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On the convergence in the sense of α-level on a fuzzy normed linear space

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Abstract

In this paper, we introduce the notions of the convergence in the sense of α-level and the fuzzy completeness on a fuzzy normed linear space. And we investigate related properties

Key Words : fuzzy norm, convergence in the sense of α-level, Cauchy sequence in the sense of α-level, fuzzy completeness

1. Introduction

Katsaras and Liu [3] introduced the notions of fuzzy vector spaces and fuzzy topological vector spaces. These ideas were modified by Katsaras [1] and in [2] Katsaras defined the fuzzy norm on a vector space. In [4] Krishna and Sarma discussed the generation of a fuzzy vector topology from an ordinary vector topology on a vector space. Also Krishna and Sarma [5] observed the convergence of sequence of fuzzy points. Rhie et. al[8] introduced the notion of fuzzy α-Cauchy sequence of fuzzy points and fuzzy completeness.

Since the concept of the completeness is essential to discribe the aspects of normed linear space relative to the closedness of a space, there may be rich applications for fuzzyfying Banach spaces if the concept of a new type of the fuzzy completeness is introduced in a fuzzy normed linear space. In this paper, we introduce the notions of the convergence of sequence in the sense of α-level and Cauchy sequence in the sense of α-level on a fuzzy normed space and fuzzy completeness, as a generalization of those in ordinary normed linear spaces. And we investigate related properties.

2. Preliminaries.

Throughout this paper, X is a vector space over the field $K(R\ or C)$.

Fuzzy subsets of X are denoted by Greek letters in general. $\chi_A$denotes the characteristic function of the set A.

Definition 2.1[3]. For two fuzzy subsets μ_1 and μ_2 of X, the fuzzy subset $\mu_1+\mu_2$ is defined by

$$(\mu_1+\mu_2)(x)=sup_{x_1+x_2=x}\min \mu_1(x_1),\mu_2(x_2)$$

And for a scalar t of K and a fuzzy subset μ of X, the fuzzy subset $t\mu$ is defined by

$$(t\mu)(x)=\begin{cases}\mu(\frac{x}{t}) & \text{if } t\neq 0\\ 0 & \text{if } t=0 \text{ and } x\neq 0\\ sup_{y\in X}\ \mu(y) & \text{if } t=0 \text{ and } x=0\end{cases}$$

Definition 2.2 [1]. $\mu\in I^X$ is said to be

i) convex if $t\mu+(1-t)\mu\leq\mu$ for each $t\in[0,1]$,

ii) balanced if $t\mu\leq\mu$ for each $t\in K$ with $|t|\leq 1$,

iii) absorbing is $sup_{t>0}t\mu(x)=1$ for all $x\in X$.

Definition2.3[1]. Let (X,τ) be a topological space and $\omega(\tau)=\{f:(X,\tau)\to[0,1]\mid f$ is lower semicontinuous $\}$. Then $\omega(\tau)$ is a fuzzy topology on X. This topology is called the fuzzy topology generated by τ on X. The fuzzy usual topology on K means the fuzzy topology generated by the usual topology of K.

Definition 2.4 [1]. A fuzzy linear topology on a vector space X over K is a fuzzy topology on X such that the two mappings

$+:X\times X\to X,\ (x,y)\to x+y$

$\cdot:K\times X\to X,\ (t,x)\to tx$

are continuous when K has the fuzzy usual topology. A linear space with a fuzzy linear topology is called a fuzzy topological linear space or a fuzzy topological vector space.

Definition 2.5[1]. Let now X be a fuzzy topological space and $x\in X$. A fuzzy set μ in X is called a neighborhood of x if there exists an open fuzzy set ρ with $\rho\leq\mu$ and $\rho(x)=\mu(x)>0$. Warren has proved in [9] that a fuzzy set μ in X is open iff μ is a neighborhood of x for each $x\in X$ with $\mu(x)>0$.

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Theorem 2.6[1]. Let μ be a neighborhood of $z_0=x_0+y_0$ in a fuzzy topological vector space X. Then, for each real number θ with $0<\theta<\mu(z_0)$ there exist open neighborhoods μ_1, μ_2 of the points x_0, y_0 respectively, such that $\mu_1+\mu_2\leq\mu$ and $\min \mu_1(x_0), \mu_2(y_0)>\theta$. In case $x_0=y_0=0$, there exists an open neighborhood μ_3 of zero with $\mu_3(0)>\theta$, $\mu_3+\mu_3\leq\mu$.

Definition 2.7[1]. Let x be a point in a fuzzy topological space X. A family F of neighborhoods of x is called a base for the system of all neighborhoods of x if for each neighborhood μ of x and each $0<\theta<\mu(x)$ there exists $\mu_1\in$ F with $\mu_1\leq\mu$ and $\mu_1(x)>\theta$.

Definition 2.8[2]. A fuzzy seminorm on X is a fuzzy set ρ in X which is convex,balanced and absorbing. If in addition $inf_{t>0}t\rho(x)=0$ for every nonzero x, then ρ is called a fuzzy norm.

Theorem 2.9[2]. If ρ is a fuzzy seminorm on X, then the family $B_\rho=\{\theta\wedge t\rho \mid 0<\theta\leq 1, t>0\}$ is a base for a fuzzy linear topology τ_ρ, where $\theta\wedge t\rho$ is the function $X\to[0,1]$ such that $\theta\wedge t\rho(x)=\min.\{\theta, \rho(\frac{x}{t})\}$

Definition 2.10[2]. Let ρ be a seminorm on a linear space. The fuzzy topology τ_ρ in Theorem2.9 is called the fuzzy topology induced by the fuzzy seminorm ρ. And a linear space equipped with a fuzzy seminorm(resp.fuzzy norm) is called a fuzzy seminormed(resp.fuzzy normed) linear space.

3. Fuzzy convergence and fuzzy completeness

In this section, we introduce the notions of the convergence of sequence in the sense of α-level and fuzzy completeness on a fuzzy normed linear space. And we investigate related properties.

Definition 3.1. Let (X,ρ) be a fuzzy normed linear space and $\alpha\in(0,1)$. A sequence $\{x_n\}\subset X$ is said to converge to a point $x\in X$ in the sense of α-level if and only if for every neighborhood N of zero with $N(0)>\alpha$, there exists a positive integer M such that $n\geq M$ implies $N(x_n-x)>\alpha$ (denoted by $x_n \xrightarrow{\alpha} x$) and x is said to be a limit of $\{x_n\}$ in the sense of α-level.

Theorem 3.2. Let (X,ρ) be a fuzzy normed linear space and $\alpha\in(0,1)$. Then

(a) If $x_n \xrightarrow{\alpha} x$ and $y_n \xrightarrow{\alpha} y$, then

$$x_n+y_n \xrightarrow{\alpha} x+y.$$

(b) If $t\in K$ and $x_n \xrightarrow{\alpha} x$, then $tx_n \xrightarrow{\alpha} tx$.

Proof. (a) Let N be a neighborhood of zero with $N(0)>\alpha$. Then there exists an open neighborhood N_1 such that $N_1+N_1\leq N$ and $N(0)>\alpha$ by Theorem 2.6. Since $x_n \xrightarrow{\alpha} x$ and $y_n \xrightarrow{\alpha} y$, there exist two positive integers M_1, M_2 such that

$n\geq M_1$, implies $N_1(x_n-x)>\alpha$ and

$n\geq M_2$ implies $N_1(y_n-y)>\alpha$.

Let $M=\max M_1, M_2$and $n\geq M$. Then

$$\begin{aligned} &N((x_n+y_n)-(x+y)) \\ &\geq (N_1+N_1)((x_n+y_n)-(x+y)) \\ &=(N_1+N_1)((x_n-x)+(y_n-y)) \\ &\geq \min N_1(x_n-x), N_1(y_n-y))>\alpha \end{aligned}$$

Therefore $\{x_n+y_n\}$ converges to $x+y$ in the sense of α-level.

(b) If $t=0$, then it is clear. Let $t\neq0$. Since for every neighborhood of zero N with $N(0)>\alpha$, $\frac{1}{t}N$ is also a neighborhood of zero with $\frac{1}{t}N(0)=N(0)>\alpha$, and $\{x_n\}$ converges to x in the sense of α-level, there exists a positive integer M such that $n\geq M$ implies $N(tx_n-tx)=\frac{1}{t}N(x_n-x)>\alpha$. Therefore $\{tx_n\}$ converges to tx in the sense of α-level. This completes the proof.

Now, we will prove that the limit in the sense of α-level is unique. For the proof, we begin with following two lemmas.

Lemma 3.3. Let (X,ρ) be a fuzzy normed linear space, $x\in X$ and $\alpha\in(0,1)$. If for every $t>0$, $t\rho(x)>\alpha$, then x is the zero vector of the space X.

Proof. Suppose that x is not zero vector. Since for every $t>0$, $t\rho(x)>\alpha, inf_{t>0}t\rho(x)\geq\alpha>0$. This contradicts to the fact that ρ is a fuzzy norm on X. Hence x is the zero vector of X.

Lemma 3.4. Let (X,ρ)be a fuzzy normed linear space and $x\in X$. If for each neighborhood of zero N with $N(0)>\alpha$, $N(x)>\alpha$, then x is the zero vector of X.

Proof. Fix a $\theta>\alpha$. Since for every $t>0$, $\theta\wedge t\rho$ is a neighborhood of zero and $\theta\wedge t\rho(0)=\theta\wedge\rho(0)=\theta\wedge1>\alpha$, $\theta\wedge t\rho(x)>\alpha$ for all $t>0$. This implies that for every $t>0$, $t\rho(x)>\alpha$. By the above lemma, x is the zero vector of X.

Theorem 3.5. The limit in the sense of α-level of a sequence $\{x_n\}$ is unique.

.Proof. Suppose that $\{x_n\}$ converges to x and x' in the sense of α-level. If N is a neighborhood of zero with $N(0)>\alpha$, then there exist two positive integers M_1 and M_2 such that

$n\geq M_1$ implies $N(x_n-x)>\alpha$ and

$n\geq M_2$ implies $N(x_n-x')>\alpha$.

Since N is a neighborhood of zero and $N(0)>\alpha$, there exists a neighborhood of zero N_1 such that $N_1(0)>\alpha$, $N_1+N_1\leq N$ by Theorem 2.6. Now, we have

$$N(x-x') \geq (N_1+N_1)(x-x')$$
$$=(N_1+N_1)((x-x_n)+(x_n-x'))$$
$$\geq \min\{N_1(x_n-x), N_1(x_n-x')\} > \alpha \quad \text{for all}$$

$n\geq \max M_1, M_2$.

By the above lemma, we get $x-x'=0$ equivalently $x=x'$. This completes the proof.

Definition 3.6. Let (X,ρ) be a fuzzy normed linear space and $\alpha\in(0,1)$. A sequence $\{x_n\}$ is said to be a Cauchy sequence in the sense of α-level if and only if for every neighborhood of zero N with $N(0)>\alpha$, there exists a positive integer M such that $n, m\geq M$ implies $N(x_n-x_m)>\alpha$.

Theorem 3.7. Let (X,ρ) be a fuzzy normed linear space and $\alpha\in(0,1)$. Then every convergent sequence in the sense of α-level in (X,ρ) is a Cauchy sequence in the sense of α-level.

Proof. Let $\alpha\in(0,1)$ and $\{x_n\}$ converge to a point $x\in X$ in the sense of α-level. Then for every neighborhood of zero N with $N(0)>\alpha$, there exists a positive integer M such that $n\geq M$ implies $N(x_n-x)>\alpha$. Let a neighborhood of zero N be given and $N(0)>\alpha$. Then there exists a neighborhood of zero N_1 such that $N_1+N_1\leq N$ and $N_1(0)>\alpha$ by Theoreom 2.6. Since N_1 is a neighborhood of zero and $N_1(0)>\alpha$, there exists a positive integer M_1 such that

$n\geq M_1$ implies $N_1(x_n-x)>\alpha$.

Now, we have

$$N(x_n-x_m) \geq (N_1+N_1)(x_n-x_m)$$
$$=(N_1+N_1)((x_n-x)+(x-x_m))$$
$$\geq \min\{N_1(x_n-x), \quad N_1(x-x_m)\} > \alpha \quad \text{for all}$$

$n, m\geq M_1$.

Therefore $\{x_n\}$ is a Cauchy sequence in th sense of α-level. This completes the proof.

Now , we consider some relations between the fuzzy completeness and ordinary completeness on a linear space.

Definition 3.8. A fuzzy normed linear space (X,ρ) is said to be complete in the sense of α-level if and only if every Cauchy sequence in the sense of α-level $\{x_n\}$ converges to a point $x\in X$.

(X,ρ) is said to be fuzzy complete if it is complete in the sense of α-level for all $\alpha\in(0,1)$.

Lemma 3.9. Let $(X, \|\cdot\|)$ be a normed linear space and B is the closed unit ball of X. Then every Cauchy sequence in the sense of α-level on the fuzzy normed linear space (X,χ_B) is a Cauchy sequence with respect to the ordinary norm.

Proof. Let $\varepsilon>0$ be given. Since $\theta\wedge\frac{\varepsilon}{2}\chi_B(0)>\alpha$ if $\theta>\alpha$, $\theta\wedge\frac{\varepsilon}{2}\chi_B$ is a neighborhood of zero with $\theta\wedge\frac{\varepsilon}{2}\chi_B(0)>\alpha$. Hence there exists a positive integer M such that $n, m\geq M$ implies

$$\theta\wedge\frac{\varepsilon}{2}\chi_B(x_n-x_m)>\alpha$$
$$\Rightarrow \frac{\varepsilon}{2}\chi_B(x_n-x_m)>\alpha$$
$$\Rightarrow \chi_B(\frac{2}{\varepsilon}(x_n-x_m))>\alpha$$
$$\Rightarrow \chi_B(\frac{2}{\varepsilon}(x_n-x_m)) = 1$$
$$\Rightarrow \|x_n-x_m\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Therefore $\{x_n\}$ is a Cauchy sequence in $(X, \|\cdot\|)$.

Theorem 3.10. Let $(X, \|\cdot\|)$ be a Banach space. Then the fuzzy normed linear space (X,χ_B) is fuzzy complete where B is the closed unit ball of X

Proof. Fix $\alpha\in(0,1)$ and let $\{x_n\}$ be a Cauchy sequence in the sense of α-level in (X,χ_B). Then it is a Cauchy sequence with respect to the ordinary norm $\|\cdot\|$ by the above lemma. Since $(X, \|\cdot\|)$ is complete, there exists an $x\in X$ such that $\|x_n-x\|\to 0$.

Now, we show that $\{x_n\}$ converges to this x in the sense of α-level in (X,χ_B). Let N be a neighborhood of zero with $N(0)>\alpha$. Then there exist $\alpha<\theta\leq 1$, $t>0$ such that $\theta\wedge t\chi_B\leq N$ because that $\{\theta\wedge t\chi_B : t>0, 0<\theta\leq 1\}$is a base at zero. For this $t>0$, there exists a positive integer M such that

$n\geq M$ implies $\|x_n-x\|<t$

$\Rightarrow$ $n\geq M$ implies $\theta\wedge t\chi_B(x_n-x)>\alpha$

$\Rightarrow$ $n\geq M$ implies $N(x_n-x)>\alpha$.

That is $\{x_n\}$ converges to x in the sense of α-level, therefore $(X,\chi_B)$is complete in the sense of α-level. Since α is arbitrary, $(X,\chi_B)$is fuzzy complete. This completes the proof.

Corollary 3.11. The field $K(R\ or\ C)$with the fuzzy topology generated by the usual topology on K is a complete fuzzy normed linear space.

Definition 3.12 [3]. Two fuzzy seminorms ρ_1, ρ_2 on X are said to be equivalent if $\tau_{\rho_1} = \tau_{\rho_2}$.

Proposition 3.13[8]. Let $(X, \| \cdot \|)$ be a normed linear space. If ρ is a lower semicontinuous fuzzy norm on X, and have the bounded support : $\{x \in X \mid \rho(x) > 0\}$ is bounded, then ρ is equivalent to the fuzzy norm χ_B where B is the closed unit ball of X.

By Theoorem 3.10 and the above proposition, we get the following theoorem.

Theorem 3.14. If X is a Banach space and ρ is a lower semicontinuous fuzzy norm having the bounded support, then the fuzzy normed linear space (X, ρ) is fuzzy complete.

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