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ISHIKAWA ITERATIVE SEQUENCE WITH ERRORS FOR φ-STRONGLY ACCRETIVE OPERATORS

Yongjin Li

Abstract. In this paper, the iterative solution is studied for equation Tx = f with a uniformly continuous φ-strongly accretive operators in arbitrary real Banach spaces. Our results extend, generalize and improve the corresponding results obtained by Zeng [11].

1. Introduction

Let X be an arbitrary real Banach space with norm $\|\cdot\|$ and X^* be the dual space of X. The duality mapping J: X $\to 2^{X^*}$ is defined by

$$Jx = \{f \in X* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\}.$$

Where ⟨x, f⟩ denotes the value of the continuous linear function f $\in X^*$ at x ∈ X.

An operator T: D(T) ⊂ X → X is said to be *accretive*, if for all x, y ∈ D(T), there exists j ∈ J(x − y) , such that

$$\langle Tx - Ty, j(x-y)\rangle \geq 0. \tag{1}$$

T is said to be strongly accretive, if for all x, y ∈ D(T), there exists j ∈ J(x − y) and a constant $k > 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq \|x-y\|^2. \tag{2}$$

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An operator T with domain $D(T)$ and range $R(T)$ in E is said to be φ − *strongly accretive*, if for all x, y ∈ D(T), there exists j ∈ J(x − y) and a strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, such that

$$\langle Tx - Ty, j(x-y)\rangle \geq \varphi(\|x-y\|)\|x-y\|. \tag{3}$$

Chidume [4] proved that the Mann iteration process converges strongly to a solution of the equation Tx = f when T is Lipschitizan and strongly accretive. A related result deal with the iterative approximation of the fixed point of the class of Lipschitizan and strongly psedocontractive mappings. In [11], Zeng proved the following theorem:

THEOREM 1.1 (See [11]) *Suppose E is an arbitrary Banach space and T: $E \to E$ be a Lipschitzian φ - strongly accretive operator, suppose the equation Tx = f has a solution. Let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0,1]$ and $\{u_n\}, \{v_n\}$ be sequences in E satisfying the following conditions:*

(1) $\sum_{n=1}^{\infty} \|u_n\| < \infty, \sum_{n=1}^{\infty} \|v_n\| < \infty$;

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(3) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;

(4) $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$.

Then for arbitrary $x_0 \in$E , the Ishikawa iteration sequence $\{x_n\}$ defined iteratively by

$$y_n = (1-\beta_n)x_n + \beta_n S x_n + v_n \tag{4}$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n + u_n \tag{5}$$

converges strongly to the unique solution x^* of the equation Tx = f, where S: $E \to E$ is defined by Sx = f + (I - T)x, for any $x \in E$.

Our objective in this paper is to consider an iterative process, which converges to a solution of Tx = f in arbitrary real Banach space. Our results improve and extend the results of Zeng [11].

The following Lemmas play an important role in proving our main results.

LEMMA 1.2 (See [9]) *Let* $\{a_n\}, \{b_n\}, \{c_n\}$ *be nonnegative sequence satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n \tag{6}$$

With $\{t_n : n = 0, 1, 2, \cdots\} \subset [0, 1]$, $\sum_{n=1}^{\infty} t_n = \infty$, $b_n = o(t_n)$ *and* $\sum_{n=1}^{\infty} c_n < \infty$ *then* $lim_{n\to\infty} a_n = 0$.

LEMMA 1.3 (See [10].) *Let X be an arbitrary real Banach space and T: X* $\to$ *X be a continuous* φ*-strongly accretive operator, then for any given f* $\in$ *X, the equation Tx = f has a unique solution.*

2. Main results

Now, we state and prove the following theorems.

THEOREM 2.1 *Let E be an arbitrary Banach space and T: E* $\to$ *E be uniformly continuous* φ *- strongly accretive operator. Suppose R(T) is bounded and* $\{u_n\}, \{v_n\}$ *be sequences in X and* $\{\alpha_n\}, \{\beta_n\}$ *be sequences in* $[0, 1]$ *such that*

(1) $\sum_{n=1}^{\infty} \|u_n\| < \infty, \|v_n\| \to 0$ as n $\to \infty$;

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty, \alpha_n \to 0$ as n $\to \infty$;

(3) $\beta_n \to 0$ as n $\to \infty$.

Then for any $x_0 \in$E , the Ishikawa iteration sequence $\{x_n\}$ with errors defined by

$$y_n = (1 - \beta_n)x_n + \beta_n[f + (I - T)x_n] + v_n \tag{7}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[f + (I - T)y_n] + u_n \tag{8}$$

converges strongly to the unique solution x^* of the equation Tx = f.

PROOF. It follows from Lemma 1.3[10] that the equation Tx = f has a unique solution x* $\in$ X. Let $Sx = f + (I - T)x$, since $Sx^* =$

$f + (I - T)x^* = x^*$, the point x^* is a fixed point of S. Thus for any $x, y \in X$, there exists $j(x-y) \in J(x-y)$ such that

$$
\begin{aligned}
< (I-S)x - (I-S)y, j(x-y) > &=< Tx - Ty, j(x-y) > \\
&\geq \varphi(\|x-y\|)\|x-y\| \\
&\geq \frac{\varphi(\|x-y\|)}{1+\varphi(\|x-y\|)}\|x-y\|^2 \\
&= \sigma(\|x-y\|)\|x-y\|^2.
\end{aligned} \tag{9}
$$

Where

$$\sigma(\|x-y\|) = \frac{\varphi(\|x-y\|)}{1+\varphi(\|x-y\|)} \in [0,1)$$

for all $x, y \in X$. Thus,

$$< (I - S - \sigma(\|x-y\|))x - (I - S - \sigma(\|x-y\|)y, j(x-y) > \geq 0$$

and so it follows from Lemma 1.1 of Kato[8] that

$$\|x-y\| \leq \|x-y+r[(I-S)x - \sigma(\|x-y\|)x - ((I-S)y - \sigma(\|x-y\|)y)\|.$$

From $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sy_n + u_n$ we obtain

$$
\begin{aligned}
x_n &= x_{n+1} + \alpha_n x_n - \alpha_n Sy_n - u_n \\
&= (1+\alpha_n)x_{n+1} + \alpha_n[(I-S)x_{n+1} - \sigma(\|x_{n+1} - x^*\|)x_{n+1}] \\
&- (1 - \sigma(\|x_{n+1} - x^*\|))\alpha_n x_n + (2 - \sigma(\|x_{n+1} - x^*\|))\alpha_n^2(x_n - Sy_n) \\
&+ \alpha_n(Sx_{n+1} - Sy_n) - [(2 - \sigma(\|x_{n+1} - x^*\|))\alpha_n + 1]u_n.
\end{aligned}
$$

It is easy to see that

$$x^* = (1+\alpha_n)x^* + \alpha_n[(I-S)x^* - \sigma(\|x_{n+1} - x^*\|)x^*] - (1 - \sigma(\|x_{n+1} - x^*\|))\alpha_n x^*$$

so that

$$
\begin{aligned}
x_n - x^* = {} & (1+\alpha_n)(x_{n+1} - x^*) + \alpha_n[(I-S)x_{n+1} - \sigma(\|x_{n+1} - x^*\|)x_{n+1} \\
&+ (I-S)x^* - \sigma(\|x_{n+1} - x^*\|)x^*] \\
&- (1 - \sigma(\|x_{n+1} - x^*\|))\alpha_n(x_n - x^*) \\
&+ (2 - \sigma(\|x_{n+1} - x^*\|))\alpha_n^2(x_n - Sy_n) \\
&+ \alpha_n(Sx_{n+1} - Sy_n) - [(2 - \sigma(\|x_{n+1} - x^*\|))\alpha_n + 1]u_n.
\end{aligned}
$$

Hence

$$\begin{aligned}
\|x_n - x^*\| \geq{}& (1+\alpha_n)\|(x_{n+1} - x^*) \\
&+ \frac{\alpha_n}{1+\alpha_n}[(I-S)x_{n+1} - \sigma(\|x_{n+1} - x^*\|)x_{n+1} \\
&+ (I-S)x^* - \sigma(\|x_{n+1} - x^*\|)x^*]\| \\
&- (1 - \sigma(\|x_{n+1} - x^*\|))\alpha_n\|x_n - x^*\| \\
&- (2 - \sigma(\|x_{n+1} - x^*\|))\alpha_n^2\|x_n - Sy_n\| \\
&- \alpha_n\|Sx_{n+1} - Sy_n\| - [(2 - \sigma(\|x_{n+1} - x^*\|))\alpha_n + 1]\|u_n\| \\
&\geq (1+\alpha_n)\|(x_{n+1} - x^*)\| \\
&- (1 - \sigma(\|x_{n+1} - x^*\|))\alpha_n\|x_n - x^*\| \\
&- (2 - \sigma(\|x_{n+1} - x^*\|))\alpha_n^2\|x_n - Sy_n\| \\
&- \alpha_n\|Sx_{n+1} - Sy_n\| - [(2 - \sigma(\|x_{n+1} - x^*\|))\alpha_n + 1]\|u_n\|
\end{aligned}$$

so that

$$\begin{aligned}
\|x_{n+1} - x^*\| \leq{}& [\frac{1 + (1 - \sigma(\|x_{n+1} - x^*\|))\alpha_n}{1+\alpha_n}]\|x_n - x^*\| \\
&+ \alpha_n^2\|x_n - Sy_n\| \\
&+ \alpha_n\|Sx_{n+1} - Sy_n\| + (2\alpha_n + 1)\|u_n\| \\
\leq{}& [1 - \frac{\sigma(\|x_{n+1} - x^*\|)\alpha_n}{1+\alpha_n}]\|x_n - x^*\| \\
&+ \alpha_n^2\|x_n - Sy_n\| \\
&+ \alpha_n\|Sx_{n+1} - Sy_n\| + (2\alpha_n + 1)\|u_n\| \\
\leq{}& [1 - \frac{\sigma(\|x_{n+1} - x^*\|)\alpha_n}{2}]\|x_n - x^*\| \\
&+ \alpha_n^2\|x_n - Sy_n\| \\
&+ \alpha_n\|Sx_{n+1} - Sy_n\| + 3\|u_n\|.
\end{aligned}$$

Since R(T) is bounded, we have $\{Ty_n\}$ is bounded, let

$$\begin{aligned}
d &= sup_{n\geq 0}\{\|Ty_n - x^*\|\} + \|x_0 - x^*\| \\
M &= d + \sum_{n=0}^{\infty}\|u_n\| + 1.
\end{aligned}$$

By induction, we assert that

$$\|x_{n+1} - x^*\| \leq d + \textstyle\sum_{j=0}^{n}\|u_n\| \leq M \quad (n = 1, 2, 3, ...).$$

So $\{x_n\}$ is bounded, thus $\{Sx_n\}$ is bounded.
Therefore

$$\begin{aligned} \|x_n - Sy_n\| &\leq \|x_n - x^*\| + \|Sy_n - Sx^*\| \\ &\leq M_1 \\ \|x_n - y_n\| &\leq \beta_n\|x_n - Sx_n\| + \|v_n\| \to 0 \end{aligned}$$

Thus

$$\|x_{n+1} - y_n\| \leq (1-\alpha_n)\|x_n - y_n\| + \alpha_n\|Sy_n - y_n\| + \|u_n\| \to 0.$$

Since T is uniformly continuous, we have

$$\|Sx_{n+1} - Sy_n\| \to 0.$$

Set

$$\begin{aligned} b_n &= \alpha_n^2\|x_n - Sy_n\| + \alpha_n\|Sx_{n+1} - Sy_n\| \\ c_n &= 3\|u_n\| \\ a_n &= \|x_n - x^*\|. \end{aligned}$$

Then we have

$$a_{n+1} \leq [1 - \tfrac{\sigma(\|x_{n+1}-x^*\|)\alpha_n}{2}]a_n + b_n + c_n.$$

According to above argument, it is easy seen that

$$b_n = o(\alpha_n), \quad \textstyle\sum_{n=1}^{\infty} c_n < \infty.$$

We discern the following cases which cover all the possibilities:
1. $\inf\{\|x_n - x^*\|\} > \delta$ for some $\delta > 0$;
2. $\inf\{\|x_n - x^*\|\} = 0$.

In the case 1. Suppose that $\inf\{\|x_n - x^*\|\} > \delta$, then $\|x_n - x^*\| > \delta$ for all n.

So

$$\sigma(\|x_n - x^*\|) = \frac{\varphi(\|x_n - x^*\|)}{1+\varphi(\|x_n - x^*\|)} \geq \sigma(\delta) > 0.$$

We obtain

$$\begin{aligned} a_{n+1} &\leq [1 - \frac{\sigma(\|x_n - x^*\|)\alpha_n}{2}]a_n + b_n + c_n \\ &\leq [1 - \frac{\sigma(\delta)\alpha_n}{2}]a_n + b_n + c_n. \end{aligned}$$

Set

$$t_n = \frac{\sigma(\delta)\alpha_n}{2}.$$

Then we have

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n.$$

According to above argument, it is easy seen that

$$\sum_{k=1}^{\infty} t_n = \infty, \quad b_n = o(t_n), \quad \sum_{k=1}^{\infty} c_n \leq \infty$$

and so, by Lemma 1.2, we have $\lim a_n = \lim \|x_n - x^*\| = 0$ which contradicts the assumed $inf\{\|x_n - x^*\|\} > \delta > 0$.

In the case 2. Suppose that $\inf\{\|x_n - x^*\|\} = 0$, then there exist $\{x_{n_k}\}$ such that $\|x_{n_k} - x^*\| \to 0$.

Since $\|x_n - y_n\| \to 0$ and $\|x_{n+1} - y_n\| \to 0$, we have $\|x_{n_k+1} - x^*\| \leq \|x_{n_k+1} - y_{n_k}\| + \|y_{n_k} - x^*\| \leq \|x_{n_k+1} - y_{n_k}\| + \|x_{n_k} - y_{n_k}\| + \|x_{n_k} - x^*\| \to 0$, so $\|x_{n_k+1} - x^*\| \to 0$.

In the same way, we have

$$\|x_{n_k+2} - x^*\| \to 0.$$

By induction, we can prove that

$$\|x_{n_k+i} - x^*\| \to 0, \quad i = 1, 2, 3, \cdots.$$

Therefore, $x_n \to x^*$.

Thus, in all case, we see that the sequence $\{x_n\}$ converges strongly to the unique solution x^* of equation Tx = f. This completes the proof. □

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Department of Mathematics
Sun Yat-sen University
Guangzhou 510275, China
E-mail: stslyj@zsu.edu.cn