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Controllability of the nonlinear Fuzzy Integro-Differential Equations on E_N^n

Young-Chel Kwun*, Dong-Gun Park* and Young-Chul Ahn*

*Department of mathematics, Dong-A University, Busan, Korea

Abstract

In this paper we study the controllability for the nonlinear fuzzy integro-differential equations on E_N^n by using the concept of fuzzy number of dimension n whose values are normal, convex, upper semicontinuous and compactly supported surface in R^n. E_N^n be the set of all fuzzy numbers in R^n with edges having bases parallel to axis $X_1, X_2, \cdots, X_n$.

Key Words : fuzzy number of dimension n, fuzzy control, nonlinear fuzzy integro-differential equation

1. Introduction

Many authors have studied several concepts of fuzzy systems. Kaleva[2] studied the existence and uniqueness of solution for the fuzzy differential equation on E^n where E^n is normal, convex, upper semicontinuous and compactly supported surface in R^n. Seikkala[9] proved the existence and uniqueness of fuzzy solution for the initial value problem on E^1. Subrahmanyam and Sudarsanam[10] studied fuzzy volterra-integral equation. Park etal.[8] are proved the existence and uniqueness of fuzzy solution for the nonlinear fuzzy differential equation on E_N^n with nonlocal initial condition, Kwun etal.[6] are studied controllability for the nonlinear fuzzy control system on E_N^n, where E_N^n be the set of all fuzzy numbers in R^n with edges having bases parallel to axis $X_1, X_2, \cdots, X_n$. For example E_N^2 be the set of all fuzzy pyramidal numbers in R^2 with edges having rectangular bases parallel to the axis X_1 and X_2.

Recently, Kwun etal [7] are studied the existence and uniqueness of fuzzy solutions for the following nonlinear integrodifferential equations:

$$\begin{cases} \frac{dx(t)}{dt} = a(t)x(t) + f(t, x(t), \\ \quad \int_0^t k(t,s,x(s))ds)) + u(t), \ t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $a:[0, T] \to E_N^n$ is fuzzy coefficient, initial value $x_0 \in E_N^n$ and $f:[0, T]\times E_N^n \times E_N^n \to E_N^n$ and $k:[0, T]\times[0, T]\times E_N^n \to E_N^n$ are nonlinear regular fuzzy functions.

In this paper, we consider the controllability of fuzzy nonlinear integro-differential equation (1.1).

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2. Properties of n-dimensional fuzzy numbers and metric

In this section, we give some definitions, properties and notations of the fuzzy number of dimension n.

Definition 2.1.([6, 8]) We consider a fuzzy graph $G \subset R^n$ that is functional fuzzy relation in R^n such that its membership function $m_G(x_1, x_2, \cdots, x_n) \in [0,1]$, $(x_1, x_2, \cdots, x_n) \in R^n$ has the following properties :

(1) For all $x_i \in R (i=1,2,\cdots,n)$, $m_G(x_1, x_2, \cdots, x_n) \in [0, 1]$ is a convex membership function.
(2) For all $\alpha \in [0,1]$, $\{(x_1, x_2, \cdots, x_n) \in R^n | m_G(x_1, x_2, \cdots, x_n) \geq \alpha\}$ is convex set.
(3) There exists $(x_1, x_2, \cdots, x_n) \in R^n$, $m_G(x_1, x_2, \cdots, x_n) = 1$.

If the above conditions are satisfied, the fuzzy subset G is called a fuzzy number of dimension n.

We denote by fuzzy number in E_N^n, $A=(a_1, a_2, \cdots, a_n)$ where a_i is projection of A to axis $X_i (i=1,2,\cdots,n)$, respectively. And $a_i (i=1,2, \cdots, n)$ is fuzzy number in R

Definition 2.2. The α-level set of fuzzy number in E_N^n is defined by

$$[A]^{\alpha}=\{(x_1,x_2,\cdots,x_n)\in R^n|(x_1,x_2,\cdots, x_n)\in \prod_{i=1}^{n}[a_i]^{\alpha},\ 0<\alpha\le 1\} \tag{2.2}$$

where

$$[a_i]^{\alpha}=\{x_i\in R|m_{a_i}(x_i)\ge\alpha,\ 0<\alpha\le 1\} \tag{2.3}$$

and $\prod$ is the Cartesian product of sets.

Definition 2.3. Let $A,B\in E_N^n$, for all $\alpha\in(0,1]$,

$$A=B\Leftrightarrow [A]^{\alpha}=[B]^{\alpha}.$$

Definition 2.4. Let $A,B\in E_N^n$, for all $\alpha\in(0,1]$,

$$[A*_n B]^{\alpha}=\prod_{i=1}^{n}[a_i*b_i]^{\alpha} \tag{2.4}$$

where $*_n$ is operation in E_N^n and $*$ is operation in E_N. And $(A*_nB)_i^{\alpha}=A_i^{\alpha}*B_i^{\alpha}$.

Definition 2.5. The derivative $\frac{dx(t)}{dt}\in E_N^n$ of fuzzy process $x(t)\in E_N^n$ is defined by

$$\left[\frac{dx(t)}{dt}\right]^{\alpha}=\prod_{i=1}^{n}\left[\frac{d}{dt}x_{il}^{\alpha}(t),\ \frac{d}{dt}x_{ir}^{\alpha}(t)\right],\ 0<\alpha\le 1. \tag{2.5}$$

Definition 2.6. The fuzzy integral $\int_a^b x(t)\,dt$ is defined by

$$\left[\int_a^b x(t)\,dt\right]^{\alpha}=\prod_{i=1}^{n}\left[\int_a^b x_{il}^{\alpha}(t)\,dt,\ \int_a^b x_{ir}^{\alpha}(t)\,dt\right], \tag{2.6}$$

where $x(t)\in E_N^n, a,b\in R$ and $0<\alpha\le 1$,.

Let $\prod_{i=1}^{n}[a_i]^{\alpha}$, $0<\alpha\le 1$, be a given family of nonempty areas. If

$$\prod_{i=1}^{n}[a_i]^{\beta}\subset\prod_{i=1}^{n}[a_i]^{\alpha},\ 0<\alpha<\beta\le 1 \tag{2.7}$$

and

$$\prod_{i=1}^{n}\lim_{k\to\infty}[a_i]^{\alpha_k}=\prod_{i=1}^{n}[a_i]^{\alpha} \tag{2.8}$$

whenever (α_k) is a nondecreasing sequence converging to $\alpha\in(0,1]$, then the family $\prod_{i=1}^{n}[a_i]^{\alpha}$, $0<\alpha\le 1$, represents the α-level sets of fuzzy number $A\in E_N^n$.

Conversly, if $\prod_{i=1}^{n}[a_i]^{\alpha}$, $0<\alpha\le 1$, are the α-level sets of fuzzy number R^n, then the condition (2.7) and (2.8) hold true.

We define the metric d_∞ on E_N^n and the suprimum metric H on $C([0,T]:E_N^n)$.

Definition 2.7. Let $A,B\in E_N^n$,

$$d_\infty=\sup\{d_H([A]^{\alpha},[B]^{\alpha})\mid \alpha\in(0,1]\}$$
$$=\sup\left\{\left(\sum_{i=1}^{n}(d_H([a_i]^{\alpha},[b_i]^{\alpha}))^2\right)^{\frac{1}{2}}\mid \alpha\in(0,1]\right\}$$

where d_H is Hausdorff distance and a_i, $b_i\in E_N$.

Definition 2.8. The supremum metric H on $C([0,T]:E_N^n)$ is defined by

$$H(x,y)=\sup\{d_\infty(x(t),y(t))\mid t\in[0,T]\}$$

where $x,y\in C([0,T]:E_N^n)$.

Definition 2.9. Nonlinear regular fuzzy function $f:[0,T]\times E_N^n\times E_N^n\to E_N^n$ is satisfied, $x,y\in E_N^n$,

$$\begin{aligned} f(t,[x]^{\alpha},[y]^{\alpha}) &= f\left(t,\prod_{m=1}^{n}[x_m]^{\alpha},\prod_{m=1}^{n}[y_m]^{\alpha}\right)\\ &=\prod_{m=1}^{n}f_m(t,[x_m]^{\alpha},[y_m]^{\alpha})\\ &=\prod_{m=1}^{n}f_m^{\alpha}(t,x,y)\\ &=f^{\alpha}(t,x,y).\end{aligned}$$

3. Controllability

In this section, we show the exact controllability for (1.1).

The equation (1.1) is related to the following fuzzy integral equations :

$$\begin{cases} x(t)=S(t)x_0+\int_0^t S(t-s)\{f(s,x(s), \int_0^s k(s,t,x(\tau)))\,d\tau\}ds+\int_0^t S(t-s)\,u(s)\,ds,\\ x(0)=x_0.\end{cases} \tag{3.1}$$

where $S(t)\in E_N^n$ and

$$[S(t)]^{\alpha}=\prod_{m=1}^{n}[S_m(t)]^{\alpha}=\prod_{m=1}^{n}[S_{ml}^{\alpha}(t),\ S_{mr}^{\alpha}(t)]$$

where

$$S_{mi}^{\alpha}(t)=\exp\left\{\int_0^t a_{mi}^{\alpha}(s)\,ds\right\},\quad i=l,r$$

is continuous. That is, there exists a constant $C>0$ such that $|S_{mi}^{\alpha}(t)|\le C$ for all $t\in[0,T]$.

Definition 3.1. The (1.1) is exact controllable if, there exists $u(t)$ such that the fuzzy solution $x(t)$ of (1.1) satisfies $x(T) = {}_\alpha x^1$. That is $[x(T)]^\alpha = \prod_{i=1}^{n} [x_i(T)]^\alpha)$ $= \prod_{i=1}^{n} [(x^1)_i]^\alpha = [x^1]^\alpha$, where x^1 is target set.

Defined the fuzzy mapping $\tilde{g}: \mathcal{P}(R^n) \rightarrow E_N^n$ by

$$\tilde{g}^\alpha(v) = \begin{cases} \int_0^T S^\alpha(T-s)v(s)ds, & v \subset \overline{\Gamma_u}, \\ 0, & \text{otherwise}. \end{cases}$$

Then there exists

$\tilde{g}_i: \mathcal{P}(R) \rightarrow E_N (i = 1, 2, \cdots, n)$ such that

$$\tilde{g}_i^\alpha(v_i) = \begin{cases} \int_0^T S_i^\alpha(T-s)v_i(s)ds, & v_i(s) \subset \overline{\Gamma_{u_i}}, \\ 0, & \text{otherwise} \end{cases}$$

where u_i is projection of u to axis X_i, $(i = 1, \cdots, n)$ respectively and there exists $\tilde{g}_{ij}^\alpha$ $(j = l, r)$

$$\tilde{g}_{il}^\alpha(v_{il}) = \int_0^T S_{il}^\alpha(T-s)v_{il}(s)ds,$$
$$v_{il}(s) \in [u_{il}^\alpha(s), u_i^1(s)],$$
$$\tilde{g}_{ir}^\alpha(v_{ir}) = \int_0^T S_{ir}^\alpha(T-s)v_{ir}(s)ds,$$
$$v_{ir}(s) \in [u_i^1(s), u_{ir}^\alpha(s)].$$

We assume that $\tilde{g}_{il}^\alpha$, $\tilde{g}_{ir}^\alpha$ are bijective mappings. We can be introduced $u(s)$ of nonlinear system

$$\begin{aligned} &[u(s)]^\alpha \\ = &\prod_{i=1}^{n}[u_i(s)]^\alpha = \prod_{i=1}^{n}[u_{il}^\alpha(s), u_{ir}^\alpha(s)] \\ = &\prod_{i=1}^{n}[(\tilde{g}_{il}^\alpha)^{-1}((x^1)_{il}^\alpha - S_{il}^\alpha(T)(x_0)_{il}^\alpha \\ - &\int_0^T S_{il}^\alpha(T-s)f_{il}^\alpha(s, x_{il}^\alpha(s), \int_0^s k(s,\tau,x_{il}^\alpha(\tau))d\tau)ds), \\ &(\tilde{g}_{ir}^\alpha)^{-1}((x^1)_{ir}^\alpha - S_{ir}^\alpha(T)(x_0)_{ir}^\alpha \\ - &\int_0^T S_{ir}^\alpha(T-s)f_{ir}^\alpha(s, x_{ir}^\alpha(s), \int_0^s k(s,\tau,x_{ir}^\alpha(\tau))d\tau)ds)]. \end{aligned}$$

Then substituting this expression into the (1.1) yields α-level of $x(T)$. For each $i = 1, \cdots, n$,

$$\begin{aligned} &[x_i(T)]^\alpha \\ = &[S_{il}^\alpha(T)(x_0)_{il}^\alpha + \int_0^T S_{il}^\alpha(T-s)f_{il}^\alpha(s, x_{il}^\alpha(s), \\ &\qquad \int_0^s k(s,\tau,x_{il}^\alpha(\tau))d\tau)ds \\ + &\int_0^T S_{il}^\alpha(T-s)(\tilde{g}_{il}^\alpha)^{-1}((x^1)_{il}^\alpha - S_{il}^\alpha(T)(x_0)_{il}^\alpha \\ - &\int_0^T S_{il}^\alpha(T-s)f_{il}^\alpha(s, x_{il}^\alpha(s), \int_0^s k(s,\tau,x_{il}^\alpha(\tau))d\tau)ds)ds, \\ &S_{ir}^\alpha(T)(x_0)_{ir}^\alpha + \int_0^T S_{ir}^\alpha(T-s)f_{ir}^\alpha(s, x_{ir}^\alpha(s), \\ &\qquad \int_0^s k(s,\tau,x_{ir}^\alpha(\tau))d\tau)ds \\ + &\int_0^T S_{ir}^\alpha(T-s)(\tilde{g}_{ir}^\alpha)^{-1}((x^1)_{ir}^\alpha - S_{ir}^\alpha(T)(x_0)_{ir}^\alpha \\ - &\int_0^T S_{ir}^\alpha(T-s)f_{ir}^\alpha(s, x_{ir}^\alpha(s), \int_0^s k(s,\tau,x_{ir}^\alpha(\tau))d\tau)ds)ds] \\ = &[(x^1)_{il}^\alpha, (x^1)_{ir}^\alpha] = [(x^1)_i]^\alpha \end{aligned}$$

Therefore

$$[x(T)]^\alpha = \prod_{i=1}^{n}[x_i(T)]^\alpha = \prod_{i=1}^{n}[(x^1)_i]^\alpha = [x^1]^\alpha.$$

We now set

$$\begin{aligned} (\Phi x)(t) = &{}_\alpha S(t)x_0 + \int_0^t S(t-s)f(s, x(s), \\ &\int_0^s k(s,\tau,x(\tau))d\tau)ds \\ + &\int_0^t S(t-s)\, \tilde{g}^{-1}(x^1 - S(T))x_0 \\ - &\int_0^T S(T-s)f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau)ds)ds. \end{aligned}$$

2where the fuzzy mappings $\tilde{g}^{-1}$ satisfied above statements.

Notice that $(\Phi x)(T) = {}_\alpha x^1$, which means that the control $u(t)$ steers the (F.C.S.) from the origine to x^1 in time T provided we can obtain a fixed point of the operator Φ.

Assume that the following hypotheses :

(H1) Linear system of (1.1) $f \equiv 0$ is exact controllable.

(H2) nonlinear regular fuzzy function $f: [0, T] \times E_N^n \times E_N^n \rightarrow E_N^n$ and $k: [0, T] \times [0, T] \times E_N^n \rightarrow E_N^n$ are satisfy a global Lipschitz condition, there exist $K > 0$ and $p > 0$ such that

$$\begin{aligned} &d_H(f_i^\alpha(t, x_1, y_1), f_i^\alpha(t, x_2, y_2)) \\ &\le K(d_H([x_1]^\alpha, [x_2]^\alpha) + d_H([y_1]^\alpha, [y_2]^\alpha)), \\ &d_H(k_i^\alpha(t, s, x_1), k_i^\alpha(t, s, x_2)) \le p\, d_H([x_1]^\alpha, [x_2]^\alpha), \end{aligned}$$

where $x_i, y_i \in E_N$ $(i = 1, 2)$.

Theorem 3.1. Suppose that hypotheses (H1), (H2) are satisfied. Then the state of the (1.1) can be steered from the initial value x_0 to any final state x^1 in time T.

Proof. The continuous function from $C([0, T]: E_N^n)$ to itself defined by

$$\begin{aligned} (\Phi x)(t) = &{}_\alpha S(t)x_0 + \int_0^t S(t-s)f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau)ds \\ + &\int_0^t S(t-s)\, \tilde{g}^{-1}(x^1 - S(T)x_0 \\ - &\int_0^T S(T-s)f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau)ds)ds. \end{aligned}$$

There exist $\Phi_i (i = 1, \cdots, n)$ is continuous function from $C([0, T]: E_N)$ to itself.

Let $x, y \in C([0, T]: E_N^n)$ there exist $(i = 1, \cdots, n)$ $x_i, y_i \in C([0, T]: E_N)$.

$$\begin{aligned} &d_H([\Phi_i x_i(t)]^\alpha, [\Phi_i y_i(t)]^\alpha) \\ \le\ &d_H([\int_0^t S_i(t-s)f_i(s, x_i(s), \int_0^s k(s,\tau,x_i(\tau))d\tau)ds]^\alpha, \\ &[\int_0^t S_i(t-s)f_i(s, y_i(s), \int_0^s k(s,\tau,y_i(\tau))d\tau)ds]^\alpha) \end{aligned}$$

$$+ \quad d_H([\int_0^T S_i(T-s)\ \widetilde{g}_i^{-1}(\int_0^T S_i(T-s)$$
$$f_i(s, x_i(s), \int_0^s k(s,\tau,x_i(\tau))d\tau)ds)ds]^\alpha, [\int_0^T S_i(T-s)\ \widetilde{g}_i^{-1}$$
$$(\int_0^T S_i(T-s) f_i(s, y_i(s), \int_0^s k(s,\tau,y_i(\tau))d\tau)ds)ds]^\alpha)$$
$$\le d_H(\int_0^t [S_i(t-s) f_i(s, x_i(s), \int_0^s k(s,\tau,x_i(\tau))d\tau)]^\alpha ds,$$
$$\int_0^t [S_i(t-s) f_i(s, y_i(s), \int_0^s k(s,\tau,y_i(\tau))d\tau)]^\alpha ds)$$
$$+ \ d_H([\widetilde{g}_i(\ \widetilde{g}_i^{-1}(\int_0^T S_i(T-s) f_i(s, x_i(s),$$
$$\int_0^s k(s,\tau,x_i(\tau))d\tau)ds))]^\alpha,$$
$$[\widetilde{g}_i(\ \widetilde{g}_i^{-1}(\int_0^T S_i(T-s) f_i(s, y_i(s),$$
$$\int_0^s k(s,\tau,y_i(\tau))d\tau)ds))]^\alpha)$$
$$\le \ CK(t+\frac{pt^2}{2}) d_H([x_i(t)]^\alpha, [y_i(t)]^\alpha)$$
$$+ \ CK(T+\frac{pT^2}{2}) d_H([x_i(t)]^\alpha, [y_i(t)]^\alpha)$$
$$= \ CK\{(t+\frac{pt^2}{2}) + (T+\frac{pT^2}{2})\} d_H([x_i(t)]^\alpha, [y_i(t)]^\alpha)$$

Thus

$$d_\infty(\Phi x, \Phi y) = \sup_{\alpha\in[0,1]} d_H([\Phi x)]^\alpha, [\Phi y]^\alpha)$$
$$= \sup_{\alpha\in[0,1]} \left\{ \sum_{i=1}^n (d_H([\Phi_i x_i(t)]^\alpha, [\Phi_i y_i(t)]^\alpha))^2 \right\}^{\frac{1}{2}}$$
$$= \ CK\{(t+\frac{pt^2}{2}) + (T+\frac{pT^2}{2})\}$$
$$\times \sup_{\alpha\in[0,1]} \left\{ \sum_{i=1}^n (d_H([x_i(t)]^\alpha, [y_i(t)]^\alpha))^2 \right\}^{\frac{1}{2}}$$
$$= CK\{(t+\frac{pt^2}{2}) + (T+\frac{pT^2}{2})\} \sup_{\alpha\in[0,1]} d_\infty(x, y).$$

Hence

$$H(\Phi x, \Phi y) = \sup_{t\in[0,T]} d_\infty(\Phi x, \Phi y)$$
$$\le \ \sup_{t\in[0,T]} CK\{(t+\frac{pt^2}{2}) + (T+\frac{pT^2}{2})\}\ d_\infty(x,y)$$
$$= \ 2CK(T+\frac{pT^2}{2}) H(x,y).$$

We take sufficiently small T, $2CK(T+\frac{pT^2}{2}) < 1$. Hence Φ is a contraction mapping. By the Banach fixed point theorem, Φ has fixed point.

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저 자 소 개

Young-Chel Kwun

He received the Ph.D. degree in Mathematics from Dong-A university, Korea in 1990. Since 1992, he has a professor in department of mathematics , Dong-A University, Busan, Korea.

E-mail : yckwun@dau.ac.kr

Dong-Gun Park

He received his B.S and M.S. degree Mathematics department in Dong-A university, Busan, Korea in 1996 and 1998 respectively. Currently, he is pursuing a PH. D. degree in the department of Math. at Osaka University. His research interests include semigroup theory, control theory and fuzzy control theory.

E-mail : dgpark@dau.ac.kr.

Young-Chul Ahn
He received his B.S and M.S. degree Mathematics department in Dong-A university,
Busan, Korea in 1980 and 1982 respectively. His research interests include semigroup theory, control theory and fuzzy control theory.
E-mail : math0623@hanmail.net