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ON THE STABILITY OF THE GENERALIZED G-TYPE FUNCTIONAL EQUATIONS

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Abstract. In this paper, we obtain the generalization of the Hyers-Ulam-Rassias stability in the sense of Găvruta and Ger of the generalized G-type functional equations of the form $f(\varphi(x)) = \Gamma(x)f(x)$. As a consequence in the cases $\varphi(x) := x+p := x+1$, we obtain the stability theorem of G-functional equation : the reciprocal functional equation of the double gamma function.

1. Introduction

In 1940, the stability problem raised by S. M. Ulam [15] was solved by D. H. Hyers in [6]. The result of Hyers has been generalized to the unbounded case by Th. M. Rassias [14], and this has been extended by P. Găvruta [4] and R. Ger [5], respectively, as follows:
(*Găvruta's sense*). For a fixed function φ such as $|E_1(f) - E_2(f)| \leq \varphi$, there exists a function g such that $E_1(g) = E_2(g)$ and $|g(x) - f(x)| \leq \Phi(x)$ for some fixed function Φ.
(*Ger's sense*). For a fixed function ψ such as $|\frac{E_1(f)}{E_2(f)} - 1| \leq \psi$, there exists a function g such that $E_1(g) = E_2(g)$ and $\alpha \leq \frac{f}{g} \leq \beta$ for some fixed functions α and β.

Namely the result of Rassias is a special case of the type φ in the stability in the sense of Găvruta.

The gamma function

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \quad (x > 0)$$

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is a solution of the gamma functional equation $g(x+1) = xg(x)$, whose stability is proved by S.-M. Jung ([8], [9]) and the author ([10], [11], [12], [13]).

The G-function introduced by E.W. Barnes [2]

$$G(z) = (2\pi)^{\frac{z-1}{2}} e^{-\frac{z(z-1)}{2}} e^{-\gamma\frac{(z-1)^2}{2}} \prod_{k=1}^{\infty} \left[\left(1 + \frac{z-1}{k}\right)^k e^{1-z+\frac{(z-1)^2}{2k}}\right]$$

does satisfy the equation $G(x+1) = \Gamma(x)G(x)$, $G(1) = \Gamma(1) = 1$, and $\Gamma(x+1) = x\Gamma(x)$, where γ is the Euler-Mascheroni's constant defined by $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) \cong 0.577215664\cdots$.

The properties and values of G-function depend on those of the double gamma function Γ_2. Since the double gamma function Γ_2 is defined by the reciprocal of the G-function (see [2]), $\Gamma_2(x) = 1/G(x)$, and its functional equation can be written in the form $\Gamma_2(x+1) = \Gamma_2(x)/\Gamma(x)$. Therefore the stability problem for the G-function is equivalent to that for the reciprocal of the double gamma function.

In this paper, we will investigate a generalization of the Hyers-Ulam stability in the sense of Găvruta and Ger for the functional equations

$$f(\varphi(x)) = \Gamma(x)f(x), \tag{1.1}$$

$$f(x+p) = \Gamma(x)f(x), \tag{1.2}$$

$$f(x+1) = \Gamma(x)f(x), \tag{1.3}$$

where φ is the given function, while f is the unknown function. The equation (1.3) will be called the G-functional equation because its solution is the G-function.

In section 2, we will study the stability in the sense of Găvruta for the functional equation (1.1), and as a consequence we obtain the stability of the equations (1.2) and (1.3).

In section 3, the stability in the sense of Ger for the functional equation (1.1) will be investigated, and also its results imply the stability of the equations (1.2) and (1.3).

Throughout this paper, let R, R_+ and R_* denote the set of real numbers, the set of all positive real numbers and the set of all nonnegative real numbers, respectively. Each positive real number δ, $p > 0$ is fixed, and n_0 is a given nonnegative integer. The functions $\varepsilon : R_+ \to R_*$, $\varphi : R_+ \to R_+$ are defined. We put $\varphi_0(x) := x$ and $\varphi_n(x) := \varphi(\varphi_{n-1}(x))$ for all positive integers n and all points $x \in R_+$.

2. Stability in the sense of Găvruta

In this section, we will study the generalization of the Hyers-Ulam-Rassias stability for the generalized G-function type functional equation (1.1), and also we obtain the same stabilities for the equation (1.2) and the G-functional equation (1.3) from the obtained result.

Let φ and ε be given functions such that

$$\omega(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\Gamma(\varphi_j(x))|} < \infty, \qquad \forall x \in R_+. \tag{2.1}$$

THEOREM 1. *Let the functions φ, ε satisfy the condition* (2.1). *If a function $f : R_+ \longrightarrow R_+$ satisfies the inequality*

$$|f(\varphi(x)) - \Gamma(x) f(x)| \leq \varepsilon(x), \tag{2.2}$$

then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.1) *such that*

$$|g(x) - f(x)| \leq \omega(x). \tag{2.3}$$

PROOF. For any $x \in R_+$ and for every positive integer n, let $\omega_n : R_+ \longrightarrow R_*$ and $g_n : R_+ \longrightarrow R_+$ be the functions defined by

$$\omega_n(x) := \sum_{k=0}^{n-1} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\Gamma(\varphi_j(x))|} \quad \text{and} \quad g_n(x) := \frac{f(\varphi_n(x))}{\prod_{j=0}^{n-1} \Gamma(\varphi_j(x))}$$

for all $x \in R_+$, respectively.

By (2.2), it follows that

$$\left|\frac{f(\varphi(x))}{\Gamma(x)} - f(x)\right| \leq \frac{\varepsilon(x)}{|\Gamma(x)|} \qquad \text{for all} \quad x \in R_+.$$

Substituting $\varphi_n(x)$ for x in this inequality, and then dividing both sides of the resulting inequality by $\prod_{j=0}^{n-1} |\Gamma(\varphi_j(x))|$, we get

$$|g_{n+1}(x) - g_n(x)| \leq \frac{\varepsilon(\varphi_n(x))}{\prod_{j=0}^{n} |\Gamma(\varphi_j(x))|}. \tag{2.4}$$

By induction on n we prove that

$$|g_n(x) - f(x)| \leq \omega_n(x) \tag{2.5}$$

for all $x \in R_+$ and all positive integers n. For the case $n = 1$, the inequality (2.5) is an immediate consequence of (2.2).

Assume that the inequality (2.5) holds true for some n. Then we prove the inequality in (2.5) holds true for $n+1$. This is an immediate consequence of

$$\begin{aligned}
|g_{n+1}(x) - f(x)| &\leq |g_{n+1}(x) - g_n(x)| + |g_n(x) - f(x)| \\
&\leq \frac{\varepsilon(\varphi_n(x))}{\prod_{j=0}^{n} |\Gamma(\varphi_j(x))|} + \omega_n(x) \\
&= \omega_{n+1}(x).
\end{aligned}$$

We claim that $\{g_n(x)\}$ is a Cauchy sequence. Indeed, by (2.4) and (2.1), we have that, for $n > m$,

$$\begin{aligned}
|g_n(x) - g_m(x)| &\leq \sum_{k=m}^{n-1} |g_{k+1}(x) - g_k(x)| \\
&\leq \sum_{k=m}^{n-1} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\Gamma(\varphi_j(x))|} \longrightarrow 0
\end{aligned}$$

as $m \longrightarrow \infty$.

Hence, we can define a function $g : R_+ \longrightarrow R_+$ by

$$g(x) := \lim_{n \to \infty} g_n(x). \tag{2.6}$$

From the definition of g_n, we have $g_n(\varphi(x)) = \Gamma(x) g_{n+1}(x)$ and therefore the function g satisfies (1.1).

We show from (2.5) that g satisfies the inequality (2.3) as follows:

$$\begin{aligned}
|g(x) - f(x)| &= \lim_{n \to \infty} |g_n(x) - f(x)| \\
&\leq \lim_{n \to \infty} \omega_n(x) \\
&= \omega(x) \qquad \forall x \in R_+.
\end{aligned}$$

If $h : R_+ \longrightarrow R_+$ is another such function, which satisfies (1.1) and (2.3), then we have

$$\begin{aligned}|g(x) - h(x)| &= |g(\varphi_n(x)) - h(\varphi_n(x))| \cdot \prod_{j=0}^{n-1} \frac{1}{|\Gamma(\varphi_j(x))|} \\ &\le 2\omega_n(\varphi_n(x)) \cdot \prod_{j=0}^{n-1} \frac{1}{|\Gamma(\varphi_j(x))|} \\ &\le 2\Big(\sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_{n+k}(x))}{\prod_{j=0}^{k} |\Gamma(\varphi_{n+j}(x))|}\Big) \cdot \prod_{j=0}^{n-1} \frac{1}{|\Gamma(\varphi_j(x))|} \\ &= 2\sum_{k=n}^{\infty} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\Gamma(\varphi_j(x))|}\end{aligned}$$

for all $x \in R_+$ and all positive integers n, which tends to zero as $n \to \infty$, since $\omega(x)$ is bounded. This implies the uniqueness of g. $\square$

Setting $\varepsilon(x) = \delta$ in Theorem 1, we have the Hyers-Ulam stability of equation (1.1).

Let the functions φ satisfy

$$\mu(x) := \sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{|\Gamma(\varphi_j(x))|} < \infty \qquad \forall x \in R_+. \tag{2.7}$$

COROLLARY 1. *Let φ satisfy condition (2.7). If a function $f : R_+ \longrightarrow R_+$ satisfies the inequality*

$$|f(\varphi(x)) - \Gamma(x)f(x)| \le \delta \tag{2.8}$$

for all $x \in R_+$, then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.1) such that

$$|g(x) - f(x)| \le \delta\mu(x). \tag{2.9}$$

(1) Stability in the case $\varphi(x) = x + p$ of the equation (1.1)

For a special case of the functional equation (1.1) with $\varphi(x) = x + p$, we obtain the equation (1.2). Then, we can obtain the same results for the functional equation (1.2).

Putting $\varphi(x) = x + p$ in functional equation (1.1), the inequality (2.1) implies the following inequality

$$\omega'(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(x+kp)}{\prod_{j=0}^{k} |\Gamma(x+jp)|} < \infty \tag{2.10}$$

Theorem 2. *Let ε satisfy condition* (2.10). *If a function $f : R_+ \longrightarrow R$ satisfies the inequality*

$$|f(x+p) - \Gamma(x)f(x)| \le \varepsilon(x) \qquad \forall x > n_0, \tag{2.11}$$

then there exists a unique solution $g : R_+ \longrightarrow R$ of the equation (1.1) *such that*

$$|g(x) - f(x)| \le \omega'(x) \qquad \forall x > n_0. \tag{2.12}$$

Proof. Setting $\varphi(x) = x + p$ in Theorem 1, then the claimed result of this theorem is satisfied except for the condition that replaces $x \in R_+$ by $x > n_0$. For this, we define the new function $g_0 : (n_0, \infty) \longrightarrow R$ by

$$g_0(x) := \lim_{n \to \infty} g_n(x)$$

in substituting g defined in (2.6) for g_0.

Now, we extend the function g_0 to the domain $(0, \infty)$. We define for each $0 < x \le n_0$,

$$g(x) := \frac{g_0(x+kp)}{\prod_{n=0}^{k-1} \Gamma(x+np)},$$

where k is the smallest natural number satisfying the inequalities $x_i + kp_i > n_0$ for each i.

Then, $g(x+p) = \Gamma(x)g(x)$ for all $x > 0$ and $g(x) = g_0(x)$ for all $x > n_0$. Also the inequality

$$|g(x) - f(x)| < \omega'(x)$$

holds for all $x > 0$. $\square$

The following theorem is the Hyers-Ulam stability of the Eq. (1.2) as a corollary of Theorem 2.

COROLLARY 2. *If a function $f : R_+ \longrightarrow R_+$ satisfies the following inequality*

$$|f(x+p) - \Gamma(x)f(x)| \leq \delta \qquad \forall x > n_0. \tag{2.12}$$

Then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.2) with

$$|g(x) - f(x)| \leq \frac{\delta}{\Gamma(x)}\left\{1 + \sum_{k=1}^{\infty}\prod_{j=1}^{k}\frac{1}{\Gamma(x+jp)}\right\}.$$

In particular, if $p > 1$, the approximate remainder is less then $\frac{\delta}{\Gamma(x)}\{1 + \frac{e}{\Gamma(x+p)}\}$.

PROOF. Apply with $\varepsilon(x) = \delta$ in Theorem 1. □

(2) STABILITY OF G-FUNCTIONAL EQUATION

As a special case of the equation (1.1) with $\varphi(x) = x + 1$ or the equation (1.2) with $p = 1$, we obtain the G-functional equation (1.3), which is the functional equation of the reciprocal of the double gamma function.

COROLLARY 3. *If a mapping $f : R_+ \longrightarrow R_+$ satisfies the inequality*

$$|f(x+1) - \Gamma(x)f(x)| \leq \varepsilon(x) \qquad \forall x > n_0,$$

then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the G-functional equation (1.3) with

$$|f(x) - g(x)| \leq \omega_G(x) \qquad \forall x > n_0,$$

where $\omega_G(x) = \sum_{k=0}^{\infty} \frac{\varepsilon(x+k)}{\prod_{j=0}^{k}\Gamma(x+j)} < \infty$.

COROLLARY 4. ([11]) *If a mapping $f : R_+ \longrightarrow R_+$ satisfies the inequality*

$$|f(x+1) - \Gamma(x)f(x)| \leq \delta \qquad \forall x > n_0,$$

then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the G-functional equation (1.3) with

$$|f(x) - g(x)| \leq \frac{\delta}{\Gamma(x)}\sum_{k=0}^{\infty}\prod_{j=0}^{k}\frac{1}{n!} \leq \frac{e\delta}{\Gamma(x)} \qquad \forall x > n_0.$$

(3) EXAMPLES

We can find many examples. Namely it is enough to construct the convergent series satisfying the condition (2.10) in Theorem 2. For example,

Ex 1. All of the series $\sum_{k=0}^{\infty} \varepsilon(x+kp)$ which are convergent.

Ex 2. All of the series $\sum_{k=0}^{\infty} \varepsilon(x+kp)$ which are bounded.

Ex 3. All of the sequence $\{\varepsilon(n)\} \leq 2! \cdot 3! \cdots n!$, where n is a natural number. By ratio test, the desired condition holds.

Ex 4. As a special case $x > n_0 = 1$, $p \geq 2$, $\varepsilon(x) \leq \Gamma(x)$, in particular, if we put $x > 2$, $p = 2$, $\varepsilon(x) = \Gamma(x)$, then the condition (2.10) implies

$$\begin{aligned}\omega'(x) &= \frac{\Gamma(x)}{\Gamma(x)} + \frac{\Gamma(x+2)}{\Gamma(x)\Gamma(x+2)} + \frac{\Gamma(x+4)}{\Gamma(x)\Gamma(x+2)\Gamma(x+4)} + \cdots, \\ &= 1 + \frac{1}{\Gamma(x)} + \frac{1}{\Gamma(x)\Gamma(x+2)} + \cdots.\end{aligned}$$

3. Stability in the sense of Ger

The following theorem provides the stability in the sense of Ger for the equation (1.1).

THEOREM 3. *Let a function $f : R_+ \longrightarrow R_+$ satisfy the inequality*

$$(3.1) \qquad \left|\frac{f(\varphi(x))}{\Gamma(x)\, f(x)} - 1\right| \leq \varepsilon(x) \qquad \forall x > n_0,$$

where $\varepsilon : R_+ \longrightarrow (0,1)$ is a function such that

$$(3.2) \qquad \alpha(x) := \prod_{j=0}^{\infty}(1 - \varepsilon(\varphi_j(x))) \quad \text{and} \quad \beta(x) := \prod_{j=0}^{\infty}(1 + \varepsilon(\varphi_j(x)))$$

are bounded for all $x > n_0$.

Then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.1) with

$$(3.3) \qquad \alpha(x) \leq \frac{g(x)}{f(x)} \leq \beta(x) \qquad \forall x > n_0.$$

PROOF. For any $x \in R_+$ and for every positive integer n, we define

$$g_n(x) = \prod_{j=0}^{n-1} \frac{f(\varphi_n(x))}{\Gamma(\varphi_j(x))}.$$

For all positive integers m, n with $n > m$, we have

$$\begin{aligned}(3.4)\qquad \frac{g_n(x)}{g_m(x)} &= \frac{f(\varphi_{m+1}(x))}{\Gamma(\varphi_m(x))f(\varphi_m(x))}\\ &\quad\cdot \frac{f(\varphi_{m+2}(x))}{\Gamma(\varphi_{m+1}(x))f(\varphi_{m+1}(x))}\\ &\quad\cdots \frac{f(\varphi_n(x))}{\Gamma(\varphi_{n-1}(x))f(\varphi_{n-1}(x))}.\end{aligned}$$

It also follows from (3.1) that

$$(3.5)\qquad 0 < 1 - \varepsilon(\varphi_j(x)) \leq \frac{f(\varphi_{j+1}(x))}{\Gamma(\varphi_j(x))\; f(\varphi_j(x))} \leq 1 + \varepsilon(\varphi_j(x))$$

for all $x > n_0$ and $j = 0, 1, 2, \cdots$. From (3.4) and (3.5), we get

$$\prod_{j=m}^{n-1} (1 - \varepsilon(\varphi_j(x))) \leq \frac{g_n(x)}{g_m(x)} \leq \prod_{j=m}^{n-1} (1 + \varepsilon(\varphi_j(x)))$$

or

$$\begin{aligned}\sum_{j=m}^{n-1} \log(1 - \varepsilon(\varphi_j(x))) &\leq \log g_n(x) - \log g_m(x)\\ &\leq \sum_{j=m}^{n-1} \log(1 + \varepsilon(\varphi_j(x))).\end{aligned}$$

Since $\sum_{j=0}^{\infty} \log(1 - \varepsilon(\varphi_j(x))) = \log \alpha(x)$ and $\sum_{j=0}^{\infty} \log(1 + \varepsilon(\varphi_j(x))) = \log \beta(x)$,

it follows that

$$\begin{aligned}&\lim_{m\to\infty} \sum_{j=m}^{\infty} \log(1 - \varepsilon(\varphi_j(x)))\\ &= \lim_{m\to\infty} \sum_{j=m}^{\infty} \log(1 + \varepsilon(\varphi_j(x))) = 0\end{aligned}$$

by the boundedness of α, β. Hence, we note that $\{\log g_n(x)\}$ is a Cauchy sequence for all $x > n_0$. It is reasonable to define a function $g_0 : R_+ \to R_+$ by

$$g_0(x) = e^{L(x)} = \lim_{n\to\infty} g_n(x) \qquad \forall x > n_0, \tag{3.6}$$

where $L(x) := \lim_{n\to\infty} \log g_n(x)$.

Thus the function g_0 satisfies the equation (1.1), that is,

$$g_0(\varphi(x)) = \lim_{n\to\infty} g_n(\varphi(x)) = \lim_{n\to\infty} \Gamma(x) g_{n+1}(x) = \Gamma(x) g_0(x) \quad \forall x > n_0. \tag{3.7}$$

Since

$$\frac{g_n(x)}{f(x)} = \frac{f(\varphi(x))}{\Gamma(x)f(x)} \cdot \frac{f(\varphi_2(x))}{\Gamma(\varphi(x))f(\varphi(x))} \cdots \frac{f(\varphi_n(x))}{\Gamma(\varphi_{n-1}(x))\, f(\varphi_{n-1}(x))},$$

we get

$$\prod_{j=0}^{n-1}(1 - \varepsilon(\varphi_j(x))) \le \frac{g_n(x)}{f(x)} \le \prod_{j=0}^{n-1}(1 + \varepsilon(\varphi_j(x))) \tag{3.8}$$

for all $x \in R_+$. Due to (3.6), (3.8), and the definitions of α, β as $n \longrightarrow \infty$, we obtain the required result (3.3).

Assume $h : R_+ \longrightarrow R_+$ is another solution of equation (1.1) which satisfies inequality (3.3). By (3.7), we have

$$\frac{g(x)}{h(x)} = \frac{g(\varphi_n(x))}{h(\varphi_n(x))} = \frac{g(\varphi_n(x))}{f(\varphi_n(x))} \cdot \frac{f(\varphi_n(x))}{h(\varphi_n(x))}$$

for any $x > n_0$ and for any natural number n.

Hence, we have

$$\frac{\alpha(\varphi_n(x))}{\beta(\varphi_n(x))} \le \frac{g(x)}{h(x)} \le \frac{\beta(\varphi_n(x))}{\alpha(\varphi_n(x))}$$

for any natural number n. By the boundedness of the series ε,

$$\alpha(\varphi_n(x)) = \prod_{j=n}^{\infty}(1 - \varepsilon(\varphi_j(x))) \longrightarrow 1$$

as $n \longrightarrow \infty$. and similarly $\beta(\varphi_n(x)) \longrightarrow 1$ as $n \longrightarrow \infty$.

Therefore, it is obvious that $h(x) = g_0(x)$. The extension of the domain from (n_0, ∞) to R_+ follows the proof of the Theorem 2. $\square$

COROLLARY 5. *Let $\eta > 0$ be given. If a function $f : R_+ \longrightarrow R_+$ satisfies the inequality*

$$\left|\frac{f(\varphi(x))}{\Gamma(x)\, f(x)} - 1\right| \leq \frac{\delta}{x^{1+\eta}} \qquad \forall x > n_0,$$

where $\varepsilon : R_+ \longrightarrow (0,1)$ is a function such that $\alpha(x) := \prod_{j=0}^{\infty}(1 - \frac{\delta}{(\varphi(x))^{1+\eta}})$ and $\beta(x) := \prod_{j=0}^{\infty}(1 + \frac{\delta}{(\varphi(x))^{1+\eta}})$ are bounded for all $x > n_0$,
then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.1) *such that for any $x > \max\{n_0, \delta^{\frac{1}{1+\eta}}\}$*

$$\alpha(x) \leq \frac{g(x)}{f(x)} \leq \beta(x).$$

COROLLARY 6. *Let a function $f : R_+ \longrightarrow R_+$ satisfy the inequality*

$$\left|\frac{f(x+p)}{\Gamma(x)\, f(x)} - 1\right| \leq \varepsilon(x) \qquad \forall x > n_0,$$

where $\varepsilon : R_+ \longrightarrow (0,1)$ is a function such that

$$\alpha(x) := \prod_{j=0}^{\infty}(1 - \varepsilon(x + jp)) \quad \text{and} \quad \beta(x) := \prod_{j=0}^{\infty}(1 + \varepsilon(x + jp))$$

are bounded for all $x > n_0$.
Then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.2) *with*

$$\alpha(x) \leq \frac{g(x)}{f(x)} \leq \beta(x).$$

COROLLARY 7. *Let $\eta > 0$ be given. If a function $f : R_+ \longrightarrow R_+$ satisfies the inequality*

$$\left|\frac{f(x+p)}{\Gamma(x)\, f(x)} - 1\right| \leq \frac{\delta}{x^{1+\eta}} \qquad \forall x > n_0,$$

then there exists a unique solution $g : R_+ \longrightarrow R_+$ of the equation (1.2) *such that for any $x > \max\{n_0, \delta^{\frac{1}{1+\eta}}\}$*

$$\alpha(x) \leq \frac{g(x)}{f(x)} \leq \beta(x),$$

where $\alpha(x) := \prod_{j=0}^{\infty}(1 - \frac{\delta}{(x+j)^{1+\eta}})$ and $\beta(x) := \prod_{j=0}^{\infty}(1 + \frac{\delta}{(x+j)^{1+\eta}})$.

COROLLARY 8. ([11]) *Let a function $f : R_+ \longrightarrow R_+$ satisfy the inequality*

$$|\frac{f(x+1)}{\Gamma(x)\, f(x)} - 1| \le \varepsilon(x) \qquad \forall x > n_0,$$

where $\varepsilon : R_+ \longrightarrow (0,1)$ is a function such that

$$\alpha(x) := \prod_{j=0}^{\infty}(1-\varepsilon(x+j)) \quad \text{and} \quad \beta(x) := \prod_{j=0}^{\infty}(1+\varepsilon(x+j))$$

are bounded for all $x > n_0$

Then there exists a unique G-function $g : R_+ \longrightarrow R_+$ satisfying (1.3) with

$$\alpha(x) \le \frac{g(x)}{f(x)} \le \beta(x).$$

COROLLARY 9. ([11]) *Let $\eta > 0$ be given. If a mapping $f : R_+ \longrightarrow R_+$ satisfies the inequality*

$$|\frac{f(x+1)}{\Gamma(x) f(x)} - 1| \le \frac{\delta}{x^{1+\eta}} \qquad \forall x > n_0,$$

then there exists a unique G-function $g : R_+ \longrightarrow R_+$ satisfying (1.3) with such that for any $x > \max\{n_0, \delta^{\frac{1}{1+\eta}}\}$

$$e^{\alpha(x)} \le \frac{f(x)}{f(x)} \le e^{\beta(x)},$$

where $\alpha(x) := \prod_{j=0}^{\infty}(1 - \frac{\delta}{(x+j)^{1+\eta}})$ and $\beta(x) := \prod_{j=0}^{\infty}(1 + \frac{\delta}{(x+j)^{1+\eta}})$.

PROOF. If $x > \delta^{\frac{1}{1+\eta}}$, then $\prod_{j=0}^{\infty}(1-\frac{\delta}{(x+j)^{1+\eta}})$ and $\prod_{j=0}^{\infty}(1+\frac{\delta}{(x+j)^{1+\eta}})$ converge, respectively. Applying Corollary 8 with $\varepsilon(x) = \frac{\delta}{x^{1+\eta}}$, we get the desired result. $\square$

(4) EXAMPLES

We take that $\varepsilon : R_+ \longrightarrow (0,1)$ is a function such that

$$\sum_{j=0}^{\infty}\varepsilon(x+jq) < +\infty. \tag{3.9}$$

Then the condition (3.9) implies that $\prod_{j=0}^{\infty}(1 \pm \varepsilon(x + jq))$ converges. Hence, we can define the functions α, β for all $x \in R_+$ such that $0 < \alpha(x) := \prod_{j=0}^{\infty}(1 - \varepsilon(x+jq)) < \prod_{j=0}^{\infty}(1+\varepsilon(x+jq)) := \beta(x) < +\infty$, that is, these series are bounded.

This property gives us the following examples derived from Corollary 6 with $x = 1, q = 1$.

Ex 5. $\varepsilon(1 + j) = \frac{1}{(1+j)^p}$ for $p > 1$. Note that the $p-$series $\sum_{k=0}^{\infty} \frac{1}{k^p}$ converges for $p > 1$.

Ex 6. $\varepsilon(1 + j) = \frac{1}{(1+j)!}$. Note that $\sum_{j=0}^{\infty} \frac{1}{(1+j)!} = e - 1$.

Ex 7. Let $\varepsilon(1 + j) = \frac{(-1)^j}{1+j}$, or $\varepsilon(1 + j) = \frac{(-1)^{1+j}}{1+j}$. Then we see that $\sum_{j=0}^{\infty} \varepsilon(1 + j) = \sum_{j=0}^{\infty} \frac{(-1)^j}{1+j}$ or $\sum_{j=0}^{\infty} \varepsilon(1 + j) = \sum_{j=0}^{\infty} \frac{(-1)^{1+j}}{1+j}$ converge by the alternating series test.

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