phi}]$$
$$\times\frac{d\phi}{\sqrt{(1-k_i^2\sin^2\phi)}\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}} \tag{20b}$$

Also the equation (20) can be partially transformed to elliptic integrals. Detailed procedure is demonstrated in Appendix B.

Case 2: $\frac{2}{1-C_i}\leq 1$ or $C_i\leq 1$, $\sqrt{\frac{2}{1-C_i}}=k_i$

The equation (14) can be rewritten as:

$$\left(\frac{d\theta}{d\varepsilon_i}\right)^2 = \frac{2\lambda_i}{k_i^2}\left[1-k_i^2\sin^2\left\{\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)\right\}\right]\left\{2-\mu\left(1-\frac{2}{k_i^2}\right)\right.$$
$$\left.-\mu\left[1-2\sin^2\left\{\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta\right)\right\}\right]\right\} \quad (21)$$

Transforming variables as $(\pi/2-\alpha+\theta) = 2\varphi$ gives:

$$\left(\frac{d\varphi}{d\varepsilon_i}\right)^2 = \frac{\lambda_i}{2k_i^2}(1-k_i^2\sin^2\varphi)\left\{2-\mu\left(1-\frac{2}{k_i^2}\right)-\mu(1-2\sin^2\varphi)\right\} \quad (22)$$

$$\sqrt{\lambda_i}d\varepsilon_i = (-1)^{i-\delta}\frac{k_i d\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{1-\mu+\frac{\mu}{k_i^2}+\mu\sin^2\varphi}} \quad (23)$$

Similarly in the case 1, we introduce a new variable γ under following transformations.

$$\tan\varphi = \sqrt{1-q_i^2}\tan\gamma \quad (24)$$

$$q_i^2 = \frac{\mu}{a_i+\mu},\ a_i = 1-\mu+\frac{\mu}{k_i^2},\ p_i = q_i^2+k_i^2-q_i^2k_i^2$$

Applying the equation (24) to (23) and integrating it gives:

$$\sqrt{\lambda_i} = (-1)^{i-\delta}\frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\{F(p_i,\gamma_i)-F(p_i,\gamma_{i-1})\} \quad (25)$$

$$\tan\varphi_i = \sqrt{1-q_i^2}\tan\gamma_i,\ \tan\varphi_{i-1} = \sqrt{1-q_i^2}\tan\gamma_{i-1}$$

Similarly, we also express the infinitesimal elements of x and y in terms of the variable φ, ε_i as:

$$\frac{L\sqrt{\lambda_i}}{S_i}(\bar{x}_{i-1}-\bar{x}_i) = (-1)^{i-\delta}\int_{\varphi_{i-1}}^{\varphi_i}\{\sin\alpha\cos2\varphi-\mu\sin\alpha\cos^2 2\varphi$$
$$+\sin2\varphi\cos\alpha-\mu\cos\alpha\cos2\varphi\sin2\varphi)$$
$$\times\frac{k_i d\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{1-\mu+\frac{\mu}{k_i^2}+\mu\sin^2\varphi}} \quad (26a)$$

$$\frac{L\sqrt{\lambda_i}}{S_i}(\bar{y}_{i-1}-\bar{y}_i) = (-1)^{i-\delta}\int_{\varphi_{i-1}}^{\varphi_i}\{-\cos\alpha\cos2\varphi+\mu\cos\alpha\cos^2 2\varphi$$
$$+\sin2\varphi\sin\alpha-\mu\sin\alpha\cos2\varphi\sin2\varphi)$$
$$\times\frac{k_i d\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{1-\mu+\frac{\mu}{k_i^2}+\mu\sin^2\varphi}} \quad (26b)$$

Where, the value of δ is unity when the total number of segments is even, zero when odd. The equation (18), (20), (25) and (26) are the solutions of extensible elastica of crimped fiber under inclined loading. Similarly, the arbitrary position (x, y) can be calculated by replacing the parameters x_i, y_i and ϕ_i (or φ_i) with x, y and ϕ(or φ). Also the equations can be partially transformed to elliptic integrals. Detailed procedure is demonstrated in Appendix C. Comparison of data and the effect of extensibility μ on the solution for circular-cross-sectioned fiber with two elements will be discussed in Results and Discussions.

Results and Discussions

Algorithm of Two-element Extensible Elastica Solutions

For two elements, the values of α, EI, ρ are assumed as given values. Instead of the method in previous paper, we adopt the procedure of shooting the value of Λ so that the value of θ_2 must be zero at the given value of ω for mathematical convenience. Note that the value of μ in the equation (5) is the function of Λ, A, ρ and I as follow.

$$\mu = \frac{P}{EA} = \frac{1}{EA}\frac{EI}{\rho^2} = \frac{I}{A}\cdot\frac{\Lambda}{\rho^2} \quad (27)$$

For a fiber of circular cross-section, the area A and second moment of area I is given as:

$$A = \pi r^2,\ I = \frac{\pi r^4}{4}$$

Where, r is the radius of the fiber. Thus the equation (27) is expressed as:

$$\mu = \frac{1}{4}\left(\frac{r}{\rho}\right)^2\Lambda \quad (28)$$

We use the above relationship for two-element example. The radius ratio r/ρ is assumed as the value of 0.05 and 0.1 Detailed algorithm of solving the equation (18) and (25) is demonstrated as follows.

For 1st Element at Loaded Tip

1) Guess the value of Λ and determine the value of D_1, C_1, μ with the given parameters α, EI, ρ, ω using the equation (28) and the following equation (29) and (30).

$$D_1 = \sin(\alpha-\omega)-\frac{1}{2}\mu\sin^2(\alpha-\omega)-\frac{1}{2\Lambda} \quad (29)$$

$$C_1 = \frac{1}{\mu}(1-\sqrt{1-2\mu D_1}) \quad (\mu\neq 0) \quad (30)$$
$$= D_1 \quad (\mu = 0)$$

2)-i) If $-1\leq C_1<1$, calculate the value of k_1, ϕ_0 using the following equations.

$$k_1 = \sqrt{\frac{1-C_1}{2}} \quad (31)$$

$$\phi_0 = \arcsin\left\{\frac{1}{k_1}\sin\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\omega\right)\right\} \tag{32}$$

2)-ii) Using the equation (18), find the value of ϕ_1 satisfying the following equation by Newton Rapson Method.

$$f(\phi_1) = \sqrt{\lambda_1}-\int_{\phi_1}^{\phi_0}\frac{dx}{\sqrt{1-k_1^2\sin^2x}\sqrt{1-\mu+\mu k_1^2+\mu k_1^2\sin^2x}} \tag{33}$$

2)-iii) Calculate the value of θ_1 using the following equation and the result of the equation (33).

$$\theta_1 = 2\arcsin(k_1\sin\phi_1)+\alpha-\frac{\pi}{2} \tag{34}$$

3)-i) If $C_1 < -1$, calculate the value of k_1, φ_0 using the following equations.

$$k_1 = \sqrt{\frac{2}{1-C_1}} \tag{35}$$

$$\varphi_0 = \frac{1}{2}\left(\frac{\pi}{2}-\alpha+\omega\right) \tag{36}$$

3)-ii) Using the equation (25), find the value of φ_1 satisfying the following equation by Newton Rapson Method.

$$f(\varphi_1) = \sqrt{\lambda_1}-(-1)^{1-1}\int_{\varphi_0}^{\varphi_1}\frac{k_1 d\varphi}{\sqrt{1-k_1^2\sin^2x}\cdot\sqrt{1-\mu+\frac{\mu}{k_1^2}+\mu\sin^2x}} = 0 \tag{37}$$

3)-iii) Calculate the value of θ_1 using the following equation and the result of the equation (37).

$$\theta_1 = 2\varphi_1+\alpha-\frac{\pi}{2} \tag{38}$$

For 2nd Element at Clamped End

4) Take the value of θ_1 from the results of the former procedure 2)-iii) or 3)-iii).

5) Determine the value of D_2, C_2 using the following equations and the equation (28).

$$D_2 = \sin(\alpha-\theta_1)-\frac{1}{2}\mu\sin^2(\alpha-\theta_1) -\frac{1}{2}\left[\sqrt{2\left\{\sin(\alpha-\theta_1)-\frac{1}{2}\mu\sin^2(\alpha-\theta_1)-D_1\right\}}-\frac{2}{\sqrt{\Lambda}}\right]^2 \tag{39}$$

$$C_2 = \frac{1}{\mu}(1-\sqrt{1-2\mu D_2}) \quad (\mu\neq 0)$$
$$= D_2 \quad (\mu=0) \tag{40}$$

6)-i) If $-1\le C_2<1$, calculate the value of k_2, ϕ_1 using the following equations.

$$k_2 = \sqrt{\frac{1-C_2}{2}} \tag{41}$$

$$\phi_1 = \arcsin\left\{\frac{1}{k_2}\sin\frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta_1\right)\right\} \tag{42}$$

6)-ii) Using the equation (18), find the value of ϕ_2 satisfying the following equation by Newton-Rapson method.

$$f(\phi_2) = \sqrt{\lambda_2}-\int_{\phi_2}^{\phi_1}\frac{dx}{\sqrt{1-k_2^2\sin^2x}\sqrt{1-\mu+\mu k_2^2+\mu k_2^2\sin^2x}} = 0 \tag{43}$$

6)-iii) Calculate the value of θ_1 using the following equation and the result of the equation (43).

$$\theta_2 = 2\arcsin(k_2\sin\phi_2)+\alpha-\frac{\pi}{2} \tag{44}$$

7)-i) If $C_1<-1$, calculate the value of k_2, φ_1 using the following equations.

$$k_2 = \sqrt{\frac{2}{1-C_2}} \tag{45}$$

$$\varphi_1 = \frac{1}{2}\left(\frac{\pi}{2}-\alpha+\theta_1\right) \tag{46}$$

7)-ii) Using the equation (25), find the value of φ_1 satisfying the following equation by Newton-Rapson method. The value of δ is equal to unity because the total number of segments is two in this case.

$$f(\varphi_2) = \sqrt{\lambda_2}-(-1)^{2-1}\int_{\varphi_1}^{\varphi_2}\frac{k_2 d\varphi}{\sqrt{1-k_2^2\sin^2x}\cdot\sqrt{1-\mu+\frac{\mu}{k_2^2}+\mu\sin^2x}} = 0 \tag{47}$$

7)-iii) Calculate the value of θ_2 using the following equation and the result of the equation (47).

$$\theta_2 = 2\varphi_2+\alpha-\frac{\pi}{2} \tag{48}$$

8) Shoot the value of Λ by false-position method so that the value of θ_2 from the result of the procedure 6)-iii) or 7)-iii) must be under the allowed tolerance.

In the above algorithm, λ_1 and λ_2 is transformed as below in terms of Λ.

$$\lambda_1 = \lambda_2 = \frac{\pi^2}{4}\Lambda \tag{49}$$

In the equation (34) and (44), one should pay attention to estimating whether the calculated values of θ_1, θ_2, ω satisfy the following relationship.

$$-\frac{3\pi}{2}+\alpha\le\theta_1,\ \theta_2,\ \omega\le\frac{\pi}{2}+\alpha \tag{50}$$

Table 1. Iterated values of Λ from the case 2 - case 2 solutions with given value of ω for inextensible elastica $\mu = 0$ at $\alpha = 45\,^\circ$

θ_2 (°)	ω (°)	Λ
0	24.2227	0.1
0.0000001	42.55242	0.2
0.0000002	70.423138	0.5
0.000001	87.280039	1
0.0000119	99.253506	2
0.0000007	104.59921	3
0.0000001	107.8640652	4

Table 2a. Iterated values of Λ from given range of $-225\,^\circ \leq \omega \leq 65\,^\circ$ for inextensible c

θ_2 (°)	Segment 1	Segment 2	ω (°)	μ	Λ
0.0000046	Case 1	Case 2	-195	0	4.1754017
0.0003236	Case 1	Case 1	-195	0	4.599905
0.0015205	Case 1	Case 2	-185	0	2.626774
0.0015783	Case 1	Case 1	-185	0	3.2495879
0.0002396	Case 1	Case 1	-185	0	4.2644488
0.0039651	Case 1	Case 2	-175	0	1.8409865
0.0073286	Case 1	Case 1	-175	0	2.6542348
0.0001413	Case 1	Case 1	-175	0	2.9630178
0.00827	Case 1	Case 2	-165	0	1.3832187
0.0037929	Case 1	Case 2	-155	0	1.0913676
0.0060427	Case 1	Case 2	-145	0	0.8931723
0.0002569	Case 1	Case 2	-135	0	0.6975071
0.0042437	Case 1	Case 2	-135	0	0.7522203
0.0001673	**Case 2**	**Case 2**	**5**	**0**	**0.019636**
0.0006645	Case 1	Case 1	5	0	4.1639934
0.0001656	Case 1	Case 1	5	0	5.2478184
0.0000947	**Case 2**	**Case 2**	**15**	**0**	**0.0598442**
0.0007565	Case 1	Case 1	15	0	4.1125928
0.00013	Case 1	Case 1	15	0	5.8525122
0.0016507	**Case 2**	**Case 2**	**25**	**0**	**0.1035928**
0.0012474	Case 1	Case 1	25	0	4.2347175
0.000067	Case 1	Case 1	25	0	6.3974896
0.001989	**Case 2**	**Case 2**	**35**	**0**	**0.1542714**
0.0011102	Case 1	Case 1	35	0	4.4676117
0.0000622	Case 1	Case 1	35	0	7.6130058
0.0019431	**Case 2**	**Case 2**	**45**	**0**	**0.2167125**
0.0007981	Case 1	Case 1	45	0	4.8087316
0.000052	Case 1	Case 1	45	0	9.5462196
0.0010201	**Case 2**	**Case 2**	**55**	**0**	**0.2987137**
0.0015016	Case 1	Case 1	55	0	5.2774345
0.0021841	**Case 2**	**Case 2**	**65**	**0**	**0.414276**
0.0000366	Case 1	Case 1	65	0	3.3194896
0.0001551	Case 1	Case 1	65	0	3.9575397

Table 1 shows the iterated values of Λ from the case 2 - case 2 solutions with given value of ω for inextensible elastica of $\mu = 0$ at $\alpha = 45\,^\circ$, which shows good agreements to the table IV_A in reference [1].

Table 2b. Iterated values of Λ from given range of $65\,^\circ \leq \omega \leq 135\,^\circ$ for inextensible cases under $\alpha = 45\,^\circ$

θ_2 (°)	Segment 1	Segment 2	ω (°)	μ	Λ
0.0015642	Case 1	Case 1	65	0	5.9139148
0.0012481	**Case 2**	**Case 2**	**75**	**0**	**0.5917881**
0.0001393	Case 1	Case 1	75	0	2.4121884
0.0002254	Case 1	Case 1	75	0	5.2026061
0.0000131	Case 1	Case 1	75	0	6.7895188
0.0014155	**Case 2**	**Case 2**	**85**	**0**	**0.897703**
0.0000262	Case 1	Case 1	85	0	2.6489086
0.0000243	Case 1	Case 1	85	0	6.0096544
0.0009944	Case 1	Case 1	85	0	8.031097
0.0003942	**Case 2**	**Case 2**	**95**	**0**	**1.5194857**
0.0000866	Case 1	Case 1	95	0	2.906201
0.0001507	Case 1	Case 1	95	0	6.8333997
0.0047067	**Case 2**	**Case 2**	**105**	**0**	**3.1104413**
0.0000019	Case 2	Case 1	115	0	3.622108

Table 3a. Iterated values of Λ from given range $-225\,^\circ \leq \omega \leq 15\,^\circ$ of for extensible cases of $r/\rho = 0.1$ under $\alpha = 45\,^\circ$

θ_2 (°)	Segment 1	Segment 2	ω (°)	μ	Λ
0.0000003	Case 1	Case 2	-195	0.0103	4.13533903
0.000375	Case 1	Case 2	-185	0.0065	2.6115489
0.0038601	Case 1	Case 1	-185	0.0081	3.2311423
0.0001329	Case 1	Case 1	-185	0.0104	4.1849894
0.0108105	Case 1	Case 2	-175	0.0046	1.8340375
0.0074014	Case 1	Case 1	-175	0.0066	2.6453498
0.0001424	Case 1	Case 1	-175	0.0073	2.9334403
0.0009417	Case 1	Case 2	-165	0.0034	1.3797442
0.0015201	Case 1	Case 2	-155	0.0027	1.0894987
0.0046996	Case 1	Case 2	-145	0.0022	0.8921372
0.0002964	Case 1	Case 2	-135	0.0017	0.6968817
0.0034014	Case 1	Case 2	-135	0.0019	0.7516574
0.0001673	**Case 2**	**Case 2**	**5**	**0**	**0.0196359**
0.0003795	Case 1	Case 1	5	0.0104	4.1883158
0.0002188	Case 1	Case 1	5	0.0132	5.2936851
0.0000954	**Case 2**	**Case 2**	**15**	**0.0001**	**0.059843**
0.0000564	Case 1	Case 1	15	0.0103	4.130149
0.0001606	Case 1	Case 1	15	0.0147	5.9039782

Tables 2 and 3 show the data of iterated values of $\Lambda(0 \leq \Lambda \leq 10)$ from given value of $-225\,^\circ \leq \omega \leq 135\,^\circ$ for inextensible and extensible elastica of $r/\rho = 0.1$. under $\alpha = 45\,^\circ$. Increasing interval of ω is $10\,^\circ$. The difference of normalized load between the classical inextensible and extensible elastica solutions has the maximum value $\Lambda = 0.1$ at $\omega = 35\,^\circ$. This difference can be assumed negligible under the common range of large deflections of slender beams. We don't present other results for $\alpha = 0\,^\circ$, $30\,^\circ$, $60\,^\circ$ and $90\,^\circ$ in this paper since they show the same tendencies.

Results of case 2 – case 2 type solutions in the above Tables 2 and 3 are emphasized with bold letter. Figure 2

Table 3b. Iterated values of Λ from given range of $25\,^{\circ} \leq \omega \leq 65\,^{\circ}$ for extensible cases of $r/\rho = 0.1$ under $\alpha = 45\,^{\circ}$

θ_2 (°)	Segment 1	Segment 2	ω (°)	μ	Λ
0.0016488	**Case 2**	**Case 2**	**25**	**0.0003**	**0.1035862**
0.0009795	Case 1	Case 1	25	0.0106	4.2472441
0.0000373	Case 1	Case 1	25	0.0161	6.4587669
0.001986	**Case 2**	**Case 2**	**35**	**0.0004**	**0.1542509**
0.0014947	Case 1	Case 1	35	0.0112	4.4749834
0.0000186	Case 1	Case 1	35	0.0193	7.7107085
0.0019579	**Case 2**	**Case 2**	**45**	**0.0005**	**0.2166614**
0.0001307	Case 1	Case 1	45	0.012	4.8098119
0.0010977	**Case 2**	**Case 2**	**55**	**0.0007**	**0.298597**
0.0000518	Case 1	Case 1	55	0.0132	5.2700674
0.0021443	**Case 2**	**Case 2**	**65**	**0.001**	**0.4140143**
0.0001528	Case 1	Case 1	65	0.0082	3.2971524
0.0001525	Case 1	Case 1	65	0.0101	4.0376426
0.0015887	Case 1	Case 1	65	0.0147	5.8943386
0.0008911	**Case 2**	**Case 2**	**75**	**0.0015**	**0.5911898**
0.0001122	Case 1	Case 1	75	0.006	2.4182686
0.0002656	Case 1	Case 1	75	0.0131	5.2334741
0.0000189	Case 1	Case 1	75	0.0169	6.7508997
0.0018849	**Case 2**	**Case 2**	**85**	**0.0022**	**0.8961972**
0.0000627	Case 1	Case 1	85	0.0066	2.6536227
0.0000194	Case 1	Case 1	85	0.015	6.020286
0.0004297	Case 1	Case 1	85	0.0199	7.9604386
0.0020122	**Case 2**	**Case 2**	**95**	**0.0038**	**1.5147713**
0.0000284	Case 1	Case 1	95	0.0073	2.9108423
0.0000983	Case 1	Case 1	95	0.017	6.821674
0.00012	**Case 2**	**Case 2**	**105**	**0.0077**	**3.080586**
0.0000045	Case 2	Case 1	115	0.0091	3.655715

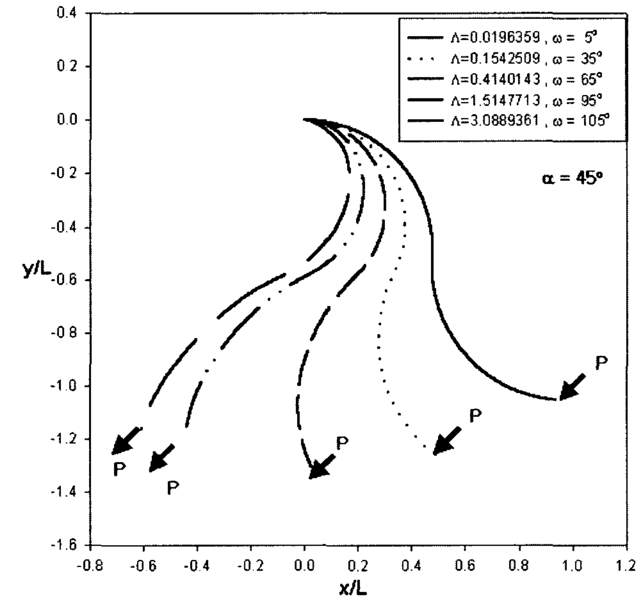

Figure 2. Shapes of case 2 - case 2 equilibrium with variable load and tip angle for extensible elastica of r/ρ = 0.1 at $\alpha = 45\,^{\circ}$.

shows some examples of case 2 - case 2 equilibrium shapes with variable load and tip angle for extensible elastica of r/ρ = 0.1 at given inclined angle $\alpha = 45\,^{\circ}$.

It is very interesting feature that there exists the region of tip angle ω that does not have equilibrium configuration in Table 2. Moreover, there exists only one root for case 2 – case 2 and case 2 – case 1 type solutions, while case 1 – case 1 and case 1 – case 2 type solutions have several multiple roots. In the case 2 – case type in Figure 2, the normalized load gradually grows larger as the tip angle ω increases up to $\omega = 95\,^{\circ}$. But the value of Λ at $\omega = 105\,^{\circ}$ becomes more than two times larger than that at $\omega = 95\,^{\circ}$. It implies that the equilibrium shape need two times larger value of normalized load to increase the tip angle by merely 10° (from 95° to 105°). Though not presented in this paper, it can be concluded that the possible region of case 2 - case 2 equilibrium is commonly about the range of $0 \leq \Lambda \leq 5$ for $\alpha = 0\,^{\circ}$, $30\,^{\circ}$, $45\,^{\circ}$, $60\,^{\circ}$ and $90\,^{\circ}$ as we confirm that from the table IV_A in reference [1]. Shapes of other type solutions with different choices of the sign in the equation (16) will be dealt with in part II.

Conclusions and Further Works

Extensible elastica solutions of large deflection of crimped fiber cantilever under inclined load were solved and several shapes of the equilibrium configurations were presented. The mathematical solutions also have the form of recursive equation of integrals, which can be transformed to elliptic integrals. During deriving the solutions, the mathematical complexity, which rendered other researchers to avoid solving the exact solution of crimped cantilever beam under inclined loading, was overcome by introducing new integral constants. Fixing the sign in the equation (16) and (19) are another two key concepts of the modeling. Where the former is involved with the multiplicity of the solutions, the later with the choosing range of the available slope angles. These three concepts are the keys to the solutions. Since the mathematical characteristics of the extensible elastica solutions are very similar to that of inextensible ones, extensible elastica solutions show similar behaviors to classical inextensible elastica ones. The difference of the normalized load Λ between the classical inextensible and extensible elastica solutions has the maximum value 0.1. Alternatively, calculated solutions have multiple roots. This topic will be discussed more precisely in part II.

References

1. J. H. Jung, T. J. Kang, and K. W. Lee, *Text. Res. J.*, **73**(1) 47 (2003).
2. A. Magnusson, M. Ristinmaa, and C. Ljung, *Int. J. Solids & Struct.*, **38**, 8441 (2001).
3. Z. X. Wang and D. R. Guo, "Special Functions", p.549, World Scientific., 1986.
4. H. Ramsey, *Int. J. Mech. Sci.*, **30**, 559 (1998).

Nomenclature

Basic geometrical and material parameters in this work are defined as follow.

E: elastic modulus of a fiber
A: cross-sectional area of a fiber
R: initial radius of curvature
θ: slope angle
α: inclined angle of load
ω: slope angle at loaded tip
ς: strain of neutral line
I: second moment of area of the cross-section
x: horizontal coordinate of arbitrary position after deflection
y: vertical coordinate of arbitrary position after deflection
(x_{i-1}, y_{i-1}): coordinates of the start point of i-th segment after deflection $(i \geq 1)$
(x_i, y_i): coordinates of the end point of i-th segment after deflection $(i \geq 1)$
X: horizontal coordinate of loaded tip after deflection
Y: vertical coordinate of loaded tip after deflection
(X, Y): coordinates of the loaded tip after deflection
s: arc length measured from the point (x_{i-1}, y_{i-1}) to (x, y) after deflection
γ: arc length measured from the point (x, y) to (x_i, y_i) after deflection
s_i: arc length measured from the point (x_{i-1}, y_{i-1}) to (x_i, y_i) after deflection
S: arc length measured from the point (x_{i-1}, y_{i-1}) to (x, y) before deflection
Γ: arc length measured from the point (x, y) to (x_i, y_i) before deflection
S_i: arc length measured from the point (x_{i-1}, y_{i-1}) to (x_i, y_i) before deflection

Appendices

Appendix A : Transformation to Elliptic Integrals Type A

$$\int_{\theta_a}^{\theta_b} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}\sqrt{A+B\sin^2\theta}} \quad (A,B>0) \tag{A-1}$$

Here we introduce the following variables as:

$$\sin^2\theta = \frac{(1-q^2)t^2}{1-q^2t^2} = \frac{1-q^2}{q^2}\left(\frac{1}{1-q^2t^2}-1\right) \quad (q^2, t^2<1)$$

$$2\sin\theta\cos\theta d\theta = \frac{1-q^2}{q^2}\frac{2q^2t}{(1-q^2t^2)^2}dt$$

$$\cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\frac{\sqrt{1-t^2}}{\sqrt{1-q^2t^2}}, \quad \sin\theta = \pm\frac{t\sqrt{1-q^2}}{\sqrt{1-q^2t^2}}$$

$$\text{or} \quad \tan\theta = \frac{t\sqrt{1-q^2}}{\sqrt{1-t^2}} \quad (-1<t<1)$$

The above signs are chosen all plus or minus Then the terms in the equation (A-1) become:

$$\therefore d\theta = \frac{1-q^2}{q^2}\frac{q^2t}{(1-q^2t^2)^2}\cdot\frac{\pm\sqrt{1-q^2t^2}}{\sqrt{1-t^2}}\cdot\frac{\pm\sqrt{1-q^2t^2}}{t\sqrt{1-q^2}}dt$$

$$= \frac{\sqrt{1-q^2}}{(1-q^2t^2)\sqrt{1-t^2}}dt \tag{A-2}$$

$$\sqrt{1-k^2\sin^2\theta} = \sqrt{1-\frac{k^2(1-q^2)t^2}{1-q^2t^2}} = \frac{\sqrt{1-(q^2+k^2-q^2k^2)t^2}}{\sqrt{1-q^2t^2}}$$

$$= \frac{\sqrt{1-p^2t^2}}{\sqrt{1-q^2t^2}}, \; p^2 = 1-(1-k^2)(1-q^2), \quad 0<p^2<1 \tag{A-3}$$

$$\sqrt{A+B\sin^2\theta} = \sqrt{A+\frac{B(1-q^2)t^2}{1-q^2t^2}} = \frac{\sqrt{A+\{B(1-q^2)-Aq^2\}t^2}}{\sqrt{1-q^2t^2}} \tag{A-4}$$

If we choose the value of q^2 so that the following relationship may be satisfied,

$$\sqrt{A+B\sin^2\theta} = \frac{\sqrt{A}}{\sqrt{1-q^2t^2}}, \text{ or } B(1-q^2)-Aq^2 = 0,$$

$$q^2 = \frac{B}{A+B} \tag{A-5}$$

Then it is satisfied that

$$\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}\sqrt{A+B\sin^2\theta}} = \frac{\sqrt{1-q^2t^2}}{\sqrt{1-p^2t^2}}\cdot\frac{\sqrt{1-q^2t^2}}{\sqrt{A}}$$

$$\cdot\frac{\sqrt{1-q^2}}{(1-q^2t^2)\sqrt{1-t^2}}dt = \frac{\sqrt{1-q^2}}{\sqrt{A}\sqrt{1-t^2}\sqrt{1-p^2t^2}}dt \tag{A-6}$$

Here, we see the form of Legendre-Jacobian elliptic integral of first kind

$$\int_{\theta_a}^{\theta_b}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}\sqrt{A+B\sin^2\theta}} = \int_{t_a}^{t_b}\frac{\sqrt{1-q^2}}{\sqrt{A}\sqrt{1-t^2}\sqrt{1-p^2t^2}}dt$$

$$= \frac{\sqrt{1-q^2}}{\sqrt{A}}\int_{\gamma_a}^{\gamma_b}\frac{dx}{\sqrt{1-p^2\sin^2x}} \tag{A-7}$$

Where, $\tan\theta_a = \frac{t_a\sqrt{1-q^2}}{\sqrt{1-t_a^2}} = \sqrt{1-q^2}\tan\gamma_a$, $\tan\theta_b = \frac{t_b\sqrt{1-q^2}}{\sqrt{1-t_b^2}}$

$= \sqrt{1-q^2}\tan\gamma_b$

Appendix B : Transformation to Elliptic Integrals Type B

We have two equations expressing infinitesimal elements of $\bar{x}$ and $\bar{y}$ from case 1.

$$\frac{L\sqrt{\lambda_i}}{S_i}(\bar{x}_{i-1}-\bar{x}_i) = \int_{\phi_i}^{\phi_{i-1}}\{1+\mu-2\mu(1-k_i^2\sin^2\phi)\}$$
$$[\sin\alpha(1-2k_i^2\sin^2\phi)+\cos\alpha\cdot 2k_i\sin\phi\cdot\sqrt{1-k_i^2\sin^2\phi}]$$
$$\times\frac{d\phi}{\sqrt{(1-k_i^2\sin^2\phi)}\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}} \quad \text{(B-1a)}$$

$$\frac{L\sqrt{\lambda_i}}{S_i}(\bar{y}_{i-1}-\bar{y}_i) = \int_{\phi_i}^{\phi_{i-1}}\{1+\mu-2\mu(1-k_i^2\sin^2\phi)\}$$
$$[-\cos\alpha(1-2k_i^2\sin^2\phi)+\sin\alpha\cdot 2k_i\sin\phi\cdot\sqrt{1-k_i^2\sin^2\phi}]$$
$$\times\frac{d\phi}{\sqrt{(1-k_i^2\sin^2\phi)}\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}} \quad \text{(B-1b)}$$

From the above two equations, we can find five different types of integrals as

$$I=\int_{\phi_i}^{\phi_{i-1}}\frac{d\phi}{\sqrt{1-k_i^2\sin^2\phi}\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}}$$

$$II=\int_{\phi_i}^{\phi_{i-1}}\frac{(1-k_i^2\sin^2\phi)d\phi}{\sqrt{1-k_i^2\sin^2\phi}\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}}$$

$$III=\int_{\phi_i}^{\phi_{i-1}}\frac{(1-k_i^2\sin^2\phi)^2d\phi}{\sqrt{1-k_i^2\sin^2\phi}\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}}$$

$$IV=\int_{\phi_i}^{\phi_{i-1}}\frac{\sin\phi d\phi}{\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}}$$

$$V=\int_{\phi_i}^{\phi_{i-1}}\frac{\sin\phi(1-k_i^2\sin^2\phi)d\phi}{\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}}$$

The integrals I, II, III can be transformed to elliptic integrals under the following transformation

$$\sin^2\phi=\frac{(1-q_i^2)\sin^2\gamma}{1-q_i^2\sin^2\gamma},\ \text{or}\ \tan\phi=\sqrt{1-q_i^2}\tan\gamma \quad \text{(B-2)}$$

$$a_i=1-\mu+\mu k_i^2,\ b_i=\mu k_i^2,\ q_i^2=\frac{b_i}{a_i+b_i}$$
$$p_i^2=q_i^2+k_i^2-q_i^2k_i^2 \quad \text{(B-3)}$$

Here are the results.

$$I=\int_{\phi_i}^{\phi_{i-1}}\frac{d\phi}{\sqrt{1-k_i^2\sin^2\phi}\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}}$$
$$=\frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\{F(p_i,\gamma_{i-1})-F(p_i,\gamma_i)\} \quad \text{(B-4)}$$

$$II=\int_{\phi_i}^{\phi_{i-1}}\frac{(1-k_i^2\sin^2\phi)d\phi}{\sqrt{1-k_i^2\sin^2\phi}\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}}$$
$$=\int_{\gamma_i}^{\gamma_{i-1}}\frac{1-\frac{k_i^2(1-q_i^2)}{q_i^2}\left(\frac{1}{1-q_i^2\sin^2\gamma}-1\right)}{\frac{\sqrt{1-p_i^2\sin^2\gamma}\quad\sqrt{a_i}}{\sqrt{1-q_i^2\sin^2\gamma}\sqrt{1-q_i^2\sin^2\gamma}}}\frac{\sqrt{1-q_i^2}}{(1-q_i^2\sin^2\gamma)}d\gamma$$
$$=\frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\int_{\gamma_i}^{\gamma_{i-1}}\left[\left\{1+\frac{k_i^2(1-q_i^2)}{q_i^2}\right\}\frac{1}{\sqrt{1-p_i^2\sin^2\gamma}}\right.$$
$$\left.-\frac{k_i^2(1-q_i^2)}{q_i^2}\frac{1}{(1-q_i^2\sin^2\gamma)\sqrt{1-p_i^2\sin^2\gamma}}\right]d\gamma$$
$$=\frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\left[\left\{1+\frac{k_i^2(1-q_i^2)}{q_i^2}\right\}\{F(p_i,\gamma_{i-1})-F(p_i,\gamma_i)\}\right.$$
$$\left.-\frac{k_i^2(1-q_i^2)}{q_i^2}\{G(p_i,\gamma_{i-1})-G(p_i,\gamma_i)\}\right] \quad \text{(B-5)}$$

$$III=\int_{\phi_i}^{\phi_{i-1}}\frac{(1-k_i^2\sin^2\phi)^2d\phi}{\sqrt{1-k_i^2\sin^2\phi}\sqrt{1-\mu+\mu k_i^2+\mu k_i^2\sin^2\phi}}$$
$$=\frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\int_{\gamma_i}^{\gamma_{i-1}}\frac{\left\{1+\frac{k_i^2(1-q_i^2)}{q_i^2}-\frac{k_i^2(1-q_i^2)}{q_i^2}\frac{1}{1-q_i^2\sin^2\gamma}\right\}^2}{\sqrt{1-p_i^2\sin^2\gamma}}d\gamma$$
$$=\frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\int_{\gamma_i}^{\gamma_{i-1}}\frac{\left\{1+\frac{k_i^2(1-q_i^2)}{q_i^2}\right\}^2-2\left\{1+\frac{k_i^2(1-q_i^2)}{q_i^2}\right\}}{\sqrt{1-p_i^2\sin^2\gamma}}$$
$$\frac{\frac{k_i^2(1-q_i^2)}{q_i^2}\frac{1}{1-q_i^2\sin^2\gamma}+\left\{\frac{k_i^2(1-q_i^2)}{q_i^2}\right\}^2\left(\frac{1}{1-q_i^2\sin^2\gamma}\right)^2}{\sqrt{1-q_i^2\sin^2\gamma}}d\gamma$$
$$=\frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\left[\left\{1+\frac{k_i^2(1-q_i^2)}{q_i^2}\right\}^2[F(p_i,\gamma_{i-1})-F(p_i,\gamma_i)]\right.$$

$$-2\left\{1+\frac{k_i^2(1-q_i^2)}{q_i^2}\right\}\frac{k_i^2(1-q_i^2)}{q_i^2}[G(p_i,q_i,\gamma_{i-1})-G(p_i,q_i,\gamma_i)]$$

$$+\left\{\frac{k_i^2(1-q_i^2)}{q_i^2}\right\}^2[H(p_i,q_i,\gamma_{i-1})-H(p_i,q_i,\gamma_i)]\Bigg\} \quad \text{(B-6)}$$

where $\tan\phi_i = \sqrt{1-q_i^2}\tan\gamma_i$, $\tan\phi_{i-1} = \sqrt{1-q_i^2}\tan\gamma_{i-1}$ and $F(p,\gamma), G(p,q,\gamma), H(p,q,\gamma)$ are defined as

$$F(p,\gamma) = \int_0^{\gamma}\frac{dx}{\sqrt{1-p^2\sin^2 x}}$$

Legendre-Jacobian elliptic integral of first kind

$$G(p,q,\gamma) = \int_0^{\gamma}\frac{dx}{(1-q^2\sin^2 x)\sqrt{1-p^2\sin^2 x}}$$

Legendre-Jacobian elliptic integral of third kind

$$H(p,q,\gamma) = \int_0^{\gamma}\frac{dx}{(1-q^2\sin^2 x)^2\sqrt{1-p^2\sin^2 x}}$$

Integral function similart to Legendre-Jacobian elliptic integral of third kind

The integral IV and V can be calculated analytically.

$$IV = \int_{\phi_i}^{\phi_{i-1}}\frac{\sin\phi d\phi}{\sqrt{1-\mu+2\mu k_i^2-\mu k_i^2\cos^2\phi}} = \frac{1}{\sqrt{\mu k_i^2}}\int_{t_i}^{t_{i-1}}\frac{-dt}{\sqrt{\frac{1}{q_i^2}-t^2}}$$

$$= \frac{1}{\sqrt{\mu k_i^2}}\{\arcsin(q_i\cos\phi_i)-\arcsin(q_i\cos\phi_{i-1})\} \quad \text{(B-7)}$$

$$V = \int_{\phi_i}^{\phi_{i-1}}\frac{\sin\phi(1-k_i^2\sin^2\phi)d\phi}{\sqrt{1-\mu+2\mu k_i^2-\mu k_i^2\cos^2\phi}} = \frac{1}{\sqrt{\mu k_i^2}}\int_{t_i}^{t_{i-1}}\frac{-\{1-k_i^2(1-t^2)\}dt}{\sqrt{\frac{1}{q_i^2}-t^2}}$$

$$= \frac{1}{\sqrt{\mu k_i^2}}\left[\int_{t_i}^{t_{i-1}}\frac{-\left(1-k_i^2+\frac{k_i^2}{q_i^2}\right)}{\sqrt{\frac{1}{q_i^2}-t^2}}dt+\int_{t_i}^{t_{i-1}}k_i^2\sqrt{\frac{1}{q_i^2}-t^2}dt\right]$$

$$= \frac{1}{\sqrt{\mu k_i^2}}\Bigg\{\left\{1-k_i^2+\frac{k_i^2}{q_i^2}\right\}\{\arcsin(q_i\cos\phi_i)-\arcsin(q_i\cos\phi_{i-1})\}$$

$$+\frac{k_i^2}{q_i^2}\left[\frac{1}{2}\{\arcsin(q_i\cos\phi_{i-1})-\arcsin(q_i\cos\phi_i)\}\right.$$

$$\left.+\frac{1}{4}[\sin\{2\arcsin(q_i\cos\phi_{i-1})\}-\{\sin2\arcsin(q_i\cos\phi_i)\}]\right]\Bigg\} \quad \text{(B-8)}$$

Where, $t_i = \frac{1}{q_i}\cos\phi_i$, $t_{i-1} = \frac{1}{q_i}\cos\phi_{i-1}$

Appendix C : Transformation to Elliptic Integrals Type C

We also have two equations expressing infinitesimal elements of $\bar{x}$ and $\bar{y}$ from case 2

$$\frac{L\sqrt{\lambda_i}}{S_i}(\bar{x}_{i-1}-\bar{x}_i) = (-1)^{i-\delta}\int_{\varphi_{i-1}}^{\varphi_i}\{\sin\alpha\cos2\varphi-\mu\sin\alpha\cos^2 2\varphi$$

$$+\sin2\varphi\cos\alpha-\mu\cos\alpha\cos2\varphi\sin2\varphi)$$

$$\times\frac{k_i d\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{1-\mu+\frac{\mu}{k_i^2}+\mu\sin^2\varphi}} \quad \text{(C-1a)}$$

$$\frac{L\sqrt{\lambda_i}}{S_i}(\bar{y}_{i-1}-\bar{y}_i) = (-1)^{i-\delta}\int_{\varphi_{i-1}}^{\varphi_i}\{-\cos\alpha\cos2\varphi+\mu\cos\alpha\cos^2 2\varphi$$

$$+\sin2\varphi\cos\alpha-\mu\sin\alpha\cos2\varphi\sin2\varphi)$$

$$\times\frac{k_i d\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{1-\mu+\frac{\mu}{k_i^2}+\mu\sin^2\varphi}} \quad \text{(C-1b)}$$

Similarly, four different types of integrals can be found as

$$I = \int_{\varphi_{i-1}}^{\varphi_i}\frac{\cos2\varphi d\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{a_i+\mu\sin^2\varphi}}$$

$$II = \int_{\varphi_{i-1}}^{\varphi_i}\frac{\cos^2 2\varphi d\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{a_i+\mu\sin^2\varphi}}$$

$$III = \int_{\varphi_{i-1}}^{\varphi_i}\frac{\sin2\varphi d\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{a_i+\mu\sin^2\varphi}}$$

$$IV = \int_{\varphi_{i-1}}^{\varphi_i}\frac{\sin2\varphi\cos2\varphi d\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{a_i+\mu\sin^2\varphi}}$$

The integrals I, II can be transformed to elliptic integrals under the following transformation

$$\tan\varphi = \sqrt{1-q_i^2}\tan\gamma \quad \text{(C-2)}$$

$$q_i^2 = \frac{\mu}{a_i+\mu}, a_i = 1-\mu+\frac{\mu}{k_i^2}, p_i^2 = q_i^2+k_i^2-q_i^2k_i^2 \quad \text{(C-3)}$$

Here are the results.

$$I = \int_{\varphi_{i-1}}^{\varphi_i}\frac{1-2\sin^2\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{a_i+\mu\sin^2\varphi}}d\varphi$$

$$= \frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\int_{\gamma_{i-1}}^{\gamma_i}\frac{1}{\sqrt{1-p_i^2\sin^2\gamma}}\left\{1-2\frac{1-q_i^2}{q_i^2}\left(\frac{1}{1-q_i^2\sin^2\gamma}-1\right)\right\}d\varphi$$

$$= \frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\left[\left\{1+\frac{2(1-q_i^2)}{q_i^2}\right\}\int_{\gamma_{i-1}}^{\gamma_i}\frac{d\gamma}{\sqrt{1-p_i^2\sin^2\gamma}}\right.$$

$$-\left\{\frac{2(1-q_i^2)}{q_i^2}\right\}\int_{\gamma_{i-1}}^{\gamma_i}\frac{d\gamma}{(1-q_i^2\sin^2\gamma)\sqrt{1-p_i^2\sin^2\gamma}}\Bigg]$$

$$=\frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\Bigg[\left\{1+\frac{2(1-q_i^2)}{q_i^2}\right\}\{F(p_i,\gamma_i)-F(p_i,\gamma_{i-1})\}$$

$$-\left\{\frac{2(1-q_i^2)}{q_i^2}\right\}\{G(p_i,q_i,\gamma_i)-G(p_i,q_i,\gamma_{i-1})\}\Bigg] \qquad \text{(C-4)}$$

$$II=\int_{\varphi_{i-1}}^{\varphi_i}\frac{(1-2\sin^2\varphi)^2d\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{a_i+\mu\sin^2\varphi}}=\frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\int_{\gamma_{i-1}}^{\gamma_i}\frac{1}{\sqrt{1-p_i^2\sin^2\gamma}}$$

$$\left\{1+\frac{2(1-q_i^2)}{q_i^2}-\frac{2(1-q_i^2)}{q_i^2}\frac{1}{1-q_i^2\sin^2\gamma}\right\}^2d\gamma$$

$$=\frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\int_{\gamma_{i-1}}^{\gamma_i}\frac{1}{\sqrt{1-p_i^2\sin^2\gamma}}\Bigg[\left\{1+\frac{2(1-q_i^2)}{q_i^2}\right\}^2-2\left\{1+\frac{2(1-q_i^2)}{q_i^2}\right\}$$

$$\frac{2(1-q_i^2)}{q_i^2}\frac{1}{1-q_i^2\sin^2\gamma}+\left\{\frac{2(1-q_i^2)}{q_i^2}\right\}^2\frac{1}{(1-q_i^2\sin^2\gamma)^2}\Bigg]d\gamma$$

$$=\frac{\sqrt{1-q_i^2}}{\sqrt{a_i}}\Bigg[\left\{1+\frac{2(1-q_i^2)}{q_i^2}\right\}^2\{F(p_i,\gamma_i)-F(p_i,\gamma_{i-1})\}$$

$$-2\left\{1+\frac{2(1-q_i^2)}{q_i^2}\right\}\frac{2(1-q_i^2)}{q_i^2}\{G(p_i,q_i,\gamma_i)-G(p_i,q_i,\gamma_{i-1})\}$$

$$+\left\{\frac{2(1-q_i^2)}{q_i^2}\right\}^2\{H(p_i,q_i,\gamma_i)-H(p_i,q_i,\gamma_{i-1})\}\Bigg] \qquad \text{(C-5)}$$

where $\tan\varphi_i=\sqrt{1-q_i^2}\tan\gamma_i$, $\tan\varphi_{i-1}=\sqrt{1-q_i^2}\tan\gamma_{i-1}$. The integral III and IV can be calculated analytically.

$$III=\int_{\varphi_{i-1}}^{\varphi_i}\frac{\sin2\varphi d\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{a_i+\mu\sin^2\varphi}}=\int_{t_{i-1}}^{t_i}\frac{dt}{\sqrt{(1-k_i^2t)(a_i+\mu t)}}$$

$$=\frac{1}{\sqrt{uk_i^2}}\int_{t_{i-1}}^{t_i}\frac{dt}{\sqrt{r_i^2-\left\{t-\frac{1}{2}\left(\frac{1}{k_i^2}-\frac{a_i}{\mu}\right)\right\}^2}}=\frac{1}{\sqrt{uk_i^2}}(\chi_i-\chi_{i-1}) \qquad \text{(C-6)}$$

$$IV=\int_{\varphi_{i-1}}^{\varphi_i}\frac{\sin2\varphi\cos2\varphi d\varphi}{\sqrt{1-k_i^2\sin^2\varphi}\cdot\sqrt{a_i+\mu\sin^2\varphi}}=\int_{t_{i-1}}^{t_i}\frac{(1-2t)dt}{\sqrt{1-k_i^2t}\cdot\sqrt{a_i+\mu t}}$$

$$=\frac{1}{\sqrt{uk_i^2}}\int_{t_{i-1}}^{t_i}\frac{(1-2t)dt}{\sqrt{r_i^2-\left\{t-\frac{1}{2}\left(\frac{1}{k_i^2}-\frac{a_i}{\mu}\right)\right\}^2}}$$

$$=\frac{1}{\sqrt{uk_i^2}}\int_{\chi_{i-1}}^{\chi_i}\left(1-\frac{1}{k_i^2}+\frac{a_i}{u}-2r_i\sin\chi\right)d\chi$$

$$=\frac{1}{\sqrt{uk_i^2}}\left\{\left(1-\frac{1}{k_i^2}+\frac{a_i}{u}\right)(\chi_i-\chi_{i-1})+2r_i(\cos\chi_i-\cos\chi_{i-1})\right\} \qquad \text{(C-7)}$$

where $\chi_i=\arcsin\left[\frac{1}{r_i}\left\{\sin^2\varphi_i-\frac{1}{2}\left(\frac{1}{k_i^2}-\frac{a_i}{u}\right)\right\}\right]$,

$$\chi_{i-1}=\arcsin\left[\frac{1}{r_i}\left\{\sin^2\varphi_{i-1}-\frac{1}{2}\left(\frac{1}{k_i^2}-\frac{a_i}{u}\right)\right\}\right]$$

$$\sin^2\varphi=t,\quad t-\frac{1}{2}\left(\frac{1}{k_i^2}-\frac{a_i}{u}\right)=r_i\sin\chi\left(-\frac{\pi}{2}\le\chi\le\frac{\pi}{2}\right),$$

$$r_i=\sqrt{\frac{a_i}{\mu k_i^2}+\frac{1}{4}\left(\frac{1}{k_i^2}-\frac{a_i}{\mu}\right)^2} \qquad \text{(C-8)}$$