Commun. Korean Math. Soc. **20** (2005), No. 1, pp. 51–62

# BOUND FOR 2-EXPONENTS OF PRIMITIVE EXTREMAL MINISTRONG DIGRAPHS

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ABSTRACT. We consider 2-colored digraphs of the primitive ministrong digraphs having given exponents. In this paper we give bounds for 2-exponents of primitive extremal ministrong digraphs.

## 1. Introduction

We use the notation and terminology for digraphs as are in [4]. In this paper we let $D = (V, E)$ be a digraph and $\mathcal{D}$ be a 2-colored digraph of $D = (V, E)$. A nonnegative square matrix $A$ is *primitive* provided there is a nonnegative integer $k$ such that $A^k$ is entrywise positive (some authors say *strictly positive*), denoted by $A^k \gg 0$. If $A$ is primitive, the smallest integer $k$ such that $A^k$ has only positive entries is called the *exponent* of $A$, denoted by $\exp(A)$.

As defined in [6], a positive discrete homogeneous 2D-system is described by the equation.

$$x(h+1, k+1) = Ax(h, k+1) + Bx(h+1, k), h, k \in \mathbb{Z}, h+k \geq 0, \tag{1.1}$$

where $A$ and $B$ are $n$ by $n$ nonnegative matrices and the initial conditions $x(h, -h)$ $(h \in \mathbb{Z})$ are nonnegative $n$ by 1 vectors. System (1.1) is called the 2D-system associated with the nonnegative matrix pair $(A, B)$. Positive discrete homogeneous 2D-dynamical systems are used in [5] to discretely model diffusion processes. A component of the vector $x(h, k)$ typically represents a quantity such as pressure, concentration or density at a particular site along a stream. We can view this stream as flowing left-to-right along the line $y = -x$. The points $(h, -h)$ $(h \in \mathbb{Z})$ correspond to the discrete sites $h (h \in \mathbb{Z})$ along the stream. The vector

Received May 19, 2004.
2000 Mathematics Subject Classification: 05C20, 05C50.
Key words and phrases: Ministrong digraphs, 2-primitive, 2-exponent.
This work is supported by Faculty Research Fund at Sungkyunkwan University 2003 and Com$^2$MaC-KOSEF.

$x(h,k)$ represents the conditions at site $h$ after $h+k$ time-steps. Thus, $x(h,-h)$ describes the initial conditions at site $h$, the vector $x(h,-h+1)$ describes the conditions at site $h$ after 1 time-step, etc. Note that by setting $t=h+k+1$, the equation (1.1) indicates that conditions at site $h+1$ after $t+1$ time-steps are determined in a linear, time and location autonomous fashion from the conditions at site $h+1$ after $t$ time-steps and the conditions at site $h$ after $t$ time-steps. Thus, at each time-step the conditions of a site are determined by its previous conditions and the conditions of the site directly upstream from it [8].

DEFINITION 1.1. [8] For nonnegative integers $h$ and $k$, the $(h,k)$-*Hurwitz product*, $(A,B)^{(h,k)}$, of $A$ and $B$ is the sum of all matrices that are a product of $h$ $A$'s and $k$ $B$'s.

For example, $(A,B)^{(1,0)} = A$ and

$$(A,B)^{(2,2)} = A^2B^2 + ABAB + AB^2A + BA^2B + BABA + B^2A^2.$$

Define the pair $(A,B)$ of nonnegative matrices to be *2-primitive* provided there exist nonnegative integers $h$ and $k$ such that $h+k>0$ and $(A,B)^{(h,k)} \gg 0$. The *2-exponent* of the primitive pair $(A,B)$ of matrices is defined to be the minimum value of $h+k$ taken over all pairs $(h,k)$ such that $(A,B)^{(h,k)} \gg 0$. We write $\exp(A,B)$ for the 2-exponent of the pair $(A,B)$.

In [6], it is shown that the nonnegative matrix pair $(A,B)$ has both $A$ and $B$ nonzero and is primitive if and only if the solutions to (1.1) are eventually strictly positive. This shows that the definition of 2-primitivity of matrix pairs truly generalizes the primitivity for a single nonnegative matrices.

A *two-colored digraph* is a digraph whose arcs are colored red or blue. We allow loops and multiple colored arcs from $i$ to $j$. There is a natural correspondence between two-colored digraphs and nonnegative matrix pairs. With each two-colored digraph $\mathcal{D}$ we associate a pair $(A,B)$ of $(0,1)$-matrices where the $(i,j)$-entry of $A$ is 1 if and only if there is a red arc from $i$ to $j$, and the $(i,j)$-entry of $B$ is 1 if and only if there is a blue arc from $i$ to $j$. For each pair $(A,B)$ of nonnegative $n$ by $n$ matrices, we associate the 2-colored digraph, $\mathcal{D}(A,B)$, with vertices $1,2,\ldots,n$, a red arc from $i$ to $j$ if $a_{ij}>0$ and a blue arc from $i$ to $j$ if $b_{ij}>0$. An $(h,k)$-*walk* from $i$ to $j$ in $\mathcal{D}$ is a walk from $i$ to $j$ consisting of $h$ red arcs, and $k$ blue arcs. One can easily show that the $(i,j)$-entry of $(A,B)^{(h,k)}$ is strictly positive if and only if there is an $(h,k)$-walk in $\mathcal{D}(A,B)$ from $i$ to $j$. Given a walk $w$ in $\mathcal{D}$, we write $r(w)$ and $b(w)$ for the numbers of red and blue arcs that $w$ has in it, and we call the column vector

$$(1.2) \qquad \begin{bmatrix} r(w) \\ b(w) \end{bmatrix} = (r(w), b(w))$$

the *composition* of $w$. The two-colored digraph $\mathcal{D}$ is *strongly connected* provided for each pair $(i, j)$ of vertices there is a walk in $\mathcal{D}$ from $i$ to $j$. The matrix pair $(A, B)$ is 2-primitive if and only if there exist nonnegative integers $h$ and $k$ with $h + k > 0$ such that for each pair $(i, j)$ of vertices there exists an $(h, k)$-walk in $\mathcal{D}$ from $i$ to $j$. We say the 2-colored digraph $\mathcal{D}$ is *2-primitive* provided the associated matrix pair $(A, B)$ is 2-primitive and the *2-exponent* of $\mathcal{D}$ is defined to be the 2-exponent of $(A, B)$. The definition for cycle matrix of $\mathcal{D}$ is given as following.

DEFINITION 1.2. [8] Let $\mathcal{D}$ be a 2-colored digraph and let $C = \{\gamma_1, \gamma_2, \ldots, \gamma_c\}$ be the set of all cycles of $\mathcal{D}$. Set $M$ to be the 2 by $c$ matrix whose $i$th column is the composition of $\gamma_i$, and $\langle M \rangle$ to be the additive subgroup of $\mathbb{Z}^2$ generated by the columns of $M$. We call $M$ the *cycle matrix* of $\mathcal{D}$.

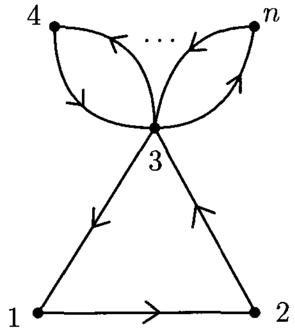
In Definition 1.2, $\langle M \rangle = \mathbb{Z}^2$ implies that $M$ has at least two linearly independent columns. Moreover, since each closed walk of $\mathcal{D}$ can be decomposed into cycles of $\mathcal{D}$, the composition of each closed walk of $\mathcal{D}$ belongs to $\langle M \rangle$.

Let $n \leq c$. The *content* of the $n$ by $c$ matrix $M$, denoted by content$(M)$, is defined to be 0 if the rank of $M$ is less than $n$ and to be the greatest common divisor of the determinants of the $n$ by $n$ submatrices of $M$, otherwise.

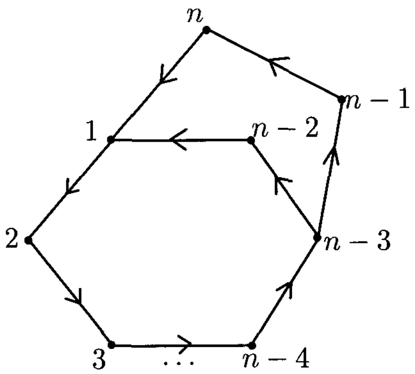
DEFINITION 1.3. [4] A strongly connected digraph $D$ is called *ministrong* (or minimally strongly connected) if each digraph obtained from $D$ by the removal of an arcs is not strongly connected.

By definition, a ministrong digraph has no loops. Each arc of a digraph $D$ corresponds to 1 in its adjacency matrix $A$. Thus the removal of an arc in $D$ means the replacement of a 1 with a 0 in $A$. An irreducible matrix $A$ is called *nearly reducible* provided each matrix obtained from $A$ by replacing of a 1 with a 0 is a reducible matrix. Thus the digraph $D$ is ministrong if and only if its adjacency matrix $A$ is nearly reducible. More generally we say that an arbitrary matrix $A$ of order $n$ is *nearly reducible* if its digraph $D$ is ministrong [4].

In section 2, we introduce some known results on ministrong digraphs and the 2-primitivity of 2-colored digraphs.




(a) $\exp(D) = 6$




(b) $\exp(D) = n^2 - 4n + 6$

FIGURE 1. Primitive ministrong digraphs with $6 \leq \exp(D) \leq n^2 - 4n + 6$

In section 3, we consider a primitive ministrong digraph $D$ with the largest exponent $n^2 - 4n + 6$ and find bounds for 2-exponent of $\mathcal{D}$ and $(h, k)$-walks from $i$ to $j$ for all $i, j \in V(\mathcal{D})$ satisfying our bounds for 2-exponent. Also, we find the 2-exponent of 2-primitive ministrong digraph on the primitive ministrong digraph with the least exponent 6.

Many applications of the 2-primitivity can be found in [2, 5, 8].

## 2. Preliminaries

Let $D = (V, E)$ be a primitive ministrong digraph of order $n$. In [3], Brualdi and Ross showed the following lemma.

LEMMA 2.1. [3] *Let $D$ be a primitive ministrong digraph with vertices. Then*

$$6 \leq \exp(D) \leq n^2 - 4n + 6$$

*with equality if and only if $D$ is isomorphic to the digraph in Figure 1.*

In [7], Ross showed that $\exp(D) \leq n + s(n - 3)$ for any ministrong digraph $D$ with girth $s$ and gave a digraph with equality.

The following Propositions 2.2 and Remark 1 are important for testing whether the nonnegative matrix pair $(A, B)$ is primitive or not.

PROPOSITION 2.2. [8] *Let $(A, B)$ be a pair of $n$ by $n$ nonnegative matrices with both $A$ and $B$ nonzero and let $M$ be the cycle matrix of $\mathcal{D}(A, B)$. Then the following are equivalent:*

(a) $(A, B)$ *is primitive,*
(b) $\mathcal{D}(A, B)$ *is strongly connected and* $\langle M \rangle = \mathbb{Z}^2$,
(c) $\mathcal{D}(A, B)$ *is strongly connected and* $\text{content}(M) = 1$.

One may rephrase this proposition as follows:

REMARK 1. A matrix pair $(A, B)$ is primitive if and only if the greatest common divisor of 2 by 2 minors of the cycle matrix $M$ of $\mathcal{D}(A, B)$ is 1.

Recently, Beasley and Kirkland showed the following proposition.

PROPOSITION 2.3. [1] *If $D$ is a primitive directed graph, then there is a 2-coloring of $D$ with which the corresponding 2-colored digraph $\mathcal{D}$ is 2-primitive.*

Shader and Suwilo gave the bounds on the 2-exponent of each 2-primitive Wielandt digraphs $\mathcal{D}$ as $2n^2 - 4n + 1 \leq \exp(\mathcal{D}) \leq 2n^2 - 3n + 1$ and then gave examples of 2-primitive digraphs with 2-exponent $(n^3 - 2n^2 + 1)/2$ for each odd integer $n \geq 5$ and $(n^3 - 5n^2 + 7n - 2)/2$ for each even integer $n \geq 6$ in [8].

## 3. Exponents of primitive extremal ministrong digraphs

In this section we consider a primitive ministrong digraph $D$ with the largest exponent on $n$ vertices as one can see $(b)$ of Figure 1. After we consider a 2-coloring of arcs on $D$, we show that the 2-colored $\mathcal{D}$ is 2-primitive and that $2n^2 - 8n + 7 \leq \exp(\mathcal{D}) \leq 2n^2 - 5n + 3$. Also, we give a 2-coloring to arcs on a primitive ministrong digraph with the smallest exponent 6 on $n$ vertices as one can see $(a)$ of Figure 1. Then it has 2-exponent 12.

We can find an upper bound for a 2-exponent of 2-primitive digraphs from Lemma 3.1.

LEMMA 3.1. [8, Lemma 1] *Let $\mathcal{D}$ be a strongly connected, 2-colored digraph with cycle matrix $M$. For each pair $(i, j)$ of vertices let $p_{ij}$ be a path from $i$ to $j$ and let $w$ be a closed walk that goes through each vertex of $\mathcal{D}$. Suppose that $\mathbf{z} = (z_1, z_2, \ldots, z_c)^T$ is a nonnegative, integer vector such that each system*

$$M\mathbf{x}_{ij} = \begin{bmatrix} r(p_{ij}) \\ b(p_{ij}) \end{bmatrix}$$

*has an integer solution* $\mathbf{x}_{ij}$ *with* $\mathbf{z} \geq \mathbf{x}_{ij}$. *Then* $\mathcal{D}$ *is 2-primitive and* $\exp(\mathcal{D}) \leq h + k$, *where* $h$ *and* $k$ *are defined by*

$$\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} r(w) \\ b(w) \end{bmatrix} + M\mathbf{z}.$$

We start with two-colored digraphs obtained by coloring arcs of the primitive ministrong digraph $D$ which has the largest exponent $n^2-4n+6$ in Lemma 2.1. We give a bound for 2-exponents of the two-colored digraphs of $D$.

THEOREM 3.2. *Let* $D$ *be a primitive ministrong digraph on* $n$ *vertices with the largest exponent* $n^2-4n+6$. *Let* $\mathcal{D}$ *be a 2-primitive ministrong digraph on* $n$ *vertices with at least one red and one blue arc. Then*

$$2n^2 - 8n + 7 \leq \exp(\mathcal{D}) \leq 2n^2 - 5n + 3$$

*First equality is obtained by the coloring* $D$ *like the* Figure 2-(b) (*blue arc is only in* $(n-2)$*-cycle*). *The second equality is obtained by the coloring* $D$ *like the* Figure 2-(a) (*one blue arc is in* $(n-1)$*-cycle and the other is in* $(n-2)$*-cycle*).

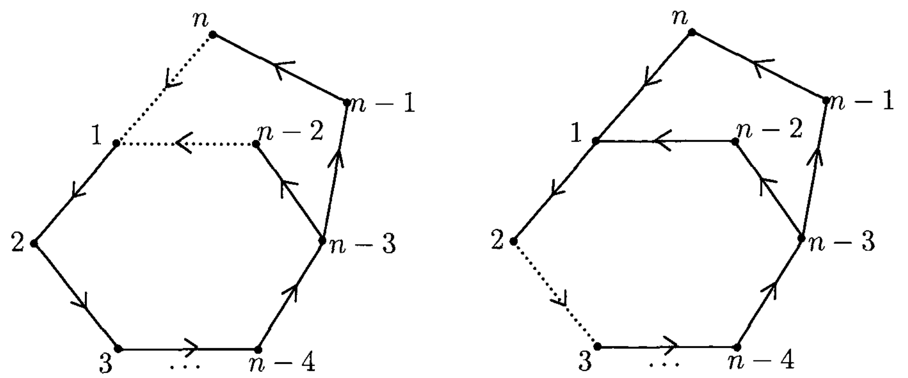
PROOF. We give red and blue colors to the arcs of $D$. Let $\mathcal{D}$ be the two-colored ministrong digraph of $D$ and assume that $\mathcal{D}$ has at least one red and one blue arc.

Let $M = \begin{bmatrix} n-1-a & n-2-b \\ a & b \end{bmatrix}$ be a cycle matrix of $\mathcal{D}$. Since $\det M \equiv 2a - b \equiv 1 \pmod{n}$ for $\mathcal{D}$ to be 2-primitive, we have $a = b = 1$. One can take $a = n-2$ and $b = n-3$, but this is just changing color between red and blue. So this argument has no loss of generality because even though we choose any other $a \neq 1, b \neq 1$ which satisfy $2a - b \equiv 1 \pmod{n}$, it is easy to check that there is no difference in getting the upper bound for 2-exponent. So one can take the cycle matrix $M$ of $\mathcal{D}$

$$M = \begin{bmatrix} n-2 & n-3 \\ 1 & 1 \end{bmatrix}$$

and $\mathcal{D}$ is 2-primitive by Proposition 2.2.

This cycle matrix $M$ implies that $(n-1)$-cycle of $\mathcal{D}$ contains exactly one blue arc. There are two possibilities. We say $\mathcal{D}$ is of type I (Figure 2-(a)), if the blue arc is one of $(n-3) \to (n-1)$ and $(n-1) \to n$ and $n \to 1$, and $\mathcal{D}$ is type II (Figure 2-(b)), otherwise. If $\mathcal{D}$ is type I, one of $(n-3) \to (n-2)$ and $(n-2) \to 1$ is blue. If $\mathcal{D}$ is type II, the arcs $(n-3) \to (n-2)$ and $(n-2) \to 1$ are both red.




*Thick arcs: red / Dotted arcs: blue*

(a) Type I-upper bound case (b) Type II-lower bound case

FIGURE 2. The 2-primitive ministrong digraphs

Let $p_{ij}$ be the path from $i$ to $j$ in $\mathcal{D}$. Let

$$\begin{bmatrix} u \\ v \end{bmatrix} = M^{-1} \begin{bmatrix} r(p_{ij}) \\ b(p_{ij}) \end{bmatrix}$$

for $\begin{bmatrix} r(p_{ij}) \\ b(p_{ij}) \end{bmatrix}$ as is in (1.2). Note that for each pair of vertices $i$ and $j$ of $\mathcal{D}$ there exists a path $p_{ij}$ from $i$ to $j$, with $r(p_{ij}) \leq n-2$ and $b(p_{ij}) \leq 1$. After analyzing all paths $p_{ij}$, we know that $\mathbf{x}_{ij} := (u, v)^T \leq (n-2, n-2)^T$. Let $w$ be a closed walk that goes through each vertex in $V(\mathcal{D})$. In other words, $w$ is a combined walk from 1 to 1 using $(n-1)$-cycle and $(n-2)$-cycle. By Lemma 3.1,

$$\begin{aligned} \begin{bmatrix} h \\ k \end{bmatrix} &= \begin{bmatrix} r(w) \\ b(w) \end{bmatrix} + M\mathbf{z} \text{ (with } \mathbf{z} \geq \mathbf{x}_{ij}) \\ &= \begin{bmatrix} 2n-5 \\ 2 \end{bmatrix} + \begin{bmatrix} n-2 & n-3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} n-2 \\ n-2 \end{bmatrix} \\ &= \begin{bmatrix} 2n^2-7n+5 \\ 2n-2 \end{bmatrix}. \end{aligned}$$

Therefore, $\exp(\mathcal{D}) \leq h+k = 2n^2-5n+3$. We will later find a $(2n^2-7n+5, 2n-2)$-walk between each pair $(i,j)$ of vertices of $\mathcal{D}$ in Remark 2.

Now we think the case of lower bound on the 2-exponent of $\mathcal{D}$ (Figure 2-(b)). Suppose that $(h,k)$ is a pair of nonnegative integers such that for all pairs $(i,j)$ of vertices there is an $(h,k)$-walk from $i$ to $j$. When $i=j$, we see that there exist nonnegative integers $u$ and $v$ with

$$\begin{bmatrix} h \\ k \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}.$$

This walk is the type of walk that are using only cycles. Next choose a blue arc on the $(n-1)$-cycle as like taking $i$ and $j$ to be the initial vertex and terminal vertex of the blue arc, respectively. Then the arcs of each walk from $i$ to $j$ can be decomposed into the arc from $i$ to $j$ and cycles. Hence,

$$\begin{bmatrix} h \\ k-1 \end{bmatrix} = M\mathbf{z}$$

has a nonnegative integer solution.

$$\begin{aligned} \mathbf{z} &= M^{-1}\begin{bmatrix} h \\ k-1 \end{bmatrix} \\ &= \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} n-3 \\ -n+2 \end{bmatrix} \\ &= \begin{bmatrix} u+n-3 \\ v-n+2 \end{bmatrix} \geq 0 \end{aligned}$$

Therefore,

$$v \geq n-2. \tag{3.1}$$

Now choose a blue arc on the $(n-1)$-cycle as like taking $i$ and $j$ to be the terminal vertex and initial vertex of the blue arc, respectively. Since the only paths from $i$ to $j$ have composition either $(n-2,0)$ or $(n-3,0)$, either

(i) $\begin{bmatrix} h-(n-2) \\ k \end{bmatrix} = M\mathbf{z}$ or (ii) $\begin{bmatrix} h-(n-3) \\ k \end{bmatrix} = M\mathbf{z}$

In the case of (i), we have

$$\begin{aligned} 0 \leq \mathbf{z} &= M^{-1}\begin{bmatrix} h-(n-2) \\ k \end{bmatrix} \\ &= \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1}\begin{bmatrix} n-2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} u-n+2 \\ v+n-2 \end{bmatrix}. \end{aligned}$$

Therefore, $u \geq n-2$. Similarly, we have $u \geq n-3$ for the case (ii). Thus

$$u \geq n-3, \tag{3.2}$$

and

$$h+k = \begin{bmatrix} 1 & 1 \end{bmatrix} M \begin{bmatrix} u \\ v \end{bmatrix} \geq \begin{bmatrix} n-1 & n-2 \end{bmatrix} \begin{bmatrix} n-3 \\ n-2 \end{bmatrix} = 2n^2-8n+7.$$

Therefore, we now have the interval for the 2-exponent of the 2-primitive ministrong digraph with the largest exponent;

$$\exp(\mathcal{D}) \in \left[2n^2-8n+7, 2n^2-5n+3\right].$$

□

REMARK 2. We can find a $(2n^2-7n+5, 2n-2)$-walk of length $2n^2-5n+3$ from $i$ to $j$ for all $i, j \in V$, which is the upper bound for 2-exponent in Theorem 3.2, as follows:

If $(i,j) = (n,n)$, then the walk that starts at vertex $i$, goes to vertex 1, goes $(n-1)$-times around the $(n-2)$-cycle, goes back to vertex $j$ and then goes $(n-2)$-times around the $(n-1)$-cycle has composition

$$(n-1)\begin{bmatrix} n-2 \\ 1 \end{bmatrix} + (n-1)\begin{bmatrix} n-3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2n^2-7n+5 \\ 2n-2 \end{bmatrix}.$$

Similarly, we can find a $(2n^2-7n+5, 2n-2)$-walk from $i$ to $j$ when $i=j$.

For the case of $(i,j) = (n-1, n)$, the walk that starts at vertex $i$, goes to 1, goes $n$-times around the $(n-2)$-cycle, goes to vertex $j$ and goes $(n-2)$-times around the $(n-1)$-cycle has composition

$$M\begin{bmatrix} n-2 \\ n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2n^2-7n+5 \\ 2n-2 \end{bmatrix}.$$

For the case of $(i,j) = (n, n-1)$, the walk that starts at vertex $i$, goes to 1, goes $(n-2)$-times around $(n-2)$-cycle, goes to vertex $j$ and goes $(n-1)$-times around $(n-1)$-cycle has composition

$$M\begin{bmatrix} n-1 \\ n-2 \end{bmatrix} + \begin{bmatrix} n-3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2n^2-7n+5 \\ 2n-2 \end{bmatrix}.$$

Otherwise, $p_{ij}$ contains vertices of the $(n-1)$-cycle and the $(n-2)$-cycle, and hence there is a walk that starts at vertex $i$, follows $p_{ij}$ to vertex $j$ and along the way and goes around the $(n-1)$-cycle $(n-1-u)$-times and the $(n-1)$-cycle $(n-1-v)$-times. Such a walk has composition

$$\begin{bmatrix} r(p_{ij}) \\ b(p_{ij}) \end{bmatrix} + M\begin{bmatrix} n-1-u \\ n-1-v \end{bmatrix} = M\begin{bmatrix} n-1 \\ n-1 \end{bmatrix} = \begin{bmatrix} 2n^2-7n+5 \\ 2n-2 \end{bmatrix}$$

For all choices of $i, j \in V$, we can find a $(2n^2-7n+5, 2n-2)$-walk from $i$ to $j$.

THEOREM 3.3. *Let $D$ be a primitive ministrong digraph on $n \geq 4$ with the smallest exponent* 6 *as in Lemma 2.1.(See* $(a)$ *of Figure 1.) Let $\mathcal{D}$ be a 2-primitive ministrong digraph on $n \geq 4$ vertices. Then* $\exp(\mathcal{D})$ *is* 12.

PROOF. As one can see $(a)$ of Figure 1, $D$ has one 3-cycle and $(n-3)$ 2-cycles. So we now consider the submatrix of the cycle matrix of $\mathcal{D}$ as following.

$$M = \begin{bmatrix} 3-a & 2-b \\ a & b \end{bmatrix}.$$

Since $\mathcal{D}$ is 2-primitive, $\det(M) \equiv 1$ which implies $a = b = 1$. So we can give a coloring to arcs on $D$ as blue color to the arcs from 1 to 2 and from 3 to $k$ for $4 \leq k \leq n$ and red color to others. Then the 2-colored $\mathcal{D}$ has one $(2,1)$-cycle and $(n-3)$ $(1,1)$-cycles. Let $M$ be the cycle matrix of $\mathcal{D}$,

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

One can easily check all path $p_{ij}$ from $i$ to $j$ in $\mathcal{D}$ have lengths at most 3. A composition (reds, blues) of path $p_{ij}$ is one of (1,0), (1,1), (1,2), (2,0) and (2,1). All $(h,k)$-walks from $i$ to $j$ in $\mathcal{D}$ are composed of the path $p_{ij}$ and cycles. Suppose that $(h,k)$ is a pair of nonnegative integers such that for all pairs $(i,j)$ of vertices there is an $(h,k)$-walk from $i$ to $j$. When $i = j$, we see that there exist nonnegative integers $u$ and $v$ with

$$\begin{bmatrix} h \\ k \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}.$$

This walk is the type of walk that are using only cycles. Next take paths $p_{ij}$ from $i$ to $j$ for $i \neq j$. Then the arcs of each walk from $i$ to $j$ can be decomposed into $p_{ij}$ and cycles.

For a path $(2,0)$,

$$\begin{bmatrix} h-2 \\ k \end{bmatrix} = M\mathbf{z}$$

has a nonnegative integer solution.

$$\begin{aligned} \mathbf{z} &= M^{-1} \begin{bmatrix} h-2 \\ k \end{bmatrix} \\ &= \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} u-2 \\ v+2 \end{bmatrix} \geq 0 \end{aligned}$$

Therefore,

$$u \geq 2. \tag{3.3}$$

Similarly, using all paths, we can easily check $(u, v)^T \geq (2, 3)$. Therefore,

$$\begin{bmatrix} h \\ k \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix} \geq \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

We now have $\exp(\mathcal{D}) = 12$ and one can find $(7,5)$-walks from $i$ to $j$ for all $i, j \in V(\mathcal{D})$. □

Acknowledgements. The authors would like to thank the anonymous referee for a number of helpful suggestions.

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