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Further Approximate Optimum Inspection Intervals

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Abstract. The author derives a general explicit formula and presents an heuristic algorithm for solving Baker's model. The examples show that this new approximate solution procedure for determining near optimum inspection intervals is more accurate than the ones suggested by Chung (1993) and Vaurio (1994), and is more efficient computationally than the one suggested by Hariga (1996). The construction and solution of the simplest profit model for an exponential failure distribution were presented in Baker (1990), and approximate analytical results were obtained by Chung (1993) and Vaurio (1994). The author will therefore mainly devote the following discussion to the problem of further approximating optimum inspection intervals.

Keywords: Exponential Distribution, Inspection, Repair, Cost, Profit, Machine

1. INTRODUCTION

Consider a single unit representing a manufacturing system composed of many components. In the following, the author will use the word "machine" to refer to such a single-unit or complex system. Under the superposition of the renewal processes related to the failure of the components, it is reasonable to assume that the machine's failure distribution is exponentially distributed, see Cox and Smith (1954). In fact, Drenick (1960) mathematically showed that under reasonably general conditions, distribution of the time between failures tends to the exponential as the complexity of machine structure or the time of operation increases. Moreover, many authors such as Davis (1952) and Epstein (1958) found strong empirical justification that this failure law characterizes a wide variety of devices including ball-and-roller bearings, vacuum tubes, bus engines, and many electronic systems.

Now suppose that a machine is subject to failures at random with a constant hazard λ per unit time, i.e. follows the exponential failure distribution $F(t)=1-\mathrm{e}^{-\lambda t}$ for $\lambda>0$ and $t\geq 0$, and that failures can be revealed only by periodic inspection (or testing) and then replaced. Notice that a continuous monitoring of operating states is not economically justifiable for some machines. Alternatively, inspections are useful in monitoring the machine's condition, and these can be made periodically at fixed multiples of some predetermined time interval T in order to reduce the probability of malfunction. Frequent inspection increases inspection costs while infrequent inspection leads to increasing lost production costs. Thus, an economically optimum inspection interval usually exists.

Many authors in the maintenance literature have considered different variations of this single machine inspection problem. Kamins (1960) and Coleman and Abrams (1962) extensively studied inspection procedures to maximize availability. Each considered the possibility that inspection might subject the machine to further stresses that might lead to failure. The probabilities of calling a good machine bad and a bad machine good were included in their analysis. The main difference between these two studies is that the former used the constant time T while the latter used the expected time between two successive inspections in determining machine availability. Other distinguished studies are for example, Jacobs (1968), Vaurio (1979), Voelker (1980), McWilliams and Martz (1980), Sim (1985), Lam (1995, 2003) and Banerjee and Chuiv (1996). A rather detailed literature review on inspection-scheduling problems proposed by Barlow *et al.* (1963) and Brender (1963) can be found in Leung (2001).

Baker (1990) proposed the simplest model, based on the restrictive assumption that a failure completely halts production, for finding the optimum inspection frequency that generates maximum profit. Chung (1993), Vaurio (1994) and Hariga (1996) subsequently developed approximate solution procedures for Baker's model.

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Assuming that a perfect maintenance policy is followed instead of performing just an inspection at the end of each cycle, Hariga (1996) generalized Baker's model by allowing the failure time to follow a general type of distribution such as Weibull. Under the exponential shifting time from the in-control state where items of perfect quality are produced to an out-of-control state where items of sub-standard quality are produced (see e.g. Lee and Rosenblatt 1987), Ben-Daya and Hariga (1998) and Hariga and Al-Fawzan (2000) reformulated Baker's model respectively by incorporating constant inspection and replacement times and by using the concept of discounted cash flow analysis to account for the effects of the time value of money on inspection policies.

For easy reference, the author restates the essence of Baker's model in the next section. In the rest of this article, he will: (1) propose a more near optimum solution procedure for Baker's model; (2) give three typical examples to show that this new procedure is a more accurate approximation than the ones put forward by Chung (1993) and Vaurio (1994), and more efficient computationally than the one suggested by Hariga (1996); and (3) conclude with a possible application of the procedure.

2. THE EXPECTED AND MAXIMUM PROFIT RATES, AND THE MAXIMUM CONDITION

Let a be the profit per unit time while the machine is operating and b be the cost of replacement if the machine is found to have failed, where $a, b \geq 0$. We assume that all replacements are equally expensive, that a failure completely halts production until the next inspection and replacement, and that each replacement restores the machine to the as-good-as new state. Let c be the cost of each periodic inspection, where $c \geq 0$. Now, suppose that the machine is inspected with periodic time T between two successive inspections. The expected profit $P(T)$ over one inspection interval T is given by Baker (1990):

$$P(T) = aT\int_T^{\infty} \lambda e^{-\lambda t} dt + a\int_0^T t\lambda e^{-\lambda t} dt - b\int_0^T \lambda e^{-\lambda t} dt - c .$$

Working out the integrals gives

$$P(T) = \left(\frac{a}{\lambda} - b\right)\left(1 - e^{-\lambda T}\right) - c .$$

Because of the memoryless property of the exponential distribution, the expected profit rate (or per unit time) using the renewal reward theorem (see e.g. Ross 1996) is given by

$$z(T) = \frac{P(T)}{T} = \frac{1}{T}\left[\left(\frac{a}{\lambda} - b\right)\left(1 - e^{-\lambda T}\right) - c\right]. \quad (1)$$

To find the value $T = T_a$ that yields maximum profit rate, we differentiate equation (1) and set the derivation equal to zero. This yields the maximum condition of equation (1), namely

$$(1 + x_a)e^{-x_a} = 1 - d , \quad (2)$$

where $x_a = \lambda T_a$ and $d = \dfrac{c}{\frac{a}{\lambda} - b}$.

The machine is profitable only if the expected profit until failure is greater than the replacement cost, i.e. $\frac{a}{\lambda} > b$. To cover the inspection cost as well, we must have $\frac{a}{\lambda} > b + c$ or $0 < d < 1$. With this condition, Hariga (1996) showed the existence and uniqueness of both the break-even inspection interval T_b and the optimum inspection interval T_a. Hence, a solution $x_a \geq 0$ exists. Once x_a is found, T_a is simply $\frac{x_a}{\lambda}$, and equation (1) yields the maximum profit rate

$$z(T_a) = \frac{a - (b + c)\lambda}{1 + \lambda T_a} . \quad (3)$$

Since derivation of equation (3) is not so obvious, it is given in the Appendix.

The emphasis in this article is to find accurate approximate solutions $x_a = \lambda T_a$ of equation (2) and then determine the maximum profit rate using equation (3).

3. MORE ACCURATE APPROXIMATE OPTIMUM INSPECTION INTERVALS

A second degree Taylor series approximation for e^x is given by

$$e^x \cong 1 + x + \frac{x^2}{2} . \quad (4)$$

Putting $e^{-x} \cong 1 - x + \frac{x^2}{2}$ into equation (2) and ignoring the cubic term which is valid for small x_a yields

$$x_s = \sqrt{2d} , \quad (5)$$

which is equation (4) in Vaurio (1994).

Chung (1993) used the approximation

$$e^x \cong \frac{2 + x}{2 - x} . \quad (6)$$

By dividing, equation (6) can be written as

$$e^x \cong 1 + x + \frac{x^2}{2 - x} . \quad (7)$$

Replacing e^{-x} by the inverse of equation (6) in equation (2) yields a quadratic equation with the solution

$$x_c = \frac{d + \sqrt{d(d+8)}}{2}, \tag{8}$$

which is equation (5) in Vaurio (1994).

Vaurio (1994) used the more accurate approximation

$$e^x \cong \frac{1 + \frac{2}{3}x + \frac{1}{6}x^2}{1 - \frac{1}{3}x}. \tag{9}$$

By dividing, equation (9) can be written as

$$e^x \cong 1 + x + \frac{x^2}{2 - \frac{2}{3}x}. \tag{10}$$

Replacing e^{-x} by the inverse of equation (9) in equation (2) yields another quadratic equation with the solution

$$x_v = \frac{2d + \sqrt{2d(9-d)}}{3-d}. \tag{11}$$

Note that equations (9) and (11) correspond to equations (6) and (7) in Vaurio (1994).

The author deduces from equations (4), (7) and (10) that the general form of approximation for e^x is given by

$$e^x \cong 1 + x + \frac{x^2}{2 - fx}, \quad \text{for} \quad 0 \le f \le 1. \tag{12}$$

Equation (12) can be written as

$$e^x \cong \frac{2 + (2-f)x + (1-f)x^2}{2 - fx}. \tag{13}$$

In particular, replacing e^{-x} by the inverse of equation (13) with $f = 1$ and $\frac{2}{3}$ in equation (2), we can obtain equations (8) and (11) respectively.

In general, replacing e^{-x} by the inverse of equation (13) in equation (2) yields a quadratic equation with the solution

$$x_f = \frac{d(2-f) + \sqrt{[d(2-f)]^2 + 8d(1-d+df)}}{2(1-d+df)}. \tag{14}$$

The derivation of equation (14) is given in the Appendix.

Putting $f = 0$ in equation (14), we have

$$x_0 = \frac{d + \sqrt{d(2-d)}}{1-d}.$$

Note that the formula for x_s is not the same as that for x_0 because the latter imposes no restriction on the value of x_a in the approximation.

Since x_f is a near optimum value, from equation (2) we have

$$(1 + x_f)e^{-x_f} \cong 1 - d.$$

The following two theorems provide conditions by which an heuristic algorithm, introduced below, is devised.

Theorem 1. $g(x_f)$ is a strictly decreasing function with respect to $x_f > 0$, where

$$g(x_f) = (1 + x_f)e^{-x_f}. \tag{15}$$

The proof of Theorem 1 is given in the Appendix.

Figure 1 shows the curve of $g(x_f)$ versus x_f. A quick but quite inaccurate x_a (especially for 1–d close to 0 such as Example 3) can be obtained from the enlarged Figure 1 which is attached at the end of the Appendix.

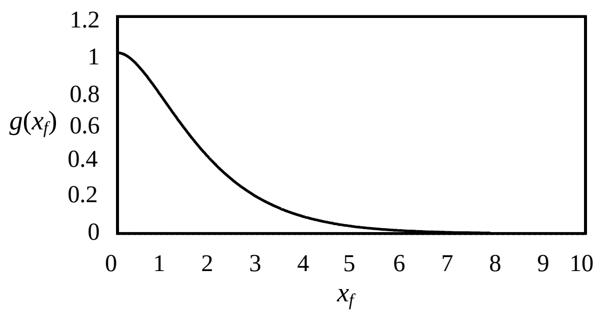


Figure 1.

Theorem 2. x_f is a strictly decreasing function with respect to f in the interval [0, 1].

The proof of Theorem 2 is given in the Appendix.

Figure 2 shows the curve of x_f versus f when d = 0.5940, see Example 1 below.

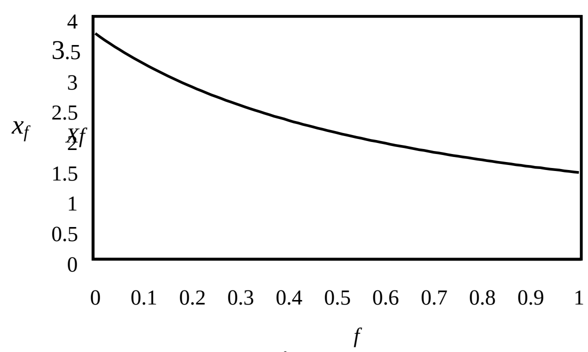


Figure 2.

4. AN HEURISTIC ALGORITHM

The procedure the author proposes to solve equation (2) works as follows:

(1) As an initial start, we may set $f = \frac{1}{2}$.

(2) Compute x_f and $g(x_f)$, correct to 4 significant figures, using equations (14) and (15) respectively.
(3) When $|g(x_f)-1+d|<10^{-3}$, stop.
(4) When $g(x_f)$ < (or >) 1 − d, set f greater (or smaller) than the value assigned in step (2). This revision is due to Theorems 1 and 2. The new f value is revised using the bisection method plus some fine adjustment. Then go to step (2).

The computations in equations (14) and (15) can easily be mechanized with a hand-held programmable calculator, with which the near optimum inspection interval x_f and the absolute error $|g(x_f)-1+d|$ can be computed "with the push of a button", and no tables and graphs (such as Table I in Baker 1990 and the enlarged Figure 1) have to be consulted. The algorithm is illustrated by the following three typical examples.

Example 1
Given that d = 0.5940; hence 1 – d = 0.4060.
From Table I in Vaurio (1994), we obtain

$$x_a = 2,\ \ x_s = 1.090,\ \ x_1 = x_c = 1.427,\ \ x_{\frac{2}{3}} = x_v = 1.807$$

Note that $x_s \neq x_0 = 3.714$.
First let $f = \frac{1}{2}$. Using equations (14) and (15) obtain

$$x_{\frac{1}{2}} = \frac{0.8910 + \sqrt{(0.8910)^2 + 4 \times 0.5940 \times 1.406}}{1.406} = 2.080$$

and

$$g(x_{\frac{1}{2}}) = (1 + x_{\frac{1}{2}})e^{-x_{\frac{1}{2}}} = (1 + 2.080)e^{-2.080} = 0.3848 < 0.4060$$

Since the error (= 0.4060 – 0.3848 = 0.0212) is not large, to speed up the convergence we perform a fine adjustment for f at once. We try f = 0.5450 and obtain

$$x_{0.5450} = 1.998 \ \text{ and } \ g(x_{0.5450}) = 0.4065 \ > 0.4060$$

Finally, we obtain $x_a \cong 1.998$.

Example 2
Let λ = 0.01 per day, a = \$1000 per day, b = \$5000, c = \$90,000. Hence, d = 0.9474 and 1 – d = 0.0526. This is the third (extreme) example solved in Baker (1990), from which we know that x_a = 4.682.
First let $f = \frac{1}{2}$. Using equations (14) and (15) obtain

$$x_{\frac{1}{2}} = \frac{1.4211 + \sqrt{(1.4211)^2 + 4 \times 0.9474 \times 1.0526}}{1.0526} = 3.6788$$

and

$$g(x_{\frac{1}{2}}) = (1 + 3.6788)e^{-3.6788} = 0.11815 \ > 0.0526$$

Next, let $f = \frac{1}{3} < \frac{1}{2}$. We have

$$x_{\frac{1}{3}} = 5.2633 \ \text{ and } \ g(x_{\frac{1}{3}}) = 0.03243 \ < 0.0526$$

Thus, $\frac{1}{3} < f < \frac{1}{2}$. We try $f = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{12}$, using the bisection method, and obtain

$$x_{\frac{5}{12}} = 4.3312 \ \text{ and } \ g(x_{\frac{5}{12}}) = 0.07011 \ > 0.0526$$

Thus, $\frac{1}{3} < f < \frac{5}{12}$. We try $f = \frac{1}{2}\left(\frac{1}{3} + \frac{5}{12}\right) = \frac{3}{8}$ and obtain

$$x_{\frac{3}{8}} = 4.7521 \ \text{ and } \ g(x_{\frac{3}{8}}) = 0.04966 \ < 0.0526$$

To speed up the convergence, we perform a fine adjustment for f. Since $\frac{3}{8} = 0.375$, we try 0.38 or 0.3825 and respectively obtain

$$x_{0.38} = 4.6973 \ \text{ and } \ g(x_{0.38}) = 0.05196 \ < 0.0526$$

or

$$x_{0.3825} = 4.6703 \ \text{ and } \ g(x_{0.3825}) = 0.05313 \ > 0.0526$$

Finally, we obtain $x_a \cong 4.6703$, $T_a \cong 467.03$ days and $z(T_a) \cong \frac{1000 - (5000 + 90{,}000) \times 0.01}{1 + 4.6703} = \$8.8179$ from equation (3).

In this example, the values of x_f and $g(x_f)$ are correct to 5 significant figures, but in practice it is sufficient for them to be correct to 4 significant figures.

Example 3
Let λ = 0.01 per day, a = \$1000 per day, b = \$5000, c = \$100. Hence, d = 0.00105 and 1 – d = 0.99895. This is the first example solved in Baker (1990), from which we know that x_a = 0.0466.
Setting $f = \frac{1}{2}$ obtains

$$x_{\frac{1}{2}} = \frac{1.575 \times 10^{-3} + \sqrt{(1.575 \times 10^{-3})^2 + 4 \times 0.00105 \times 1998.95 \times 10^{-3}}}{1998.95 \times 10^{-3}} = 0.0466$$

and

$$g(x_{\frac{1}{2}}) = (1 + 0.0466)e^{-0.0466} = 0.99895$$

Then, we obtain $x_a = 0.0466$, $T_a = 4.66$ days and $z(T_a)$ $= \frac{1000 - (5000 + 100) \times 0.01}{1 + 0.0466} = \$906.75$ from equation (3).

5. CONCLUSIONS

The three typical examples show that the formula for x_f, i.e. equation (14), is the most accurate approximation of x_a, regardless of the different values of

x_a and hence of the range of d. Moreover, the author deems that the proposed algorithm is more efficient and less tedious than the one proposed by Hariga (1996) on p.356.

As stated on p.73 in Chung (1993), the expression for x_c can be used provided the optimum x_a is very small, such as the first two examples (where x_a = 0.0466 and 0.0341, and x_c = 0.0464 and 0.0339) on p.17 in Baker (1990). However, Hariga (1996) on p.356 showed that Chung's approximate inspection interval may yield a negative profit by means of the third example (where x_a = 4.682 and x_c = 1.929) on p.18 in Baker's article. This means that x_c is a poor approximation of x_a when it is comparatively large.

The proposed algorithm can also be applied to solve equation (7) or (12) in Ben-Daya and Hariga (1998). This algorithm incorporates fixed inspection and replacement times and relaxes the strict assumption of no production during the failed (regarded as an out-of-control) state. It should be more efficient and less tedious to use than the algorithm suggested on pp.484-485 of Ben-Daya and Hariga (1998) for solving equation (7) and be more accurate than equations (13) to (15) in reaching an approximate solution of equation (12).

APPENDIX

1. Derivation of Equation (3)

From equation (2), we have

$$e^{-\lambda T_a} = \frac{1-d}{1+\lambda T_a}.$$

Adding unity to the negation of both sides and simplifying, we obtain

$$1 - e^{-\lambda T_a} = \frac{\lambda T_a + d}{1+\lambda T_a};$$

substituting this in equation (1) yields

$$\begin{aligned} z(T_a) &= \frac{1}{T_a}\left[\left(\frac{a}{\lambda}-b\right)\left(\frac{\lambda T_a + d}{1+\lambda T_a}\right) - c\right] \\ &= \frac{a-\lambda b}{1+\lambda T_a} + \frac{1}{T_a}\left[\frac{d(a-\lambda b)}{\lambda(1+\lambda T_a)} - c\right] \\ &= \frac{a-\lambda b}{1+\lambda T_a} + \frac{1}{T_a}\left(\frac{c}{1+\lambda T_a} - c\right) \end{aligned}$$

since $$d = \frac{c}{\frac{a}{\lambda}-b} = \frac{\lambda c}{a-\lambda b}$$
$$= \frac{a-\lambda b-\lambda c}{1+\lambda T_a}.$$

2. Derivation of Equation (14)

Inverting equation (13), we have

$$e^{-x} \cong \frac{2-fx}{2+(2-f)x+(1-f)x^2};$$

then substituting this in equation (2), we have

$$(1+x_a)\left[\frac{2-fx_a}{2+(2-f)x_a+(1-f)x_a^2}\right] \cong 1-d$$

or

$$(1-d+df)x_a^2 - d(2-f)x_a - 2d \cong 0.$$

Notice that the coefficient of the square term is positive since $0<d<1$.

By the quadratic formula, we have

$$x_a \cong \frac{d(2-f)+\sqrt{[d(2-f)]^2+8d(1-d+df)}}{2(1-d+df)} = x_f$$

or

$$x_a \cong \frac{d(2-f)-\sqrt{[d(2-f)]^2+8d(1-d+df)}}{2(1-d+df)},$$

which is negative and thus is rejected.

3. Proof of Theorem 1

Differentiating equation (15) with respect to x_f, we have

$$\frac{dg(x_f)}{dx_f} = -x_f e^{-x_f},$$

which is negative for $x_f > 0$. Hence, $g(x_f)$ is a strictly decreasing function with respect to x_f.

4. Proof of Theorem 2

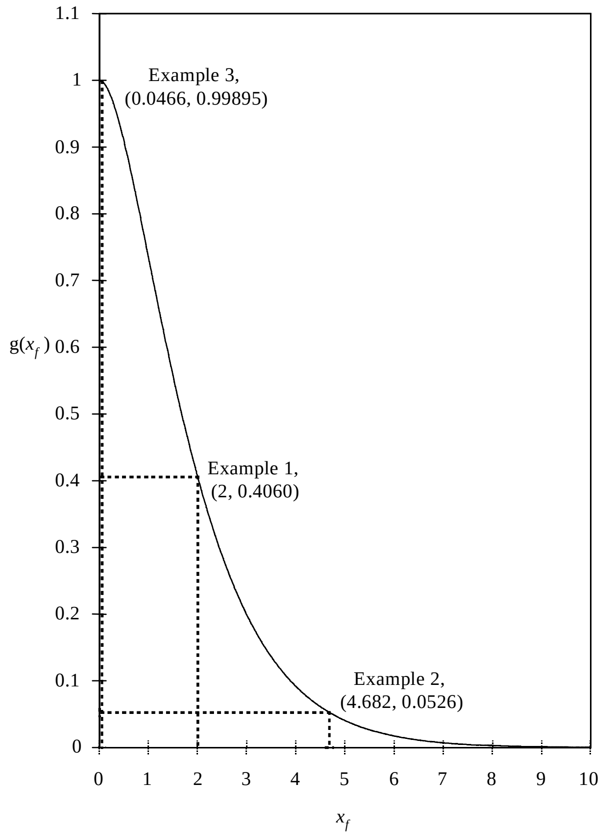
Implicitly differentiating the quadratic equation

$$(1-d+df)x_f^2 - d(2-f)x_f - 2d = 0,$$

which is obtained by transposing equation (14), with respect to f and solving for $\frac{dx_f}{df}$ yields

$$\frac{dx_f}{df} = \frac{-dx_f(x_f+1)}{\sqrt{\nabla}},$$

where $\nabla \equiv [d(2-f)]^2 + 8d(1-d+df)$ denotes the discriminant of the quadratic equation. The derivative is negative for $x_f > 0$ and $0<d<1$. Hence, x_f is a strictly decreasing function with respect to f.



Enlarged Figure 1.

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