CORESTRICTION MAP ON BRAUER SUBGROUPS

Eunmi Choi

ABSTRACT. For an extension field K of k, a restriction homomorphism on Brauer k-group $B(k)$ maps Brauer k-algebras to Brauer K- algebras by tensor product. A purpose of this work is to study the restriction map that sends radical (Schur) k-algebras to radical (Schur) K-algebras. And we ask an analogous question with respect to corestriction map on Brauer group $B(K)$ that whether the corestriction map sends radical K-algebras to radical k-algebras.

1. Introduction

Let k be a field. A central simple k-algebra is a Brauer algebra, and a Brauer algebra that is a homomorphic image of a group algebra kG [resp. twisted group algebra $k^\alpha G$ with a 2-cocycle $\alpha \in Z^2(G, k^*)$] for some finite group G is called a Schur [resp. projective Schur] algebra. The similarity class containing a Brauer k-algebra A is denoted by $[A]$, and they form a Brauer group $B(k)$. The Schur group $S(k)$ [resp. projective Schur group $P(k)$] is a subgroup of $B(k)$ consisting of similarity classes which are represented by Schur [resp. projective Schur] algebras.

Let L/k be a finite Galois extension with Galois group $\mathcal{G} = \mathrm{Gal}(L/k)$. Let $A = \sum_{\sigma \in \mathcal{G}} Lu_\sigma$ denote an algebra having basis $\{u_\sigma | \sigma \in \mathcal{G}\}$ such that $u_\sigma x = \sigma(x)u_\sigma$ and $u_\sigma u_\tau = \alpha(\sigma, \tau)u_{\sigma\tau}$ for $x \in L$ and $\sigma, \tau \in \mathcal{G}$ where each $\alpha(\sigma, \tau) \in L^* = L - \{0\}$. Then $\alpha \in Z^2(\mathcal{G}, L^*)$, and A is called a crossed product algebra denoted by $(L/k, \alpha)$. In particular, if L is a cyclotomic extension $k(\varepsilon)$ of k (ε : primitive root of unity) and α has values in $\langle \varepsilon \rangle$, then $(L/k, \alpha)$ is called a cyclotomic algebra. A crossed product algebra $(L/k, \alpha)$ is called a (abelian) radical algebra ([1]) if $L = k(\Gamma)$ is a finite radical $\mathcal{G}$-Galois (abelian) extension over k (i.e., Γ is a multiplicative subgroup of L^* and Γ is $\mathcal{G}$-invariant) and $\alpha \in Z^2(\mathcal{G}, L^*)$ is the image

of some $\alpha' \in Z^2(\mathcal{G}, \Gamma)$. The sets of similarity classes of cyclotomic k-algebras and of radical k-algebras form a cyclotomic group $C(k)$ and a radical group $R(k)$ respectively. Clearly, $C(k) \leq S(k) \leq R(k) \leq P(k) \leq B(k)$.

If $[A] \in B(k)$ then there is a finite Galois extension E of k such that $[E \otimes A] = 1$ in $B(E)$. That is, E is a splitting field for A, and A is similar to a crossed product algebra $(E/k, \alpha)$ for some 2-cocycle $\alpha \in Z^2(E/k, E^*)$[10, p.28]. A subgroup $B(E/k)$ of $B(k)$ consisting of Brauer algebras which are split by E is called a relative Brauer group [4]. By the crossed product theorem [10, p.28], $B(E/k)$ is identified with $H^2(E/k, E^*)$ and

$$B(k) = \varinjlim_E H^2(E/k) = \bigcup_E H^2(E/k) = H^2(*/k)$$

where the direct limit $\varinjlim_E H^2(*/k)$ runs over all the finite Galois extensions E of k.

Let K/k be an extension field. Then the homomorphism

$$\mathrm{Res}_{k \to K} : B(k) \to B(K) \quad \text{defined by} \quad [A] \mapsto [K \otimes A]$$

is the restriction map on Brauer group, and the kernel of $\mathrm{Res}_{k \to K}$ is composed of Brauer k-algebras split by K. Thus $\ker(\mathrm{Res}_{k \to K})$ is equal to $B(K/k)$. If E/k is a finite Galois extension containing K/k, then $\mathrm{Res}_{k \to K}$ is also defined on $B(E/k) \to B(E/K)$ by $[A] \mapsto [K \otimes A]$. This yields the commutative diagram that

$$
\begin{array}{ccc}
H^2(E/k) & \overset{\mathrm{res}_{k \to K}}{\to} & H^2(E/K) \\
\phi \downarrow & & \phi \downarrow \\
B(E/k) & \overset{\mathrm{Res}_{k \to K}}{\to} & B(E/K)
\end{array}
$$

(refer to [7] p.252), where $\phi(f) = [(E/k, f)] \in B(E/k)$ is an isomorphism for $f \in Z^2(E/k)$, and $\mathrm{res}_{k \to K}$ is the cohomological restriction defined in the following manner: for any groups $H < G$ and any G-module M, res : $H^2(G, M) \to H^2(H, M)$ satisfies $(\mathrm{res}\alpha)(h_1, h_2) = \alpha(h_1, h_2)$ with $\alpha \in Z^2(G, M)$ and $h_1, h_2 \in H$. Thus with a crossed product algebra $[(E/k, \alpha)]$ in $B(E/k)$, we have

$$\mathrm{Res}_{k \to K}\big[(E/k, \alpha)\big] = \big[K \otimes (E/k, \alpha)\big] = \big[(E/K, \mathrm{res}_{k \to K}\alpha)\big].$$

In this paper we first ask whether the restriction map Res : $B(k) \to B(K)$ on Brauer groups sends Schur k-algebras to Schur K-algebras, and radical k-algebras to radical K-algebras. In Section 2 we show that $\mathrm{Res}(S(k))$ and $\mathrm{Res}(R(k))$ are contained in $S(K)$ and $R(K)$ respectively. We then ask the similar question with respect to the corestriction map

Cor $: B(K) \to B(k)$. While restriction maps are defined explicitly by tensor product, corestriction maps are not much known. In Section 3 we adapt the notion of class formation introduced in class field theory [2]. The Galois extension K/k produces a formation, and the situation of cohomology group in the class formation is entirely analogous to that of Brauer group. We will define corestriction maps on Schur group and Radical group precisely (in Theorem 7) and prove that $\mathrm{Cor}(S(K))$, $\mathrm{Cor}(R(K))$ and $\mathrm{Cor}(R(E/K))$ are contained in $S(k)$, $R(k)$ and $R(E/k)$ respectively.

In what follows, let $Z^2(K/k, M) = Z^2(\mathcal{G}, M)$ denote the cocycle group defined over $\mathrm{Gal}(K/k) = \mathcal{G}$ and a $\mathcal{G}$-module M, and $H^2(K/k, M)$ the cohomology group. When $M = K^*$, we write $Z^2(K/k, K^*) = Z^2(K/k)$. We shall denote $\bar{\alpha} \in H^2(K/k)$ for the cohomologous class of $\alpha \in Z^2(K/k)$. Denote Res, Inf and Cor the restriction, inflation and corestriction map on Brauer groups respectively, while res, inf and cor the cohomological ones. We write by ε a primitive root of unity.

2. Preliminaries and restriction map on Brauer groups

Throughout the paper we always assume that K is a Galois extension field of k. We begin with a well known fact about crossed product algebras.

REMARK 2.1. ([7], (29.6)) Two 2-cocycles α and β in $Z^2(K/k)$ are cohomologous if and only if the crossed product algebras $(K/k, \alpha)$ and $(K/k, \beta)$ are k-isomorphic.

We shall verify that $\mathrm{Res}_{k \to K} : B(k) \to B(K)$ maps the subgroups $S(k)$, $C(k)$, $P(k)$ and $R(k)$ of $B(k)$ to $S(K)$, $C(K)$, $P(K)$ and $R(K)$ respectively.

THEOREM 2.2. *The restriction* $\mathrm{Res}_{k \to K}$ *maps (projective) Schur k-algebras to (projective) Schur K-algebras, that is,* $\mathrm{Res}_{k \to K}|_{P(k)} : P(k) \to P(K)$ *and* $\mathrm{Res}_{k \to K}|_{S(k)} : S(k) \to S(K)$ *are homomorphisms.*

PROOF. Let $[S] \in P(k)$. Then there is $A \in [S]$ that is an image of $k^\alpha G$ by a surjective homomorphism ϕ, for some finite group G and $\alpha \in Z^2(G, k^*)$. Regarding $[A]$ as in $B(k)$, $\mathrm{Res}_{k \to K}[A] = [K \otimes A] \in B(K)$. Hence if let $\bar{\phi} = K \otimes \phi$ then $\bar{\phi}$ is a surjection of $K \otimes k^\alpha G = K^\alpha G$ onto $K \otimes A$. And the center $Z(K \otimes A) = K \otimes Z(A) = K$, because $Z(A) = k$. This shows that $K \otimes A$ is a central simple K-algebra which

is a homomorphic image of $K^{\alpha}G$. Thus $\mathrm{Res}_{k \to K}[S] = \mathrm{Res}_{k \to K}[A] = [K \otimes A] \in P(K)$. When $\alpha = 1$, $\mathrm{Res}_{k \to K}|_{S(k)} : S(k) \to S(K)$ is a homomorphism. $\square$

THEOREM 2.3. *The $\mathrm{Res}_{k \to K}$ maps cyclotomic k-algebras to cyclotomic K-algebras, that is, $\mathrm{Res}_{k \to K}|_{C(k)} : C(k) \to C(K)$ is a homomorphism.*

PROOF. Let $[S] \in C(k)$ and $A \in [S]$ be a cyclotomic k-algebra $(k(\varepsilon)/k, \alpha)$ with $\alpha \in Z^2(k(\varepsilon)/k, \langle \varepsilon \rangle)$. Due to [7](29.13), $K \otimes (k(\varepsilon)/k, \alpha) \cong (K(\varepsilon)/K, \gamma)$ where γ is obtained from α by restriction to $\mathrm{Gal}(K(\varepsilon)/K) \cong \mathrm{Gal}(k(\varepsilon)/(K \cap k(\varepsilon))) < \mathrm{Gal}(k(\varepsilon)/k)$. Since $\gamma(x, y) = \alpha(x, y)$ for $x, y \in \mathrm{Gal}(K(\varepsilon)/K)$, γ has values in $\langle \varepsilon \rangle$ and $(K(\varepsilon)/K, \gamma)$ is a cyclotomic K-algebra. Therefore $[S] = [A]$ and

$$\mathrm{Res}_{k \to K}[S] = K \otimes \big[(k(\varepsilon)/k, \alpha)\big] = \big[(K(\varepsilon)/K, \gamma\big] \in C(K).$$

$\square$

REMARK 2.4. By virtue of cohomological homomorphisms

$$H^2\big(k(\varepsilon)/k\big) \overset{\mathrm{res}_{k \to K \cap k(\varepsilon)}}{\to} H^2\big(k(\varepsilon)/(K \cap k(\varepsilon))\big) \overset{I}{\cong} H^2\big(K(\varepsilon)/K\big),$$

(*I* an isomorphism [5] p.268, we may consider $\gamma = I \, \mathrm{res}_{k \to K \cap k(\varepsilon)} \alpha$, so

$$\mathrm{Res}_{k \to K}[S] = \big[(K(\varepsilon)/K, \, I \quad \mathrm{res}_{k \to K \cap k(\varepsilon)} \alpha)\big]$$
$$= \big[(k(\varepsilon)/(K \cap k(\varepsilon)), \, \mathrm{res}_{k \to K \cap k(\varepsilon)} \alpha)\big].$$

In particular, if $k = K \cap k(\varepsilon)$ or k is a maximal cyclotomic extension over rational number field contained in K then $\mathrm{Res}_{k \to K}[(k(\varepsilon)/k, \alpha)] = [(K(\varepsilon)/K, \alpha)]$ (refer to [10], p.46).

THEOREM 2.5. *The $\mathrm{Res}_{k \to K}$ maps radical k-algebras to radical K-algebras. That is, $\mathrm{Res}_{k \to K}|_{R(k)} : R(k) \to R(K)$ is a homomorphism.*

PROOF. Let $[S] \in R(k)$ and $A \in [S]$ be a radical k-algebra $(k(\Gamma)/k, \alpha)$ where $k(\Gamma)$ is a finite radical Galois extension over k and $\alpha \in Z^2(k(\Gamma)/k)$ is an image of some $\alpha' \in Z^2(k(\Gamma)/k, \Gamma)$. Regarding A as a Brauer k-algebra, $\mathrm{Res}_{k \to K}$ on $B(k)$ determines

$$\mathrm{Res}_{k \to K}[S] = \mathrm{Res}_{k \to K}[A] = \big[K \otimes (k(\Gamma)/k, \alpha)\big]$$
$$= \big[(K(\Gamma)/K, \mathrm{res}_{k \to K \cap k(\Gamma)} \alpha)\big].$$

Since $k(\Gamma)/k$ is Galois, so are $k(\Gamma)/(K \cap k(\Gamma))$ and $K(\Gamma)/K$. Moreover since the quotient group $\Gamma k^*/k^*$ is finite (i.e., for any $\gamma \in \Gamma$, $\gamma^n \in k^*$ for

some $n > 0$), $\Gamma K^*/K^*$ is also finite. Hence $K(\Gamma)/K$ is a radical extension. And $\mathrm{res}_{k \to K \cap k(\Gamma)}\alpha$ is an image of $\mathrm{res}_{k \to K \cap k(\Gamma)}\alpha' \in Z^2(K(\Gamma)/K, \Gamma)$. Thus $(K(\Gamma)/K, \mathrm{res}_{k \to K \cap k(\Gamma)}\alpha)$ is a radical K-algebra and $\mathrm{Res}_{k \to K}[S] \in R(K)$. $\qquad\square$

We remark some properties of inflation maps for next use. Let $k < K < E$ be field extensions and assume that $\mathrm{Gal}(E/K)$ is a normal subgroup of $\mathrm{Gal}(E/k)$. Then $\mathrm{Gal}(E/k)/\mathrm{Gal}(E/K)$ is isomorphic to $\mathrm{Gal}(K/k)$. The cohomological inflation map $\inf_{K \to E} : H^2(K/k) \to H^2(E/k)$ is defined by, for $\beta \in Z^2(K/k)$,

$$\bigl(\inf{}_{K \to E}\beta\bigr)(x_1, x_2) = \beta\bigl(x_1\mathrm{Gal}(E/K),\ x_2\mathrm{Gal}(E/K)\bigr)$$

for any $x_i \in \mathrm{Gal}(E/k)$.

REMARK 2.6. ([7], p.253) For finite Galois field extensions $k < K < E$, we have the following commutative diagram that

$$
\begin{array}{ccc}
H^2(K/k) & \xrightarrow{\inf_{K \to E}} & H^2(E/k) \\
\downarrow & & \downarrow \\
B(K/k) & \xrightarrow{\mathrm{Inf}_{K \to E}} & B(E/k),
\end{array}
$$

where vertical arrows are isomorphisms. Moreover the crossed product algebra $(K/k, \alpha)$ is similar to $(E/k, \inf_{K \to E}\alpha)$ for $\alpha \in Z^2(K/k)$.

3. Corestriction map on Brauer groups

Alongside restrictions $\mathrm{res}_{k \to K} : H^2(E/k) \to H^2(E/K)$ are the ones in the opposite direction, called corestriction map, denoted by $\mathrm{cor}_{K \to k} : H^2(E/K) \to H^2(E/k)$. And the corestriction map $\mathrm{Cor}_{K \to k} : B(E/k) \to B(E/K)$ on Brauer groups comes from the corresponding map on cohomology groups which satisfies the following commutative diagrams:

$$
\begin{array}{ccccccc}
H^2(E/K) & \xrightarrow{\mathrm{cor}_{K \to k}} & H^2(E/k) & & \varinjlim H^2(E/K) & \xrightarrow{\mathrm{cor}_{K \to k}} & \varinjlim H^2(E/k) \\
\downarrow & & \downarrow & ; & \downarrow & & \downarrow \\
B(E/K) & \xrightarrow{\mathrm{Cor}_{K \to k}} & B(E/k) & & B(K) & \xrightarrow{\mathrm{Cor}_{K \to k}} & B(k)
\end{array}
$$

where the direct limit $\varinjlim H^2(*/k)$ runs over all the finite Galois extensions of k.

While $\mathrm{Res}_{k \to K}$ on $B(k)$ is defined explicitly by a tensor product, the corestriction $\mathrm{Cor}_{K \to k}$ on $B(K)$ is not much known. In this section

we study how the corestriction map $\mathrm{Cor}_{K \to k}$ works on both $S(K)$ and $R(K)$.

Let $H < G$ with $|G : H| = u$ and let $S = \{s_1 = 1, \cdots, s_u\}$ be a right transversal of H in G. Then any $x \in G$ can be written as $x = h s_i$ for some $h \in H$ and $1 \leq i \leq u$. Write $\tilde{x}$ for s_i, so that $x\tilde{x}^{-1} \in H$.

REMARK 3.1. ([6], (2.3.3)) Let $H < G$ and $\alpha \in Z^2(H, M)$ with a G-module M. Then $(\mathrm{cor}_{H \to G}\alpha)(x,y) = \prod_{i=1}^{u} \alpha(s_i x \widetilde{s_i x}^{-1}, \widetilde{s_i x} y \widetilde{s_i x y}^{-1})$ for any $x, y \in G$. And the composition homomorphism $\mathrm{cor}_{H \to G}\mathrm{res}_{G \to H}$ on cohomology groups maps $\beta \in H^2(G, M)$ to β^u for $u = |G : H|$.

Analogously, the composition map $\mathrm{Cor}_{K \to k}\mathrm{Res}_{k \to K}$ on Brauer groups sends $[A] \in B(k)$ to $[A]^u$ for $u = |K : k|$ ([10] p.28).

For Galois extensions $k < K < L < E$, the diagram involving restriction, corestriction and inflation maps is commutative ([9](2.3.7),(2.4.5)):

$$
\begin{array}{ccccc}
H^2(L/k) & \overset{\mathrm{res}_{k \to K}}{\to} & H^2(L/K) & \overset{\mathrm{cor}_{K \to k}}{\to} & H^2(L/k) \\
\inf_{L \to E} \downarrow & & \downarrow \inf_{L \to E} & & \downarrow \inf_{L \to E} \\
H^2(E/k) & \overset{\mathrm{res}_{k \to K}}{\to} & H^2(E/K) & \overset{\mathrm{cor}_{K \to k}}{\to} & H^2(E/k)
\end{array}
\qquad (1)
$$

Let G be a group and X be a finite indexed set. Let $\{G_F\}_{F \in X}$ be a family of subgroups of finite index in G satisfying the followings:

1. For every finite family $F_i \in X$, there is $F \in X$ such that $G_F = \cap G_{F_i}$.
2. For any G' with $G_F < G' < G$ for $F \in X$, there is $F' \in X$ such that $G' = G_{F'}$.
3. For any $g \in G$ and $F_i \in X$, there is $F_j \in X$ such that $g G_{F_i} g^{-1} = G_{F_j}$.

The system $(G, \{G_F\}_{F \in X}, M)$ with a G-module M is called a formation ([8], (XI)). If $F, E \in X$, we say that E is an extension of F (write E/F) whenever $G_E < G_F$. In particular if G_E is normal in G_F, E/F is said to be Galois, and the quotient group G_F/G_E is called the Galois group. A class formation is a formation $(G, \{G_F\}_{F \in X}, M)$ satisfying the following Axioms I and II.

I. For every Galois extension E/F, $H^1(E/F) = 0$.
II. Inv_F on $H^2(*/F)$ is injective, and $\mathrm{Inv}_L \mathrm{Res}_{F \to L} = |L : F|\mathrm{Inv}_F$ for every L/F.

Here, $\mathrm{Inv}_F : H^2(E/F) \to \mathbb{Q}/\mathbb{Z}$ is the invariant homomorphism and $H^2(*/F)$ denotes the direct limit of $H^2(E/F)$, as E runs through the set of Galois extensions of F. The group $H^2(*/F)$ is therefore the union

of $H^2(E/F)$, so the situation is entirely analogous to that of the Brauer group $B(F)$.

Let K/k be a Galois extension field with Galois group $G = \mathrm{Gal}(K/k)$, in particular K is often taken as a separable closure of k. Let X denote the set of all finite extensions of k contained in K, let $G_F = \mathrm{Gal}(K/F)$ for each $F \in X$, and let $M = K^*$. Then the ordinary Galois theory shows that $(G, \{G_F\}_{F \in X}, M)$ forms a formation of k such that $|G : G_F| = |F : k|$ and the fixed subgroup M^{G_F} of M by G_F equals F^*. Furthermore, if F_i/F_j is a Galois extension with any $F_i, F_j \in X$ then Noether's equation ([8], p.150 or [9], (1.5.4)) shows that $H^1(F_i/F_j) = 0$ which is the Axiom I. Hence a formation of k satisfying Axiom II is a class formation. It is proved in (Chapter 6, Theorem 1 at [2]) that formations of both local fields (see [7], (31.9)) and global fields (see [3], [8]) are class formations, however the verification for global class field theory is more involved with idele classes.

PROPOSITION 3.2. ([8], p.167) Let $(G, \{G_F\}_{F \in X}, M)$ be a class formation of k. Let $F < E < L$ be in X with L/F Galois. Then $res_{F \to E} : H^2(L/F) \to H^2(L/E)$ is surjective, so $Res_{F \to E}$ is a surjective homomorphism from $B(F)$ onto $B(E)$.

Next theorem which is one of our main result provides a construction of the corestriction map on Schur group according to cohomological corestriction map.

THEOREM 3.3. Let K/k be a Galois extension. If the formation $(G, \{G_F\}_{F \in X}, M)$ of k satisfies the Axiom II, then $Cor_{K \to k}$ on $B(K)$ maps Schur K-algebras to Schur k-algebras, that is, $Cor_{K \to k}|_{S(K)} : S(K) \to S(k)$ is a homomorphism.

PROOF. Let $[S] \in S(K)$. By Brauer-Witt theorem ([10], p.31), there is $A \in [S]$ which is similar to a cyclotomic algebra $(K(\varepsilon)/K, \alpha)$ with $\alpha \in Z^2(K(\varepsilon)/K, \langle \varepsilon \rangle)$.

Clearly $[S] = [A]$ and $K(\varepsilon)$ is a Galois extension over k, because both $K(\varepsilon)/K$ and K/k are Galois extensions and $K(\varepsilon)$ is a splitting field over K of a polynomial in $k[X]$ (see [5], p.268). Hence the corestriction map $\mathrm{cor}_{K \to k} : H^2(K(\varepsilon)/K) \to H^2(K(\varepsilon)/k)$ can be determined, and $\mathrm{cor}_{K \to k}\alpha$ belongs to $Z^2(K(\varepsilon)/k, \langle \varepsilon \rangle)$ due to Remark 3.1.

Denote the crossed product algebra $(K(\varepsilon)/k, \mathrm{cor}_{K \to k}\alpha)$ by A'. Then $[A'] \in B(k)$, and we define a map ϕ on $S(K)$ by $[A] \mapsto [A']$, that is,

$$\phi : S(K) \to B(k), \quad \phi\big[(K(\varepsilon)/K, \alpha)\big] = \big[(K(\varepsilon)/k, \mathrm{cor}_{K \to k}\alpha)\big].$$

Let $(L/K, \beta)$ be any crossed product algebra contained in $[A]$. Then $(L/K, \beta)$ is similar to $A = (K(\varepsilon)/K, \alpha)$. By Remark 2.6, $(LK(\varepsilon)/K,$ $\inf_{L \to LK(\varepsilon)} \beta)$ is similar to $(LK(\varepsilon)/K, \inf_{K(\varepsilon) \to LK(\varepsilon)} \alpha)$, hence $\inf_{L \to LK(\varepsilon)} \beta$ and $\inf_{K(\varepsilon) \to LK(\varepsilon)} \alpha$ are cohomologous in $H^2(LK(\varepsilon)/K)$. Applying corestriction $\operatorname{cor}_{K \to k} : H^2(LK(\varepsilon)/K) \to H^2(LK(\varepsilon)/k)$ to the commutative diagram (1), $\operatorname{cor}_{K \to k} \inf_{L \to LK(\varepsilon)} \beta = \inf_{L \to LK(\varepsilon)} \operatorname{cor}_{K \to k} \beta$ is cohomologous to $\operatorname{cor}_{K \to k} \inf_{K(\varepsilon) \to LK(\varepsilon)} \alpha = \inf_{K(\varepsilon) \to LK(\varepsilon)} \operatorname{cor}_{K \to k} \alpha$ in $H^2(LK(\varepsilon)/k)$ This shows the well definedness of ϕ that

$$
\begin{aligned}
\phi[(L/K, \beta)] &= [(L/k, \operatorname{cor}_{K \to k} \beta)] \\
&= [(LK(\varepsilon)/k, \inf_{L \to LK(\varepsilon)} \operatorname{cor}_{K \to k} \beta)] \\
&= [(LK(\varepsilon)/k, \inf_{K(\varepsilon) \to LK(\varepsilon)} \operatorname{cor}_{K \to k} \alpha)] \\
&= [(K(\varepsilon)/k, \operatorname{cor}_{K \to k} \alpha)] = \phi[(K(\varepsilon)/K, \alpha)].
\end{aligned}
$$

For $i = 1, 2$, if $A_i = (K(\varepsilon_i)/K, \alpha_i)$ are Schur K-algebras with cyclotomic extensions $K(\varepsilon_i)$ of K and $\alpha_i \in Z^2(K(\varepsilon_i)/K, \langle \varepsilon_i \rangle)$, and if we write $K(\varepsilon_1, \varepsilon_2) = K(\varepsilon_{12})$ then

$$
\begin{aligned}
&[A_1][A_2] \\
&= \left[(K(\varepsilon_{12})/K, \ \inf_{K(\varepsilon_1) \to K(\varepsilon_{12})} \alpha_1) \right] \left[(K(\varepsilon_{12})/K, \inf_{K(\varepsilon_2) \to K(\varepsilon_{12})} \alpha_2) \right] \\
&= \left[(K(\varepsilon_{12})/K, \ \inf_{K(\varepsilon_1) \to K(\varepsilon_{12})} \alpha_1 \cdot \inf_{K(\varepsilon_2) \to K(\varepsilon_{12})} \alpha_2) \right]
\end{aligned}
$$

by Remark 2.6, thus it follows from the diagram (1) that

$$
\begin{aligned}
\phi\big([A_1][A_2]\big) \\
&= \big[(K(\varepsilon_{12})/k, \ \operatorname{cor}_{K \to k}(\inf_{K(\varepsilon_1) \to K(\varepsilon_{12})} \alpha_1 \cdot \\
&\quad \inf_{K(\varepsilon_2) \to K(\varepsilon_{12})} \alpha_2)) \big] \\
&= \big[(K(\varepsilon_{12})/k, \ (\operatorname{cor}_{K \to k} \inf_{K(\varepsilon_1) \to K(\varepsilon_{12})} \alpha_1) \cdot \\
&\quad (\operatorname{cor}_{K \to k} \inf_{K(\varepsilon_2) \to K(\varepsilon_{12})} \alpha_2)) \big] \\
&= \big[(K(\varepsilon_{12})/k, \ \operatorname{cor}_{K \to k} \inf_{K(\varepsilon_1) \to K(\varepsilon_{12})} \alpha_1) \big] \\
&\quad \big[(K(\varepsilon_{12})/k, \ \operatorname{cor}_{K \to k} \inf_{K(\varepsilon_2) \to K(\varepsilon_{12})} \alpha_2) \big] \\
&= \big[(K(\varepsilon_{12})/k, \ \inf_{K(\varepsilon_1) \to K(\varepsilon_{12})} \operatorname{cor}_{K \to k} \alpha_1) \big] \\
&\quad \big[(K(\varepsilon_{12})/k, \ \inf_{K(\varepsilon_2) \to K(\varepsilon_{12})} \operatorname{cor}_{K \to k} \alpha_2) \big] \\
&= \big[(K(\varepsilon_1)/k, \ \operatorname{cor}_{K \to k} \alpha_1) \big] \\
&\quad \big[(K(\varepsilon_2)/k, \ \operatorname{cor}_{K \to k} \alpha_2) \big] \\
&= \phi\big([A_1]\big) \phi\big([A_2]\big),
\end{aligned}
$$

hence ϕ is a homomorphism on $S(K)$.

Now we claim that

$$\phi = \mathrm{Cor}_{K \to k}|_{S(K)} \; : S(K) \to B(k).$$

Since $\mathrm{Res}_{k \to K}|_{S(k)} : S(k) \to S(K)$ is a homomorphism by Theorem 2.2, both compositions $\mathrm{Cor}_{K \to k}|_{S(K)} \cdot \mathrm{Res}_{k \to K}|_{S(k)}$ and $\phi \cdot \mathrm{Res}_{k \to K}|_{S(k)}$ are defined over $S(k)$ to $B(k)$.

Let $[T] \in S(k)$ and let $B \in [T]$ be a cyclotomic k-algebra $(k(\nu)/k, \beta)$ with primitive root of unity ν and $\beta \in Z^2(k(\nu)/k, \langle \nu \rangle)$. Then $\phi \cdot \mathrm{Res}_{k \to K}|_{S(k)}$ defines

$$
\begin{aligned}
&\big(\phi \cdot \mathrm{Res}_{k \to K}|_{S(k)}\big)[T] \\
&= \big(\phi \cdot \mathrm{Res}_{k \to K}|_{S(k)}\big)[(k(\nu)/k, \beta)] \\
&= \phi\big[(K(\nu)/K, I \; \mathrm{res}_{k \to K \cap k(\nu)}\beta)\big] \\
&= \big[(K(\nu)/k, \; \mathrm{cor}_{K \to k}I \; \mathrm{res}_{k \to K \cap k(\nu)}\beta)\big]
\end{aligned}
\tag{2}
$$

where $I : H^2(k(\nu)/(K \cap k(\nu))) \cong H^2(K(\nu)/K)$. We observe that the following diagram is commutative:

$$
\begin{array}{ccc}
H^2(k(\nu)/k) & \overset{\mathrm{inf}_{k(\nu) \to K(\nu)}}{\longrightarrow} & H^2(K(\nu)/k) \\
{\scriptstyle \mathrm{res}_{k \to K \cap k(\nu)}} \downarrow & & \downarrow {\scriptstyle \mathrm{res}_{k \to K \cap k(\nu)}} \\
H^2(k(\nu)/(K \cap k(\nu))) & \overset{\mathrm{inf}_{k(\nu) \to K(\nu)}}{\longrightarrow} & H^2(K(\nu)/(K \cap k(\nu))) \\
& \overset{I}{\cong} & \downarrow {\scriptstyle \mathrm{res}_{K \cap k(\nu) \to K}} \\
& & H^2(K(\nu)/K).
\end{array}
$$

Indeed, let $x_i \in \mathrm{Gal}(K(\nu)/K)$. Then $x_i \in \mathrm{Gal}(K(\nu)/(K \cap k(\nu)))$, and we write $\bar{x}_i = x_i \mathrm{Gal}(K(\nu)/k(\nu))$ in $\mathrm{Gal}(k(\nu)/(K \cap k(\nu)))$. Then for any $\theta \in H^2(k(\nu)/(K \cap k(\nu)))$,

$$
\begin{aligned}
&\big(\mathrm{res}_{K \cap k(\nu) \to K} \mathrm{inf}_{k(\nu) \to K(\nu)}\big)(\theta)(x_1, x_2) \\
&= (\mathrm{inf}_{k(\nu) \to K(\nu)}\theta)(x_1, x_2) = \theta(\bar{x}_1, \bar{x}_2).
\end{aligned}
$$

But since $\mathrm{Gal}(K(\nu)/(K \cap k(\nu))) \to \mathrm{Gal}(k(\nu)/(K \cap k(\nu)))$, $x_i \mapsto x_i|_{k(\nu)}$ is a surjective homomorphism having kernel $\mathrm{Gal}(K(\nu)/k(\nu))$, $\bar{x}_i = x_i \mathrm{Gal}(K(\nu)/k(\nu))$ can be identified with $x_i|_{k(\nu)}$ and the isomorphism $I : H^2(k(\nu)/(K \cap k(\nu))) \to H^2(K(\nu)/K)$ is determined by

$$(I\theta)(x_1, x_2) = \theta(x_1|_{k(\nu)}, x_2|_{k(\nu)}) = \theta(\bar{x}_1, \bar{x}_2).$$

Thus we have

$$\mathrm{res}_{K \cap k(\nu) \to K} \mathrm{inf}_{k(\nu) \to K(\nu)}(\theta) = I(\theta), \quad \text{for any } \theta \in H^2(k(\nu)/(K \cap k(\nu))).$$

Hence, due to Remarks 2.6, 3.1 and (1), it follows immediately from (2) that

$$
\begin{aligned}
\big(\phi \cdot \mathrm{Res}_{k\to K}|_{S(k)}\big)[T] \\
&= \big[(K(\nu)/k, \mathrm{cor}_{K\to k} I\, \mathrm{res}_{k\to K\cap k(\nu)}\beta)\big] \\
&= \big[(K(\nu)/k, \inf_{k(\nu)\to K(\nu)}\mathrm{cor}_{K\to k}\mathrm{res}_{k\to K}\beta)\big] \\
&= \big[(k(\nu)/k, \mathrm{cor}_{K\to k}\mathrm{res}_{k\to K}\beta)\big] \\
&= \big[(k(\nu)/k, \beta)\big]^u \qquad \text{with } u = |K:k| \\
&= \big(\mathrm{Cor}_{K\to k}|_{S(K)} \cdot \mathrm{Res}_{k\to K}|_{S(k)}\big)[(k(\nu)/k, \beta)] \\
&= \big(\mathrm{Cor}_{K\to k}|_{S(K)} \cdot \mathrm{Res}_{k\to K}|_{S(k)}\big)[T],
\end{aligned}
$$

so we have $\phi \cdot \mathrm{Res}_{k\to K}|_{S(k)} = \mathrm{Cor}_{K\to k}|_{S(K)} \cdot \mathrm{Res}_{k\to K}|_{S(k)}$. This shows our claim that $\phi = \mathrm{Cor}_{K\to k}$ on $S(K)$, because the restriction map $\mathrm{Res}_{k\to K}|_{S(k)}$ on Schur group is surjective by Proposition 3.2. Now we have

$$
\begin{aligned}
\mathrm{Cor}_{K\to k}|_{S(K)}[S] \\
&= \mathrm{Cor}_{K\to k}|_{S(K)}\big[(K(\varepsilon)/K, \alpha)\big] \\
&= \big[(K(\varepsilon)/k, \mathrm{cor}_{K\to k}\alpha)\big] \in B(k).
\end{aligned}
$$

We remark that $[(K(\varepsilon)/k, \mathrm{cor}_{K\to k}\alpha)]$ is not necessarily contained in $S(k)$, for $K(\varepsilon)$ may not be a cyclotomic extension of k. Hence we will construct a cyclotomic k-algebra which is similar to the crossed product algebra $(K(\varepsilon)/k, \mathrm{cor}_{K\to k}\alpha)$, so that we can conclude $\mathrm{Cor}_{K\to k}|_{S(K)}[S] \in S(k)$ and $\mathrm{Cor}_{K\to k} : S(K) \to S(k)$ is a homomorphism.

Since $\mathrm{Gal}(K(\varepsilon)/K) \cong \mathrm{Gal}\big(k(\varepsilon)/(K\cap k(\varepsilon))\big)$, there is $\theta \in Z^2\big(k(\varepsilon)/(K\cap k(\varepsilon)), \langle\varepsilon\rangle\big)$ that corresponds to $\alpha \in Z^2(K(\varepsilon)/K, \langle\varepsilon\rangle)$. Moreover since the diagram

$$
\begin{array}{ccc}
H^2\big(k(\varepsilon)/(K\cap k(\varepsilon))\big) & \overset{\inf_{k(\varepsilon)\to K(\varepsilon)}}{\longrightarrow} & H^2\big(K(\varepsilon)/(K\cap k(\varepsilon))\big) \\
{\scriptstyle I}\,\cong & & \downarrow \mathrm{res}_{K\cap k(\varepsilon)\to K} \\
& & H^2\big(K(\varepsilon)/K\big)
\end{array}
$$

is commutative, we can say that $\mathrm{res}_{K\cap k(\varepsilon)\to K}\inf_{k(\varepsilon)\to K(\varepsilon)}\theta = \alpha$. Then owing to the transitivity of corestriction maps, we have

$$
\begin{aligned}
\mathrm{cor}_{K\to k}\alpha &= \mathrm{cor}_{K\to k}\big(\mathrm{res}_{K\cap k(\varepsilon)\to K}\inf_{k(\varepsilon)\to K(\varepsilon)}\theta\big) \\
&= \mathrm{cor}_{K\cap k(\varepsilon)\to k}\mathrm{cor}_{K\to K\cap k(\varepsilon)}\mathrm{res}_{K\cap k(\varepsilon)\to K}\inf_{k(\varepsilon)\to K(\varepsilon)}\theta \\
&= \mathrm{cor}_{K\cap k(\varepsilon)\to k}(\inf_{k(\varepsilon)\to K(\varepsilon)}\theta)^u \\
&= \mathrm{cor}_{K\cap k(\varepsilon)\to k}\inf_{k(\varepsilon)\to K(\varepsilon)}\theta^u \\
&= \inf_{k(\varepsilon)\to K(\varepsilon)}\mathrm{cor}_{K\cap k(\varepsilon)\to k}\theta^u
\end{aligned}
$$

where $u = |K : K \cap k(\varepsilon)|$, by Remark 2.6 and the commutative diagram (1). It therefore follows immediately that

$$
\begin{aligned}
\mathrm{Cor}_{K \to k}|_{S(K)}[S] &= \left[(K(\varepsilon)/k, \mathrm{cor}_{K \to k}\alpha)\right] \\
&= \left[(K(\varepsilon)/k, \inf_{k(\varepsilon) \to K(\varepsilon)} \mathrm{cor}_{K \cap k(\varepsilon) \to k}\theta^u)\right] \\
&= \left[(K(\varepsilon)/k, \inf_{k(\varepsilon) \to K(\varepsilon)} \mathrm{cor}_{K \cap k(\varepsilon) \to k}\theta)\right]^u \\
&= \left[(k(\varepsilon)/k, \mathrm{cor}_{K \cap k(\varepsilon) \to k}\theta)\right]^u.
\end{aligned}
$$

But since $\left(k(\varepsilon)/k, \, \mathrm{cor}_{K \cap k(\varepsilon) \to k}\theta\right)$ is a Schur k-algebra, $\left[(k(\varepsilon)/k, \mathrm{cor}_{K \cap k(\varepsilon) \to k}\theta)\right]$ belongs to $S(k)$, hence the corestriction $\mathrm{Cor}_{K \to k}$ maps $\left[(K(\varepsilon)/K, \alpha)\right]$ in $S(K)$ to $\left[(k(\varepsilon)/k, \mathrm{cor}_{K \cap k(\varepsilon) \to k}\theta)\right]^u$ in $S(k)$. This completes the proof. $\qquad \square$

Observe that, if K is a cyclotomic extension of k in Theorem 3.3, then $K(\varepsilon)/k$ is cyclotomic, hence $\mathrm{Cor}_{K \to k}|_{S(K)}[S] = \left[(K(\varepsilon)/k, \mathrm{cor}_{K \to k}\alpha)\right] \in S(k)$.

In the rest of the paper we assume that the formation of k satisfies Axiom II.

THEOREM 3.4. If K/k is a Galois radical extension, then $\mathrm{Cor}_{K \to k}$ maps radical K-algebras to radical k-algebras, that is, $\mathrm{Cor}_{K \to k} : R(K) \to R(k)$ is a homomorphism.

PROOF. Let $[S] \in R(K)$ and A be a radical K-algebra in $[S]$ represented by a crossed product algebra $(K(\Gamma)/K, \alpha)$, where $K(\Gamma)$ is a $\mathrm{Gal}(K(\Gamma)/K)$-radical extension of K and $\alpha \in Z^2(K(\Gamma)/K)$ is an image of some $\alpha' \in Z^2(K(\Gamma)/K, \Gamma)$.

Since $K(\Gamma)/K$ and K/k are Galois radical extensions, $K(\Gamma)$ is radical over k ([5], p.309). Indeed if $K = k(\Gamma_0)$ for some subset $\Gamma_0 < K^*$ then $K(\Gamma) = k(\Gamma_0, \Gamma)$. If $a \in \Gamma$ then $a^n \in K$ for some $n > 0$. If $a^n \in k$, then a is a root of $X^n - b \in k[X]$ for some b. If $a^n \in \Gamma_0$ then $(a^n)^m \in k$ and a is a root of $X^{nm} - b \in k[X]$ for some $m > 0$, $b \in k$. Thus $K(\Gamma)$ is a splitting field over K of family of polynomials in $k[X]$, so $K(\Gamma)/k$ is a Galois radical extension (see [5], p.268). Hence we may consider the cohomological corestriction map $\mathrm{cor}_{K \to k} : H^2(K(\Gamma)/K) \to H^2(K(\Gamma)/k)$ that sends $\alpha \in Z^2(K(\Gamma)/K)$ to $\mathrm{cor}_{K \to k}\alpha \in Z^2(K(\Gamma)/k)$. We denote the crossed product algebra $(K(\Gamma)/k, \mathrm{cor}_{K \to k}\alpha)$ by A'. Then $[A'] \in B(k)$.

Let $\phi : R(K) \to B(k)$ be the homomorphism defined by $[A] \mapsto [A']$ (refer to Theorem 3.3). Since the restriction map $\mathrm{Res}_{k \to K}|_{R(k)} : R(k) \to$

$R(K)$ is a surjective homomorphism (by Theorem 2.5 and Proposition 3.2) satisfying

$$\phi \cdot \text{Res}_{k \to K}|_{R(k)} = \text{Cor}_{K \to k}|_{R(K)} \cdot \text{Res}_{k \to K}|_{R(k)} : \ R(k) \to B(k)$$

by Theorem 3.3, it follows that

$$\text{Cor}_{K \to k}|_{R(K)}[S] = \phi([A]) = \left[(K(\Gamma)/k, \text{cor}_{K \to k}\alpha)\right] = \left[A'\right] \in B(k).$$

Now due to the commutative diagram

$$
\begin{array}{ccccccc}
Z^2(K(\Gamma)/K, \Gamma) & \hookrightarrow & Z^2(K(\Gamma)/K) & & \alpha' & \mapsto & \alpha \\
\text{cor}_{K \to k} \downarrow & & \downarrow \text{cor}_{K \to k} & ; & \downarrow & & \downarrow \\
Z^2(K(\Gamma)/k, \Gamma) & \hookrightarrow & Z^2(K(\Gamma)/k) & & \text{cor}_{K \to k}\alpha' & & \text{cor}_{K \to k}\alpha,
\end{array}
$$

all values of α' and $\text{cor}_{K \to k}\alpha'$ are in Γ (see Remark 3.1), hence $\text{cor}_{K \to k}\alpha$ can be regarded as the homomorphic image of $\text{cor}_{K \to k}\alpha' \in Z^2(K(\Gamma)/k, \Gamma)$ It therefore follows immediately that $\text{Cor}_{k \to K}[S] = \text{Cor}_{k \to K}[A] = [A'] \in R(k)$. $\qquad\square$

In the proof of Theorem 3.4, the assumption K/k is radical is necessary. The following is a special case.

COROLLARY 3.5. *Let $A = \left(K(\Gamma)/K, \alpha\right)$ be a radical K-algebra and $k < k(\Gamma) < K(\Gamma)$ be Galois extensions. If either $k(\Gamma)/k$ is radical or $K \cap k(\Gamma) = k$, then $\text{Cor}_{K \to k}$ maps $[A]$ to a class of radical k-algebras.*

PROOF. Note that $k(\Gamma)/(K \cap k(\Gamma))$ and $K(\Gamma)/k(\Gamma)$ are Galois extensions. Since $\text{res}_{k \to K \cap k(\Gamma)} : H^2\left(k(\Gamma)/k\right) \to H^2\left(k(\Gamma)/(K \cap k(\Gamma))\right)$ is surjective, the commutative diagram (refer to the diagram in Theorem 3.3)

$$
\begin{array}{ccc}
H^2\left(k(\Gamma)/k\right) & \overset{\inf_{k(\Gamma) \to K(\Gamma)}}{\longrightarrow} & H^2\left(K(\Gamma)/k\right) \\
\text{res}_{k \to K \cap k(\Gamma)} \downarrow & & \downarrow \text{res}_{k \to K} \\
H^2\left(k(\Gamma)/(K \cap k(\Gamma))\right) & \overset{I}{\cong} & H^2\left(K(\Gamma)/K\right)
\end{array}
$$

shows that there is $\theta \in Z^2(k(\Gamma)/k)$ with $\text{res}_{k \to K}\inf_{k(\Gamma) \to K(\Gamma)}\theta = \alpha$. Thus we have

$$\text{Cor}_{K \to k}\left[(K(\Gamma)/K, \alpha)\right] = \left[(K(\Gamma)/k, \text{cor}_{K \to k}\alpha)\right]$$
$$= \left[(K(\Gamma)/k, \inf_{k(\Gamma) \to K(\Gamma)}\theta)\right]^u = \left[(k(\Gamma)/k, \theta)\right]^u$$

with $u = |K : k|$. Moreover for $\alpha' \in Z^2(K(\Gamma)/K, \Gamma)$ that is mapped to α, there is $\theta' \in Z^2(k(\Gamma)/k, \Gamma)$ such that $\text{res}_{k \to K}\inf_{k(\Gamma) \to K(\Gamma)}\theta' = \alpha'$, and θ is a homomorphic image of θ'. In case $k(\Gamma)/k$ is a radical extension, it is clear that $\text{Cor}_{K \to k}\left[(K(\Gamma)/K, \alpha)\right] = \left[(k(\Gamma)/k, \theta)\right]^u$ belongs to $R(k)$.

Suppose that $K \cap k(\Gamma) = k$. If $x \in \Gamma$ then $x^n \in K$ for some $n > 0$. At the same time, since $x^n \in k(\Gamma)$, it follows that $x^n \in K \cap k(\Gamma) = k$. Hence $k(\Gamma)$ is a radical extension of k, thus all arguments go back to the above case.   □

As the relative Brauer group $B(K/k)$ is the $\ker(\mathrm{Res}_{k \to K} : B(k) \to B(K))$, we denote by $S(K/k)$, $P(K/k)$ and $R(K/k)$ the kernels of $\mathrm{Res}_{k \to K}$ on $S(k)$, $P(k)$ and $R(k)$ respectively. We finally discuss the restriction and corestriction maps on groups $S(E/k) < S(k)$ and $R(E/K) < R(K)$.

THEOREM 3.6. *Let* $k < K < E$ *be Galois extensions contained in the separable closure of* k. *Then* $\mathrm{Res}_{k \to K}|_{S(E/k)} : S(E/k) \to S(E/K)$, $\mathrm{Res}_{k \to K}|_{R(E/k)} : R(E/k) \to R(E/K)$ *and* $\mathrm{Cor}_{K \to k}|_{S(E/K)} : S(E/K) \to S(E/k)$ *are homomorphisms that make the diagram commute:*

$$
\begin{array}{ccc}
S(E/K) & \hookrightarrow & S(K) \\
\uparrow\downarrow & & \mathrm{Res}_{k \to K} \uparrow\downarrow \; \mathrm{Cor}_{K \to k} \\
S(E/k) & \hookrightarrow & S(k).
\end{array}
$$

Moreover $\mathrm{Cor}_{K \to k}|_{R(E/K)} : R(E/K) \to R(E/k)$ *is also a homomorphism if* K *is radical over* k.

PROOF. Let $[S] \in R(E/k)$. Then $[S] \in R(k)$ and $\mathrm{Res}_{k \to E}[S] = 1 \in S(E)$. Due to Theorem 2.5, $\mathrm{Res}_{k \to K}[S] \in R(K)$. Since $E \otimes \mathrm{Res}_{k \to K}[S] = \mathrm{Res}_{K \to E}(\mathrm{Res}_{k \to K}[S]) = \mathrm{Res}_{k \to E}[S] = 1$, $\mathrm{Res}_{k \to K}[S]$ belongs to $R(E/K)$ and $\mathrm{Res}_{k \to K}|_{R(E/k)} : R(E/k) \to R(E/K)$ is a homomorphism.

Assume that K/k is a radical extension. Let $[S] \in R(E/K)$ and $A \in [S]$ be a radical K-algebra $(K(\Gamma)/K, \alpha)$ where $\alpha \in Z^2(K(\Gamma)/K)$ is an image of some $\alpha' \in Z^2(K(\Gamma)/K, \Gamma)$. Then

$$
\begin{aligned}
1 = E \otimes [A] &= \left[(E(\Gamma)/E, I \, \mathrm{res}_{K \to E \cap K(\Gamma)} \alpha) \right] \\
&= \left[(K(\Gamma)/(E \cap K(\Gamma)), \mathrm{res}_{K \to E \cap K(\Gamma)} \alpha) \right]
\end{aligned}
$$

where $\mathrm{Gal}(K(\Gamma)/(E \cap K(\Gamma))) \overset{I}{\cong} \mathrm{Gal}(E(\Gamma)/E)$, thus we have

$$
1 = \mathrm{res}_{K \to E \cap K(\Gamma)} \alpha.
$$

Due to the proof of Theorem 3.4, $\mathrm{Cor}_{K \to k}[A] = \left[(K(\Gamma)/k, \mathrm{cor}_{K \to k} \alpha) \right] \in R(k)$ such that $\mathrm{cor}_{K \to k} \alpha$ is a homomorphic image of $\mathrm{cor}_{K \to k} \alpha'$. Moreover since

$$
E \otimes \mathrm{Cor}_{K \to k}[A] = \left[(E(\Gamma)/E, I \, \mathrm{res}_{k \to E \cap K(\Gamma)} \mathrm{cor}_{K \to k} \alpha) \right],
$$

we are only left to show

$\mathrm{res}_{k \to E \cap K(\Gamma)} \mathrm{cor}_{K \to k} \alpha = 1$ in $Z^2(K(\Gamma)/(E \cap K(\Gamma))) \cong Z^2(E(\Gamma)/E)$.

Consider the inflation-restriction sequence

$$0 \to H^2\big((E \cap K(\Gamma))/K, \Gamma^H\big) \overset{\mathrm{inf}}{\to} H^2(K(\Gamma)/K, \Gamma)$$
$$\overset{\mathrm{res}}{\to} H^2\big(K(\Gamma)/(E \cap K(\Gamma)), \Gamma\big)$$

where $\mathrm{inf} = \mathrm{inf}_{E \cap K(\Gamma) \to K(\Gamma)}$, $\mathrm{res} = \mathrm{res}_{K \to E \cap K(\Gamma)}$ and Γ^H is the fixed module of Γ by $H = \mathrm{Gal}(K(\Gamma)/(E \cap K(\Gamma)))$. Since $H^1(K(\Gamma)/(E \cap K(\Gamma))) = 1$ [9, (1.5.4)], the sequence is exact [9, (3-4-3)], or [8, (X.6.5)]. From $1 = \mathrm{res}_{K \to E \cap K(\Gamma)} \alpha$, we have $\bar{\alpha}$ belongs to $\ker(\mathrm{res}_{K \to E \cap K(\Gamma)}) = \mathrm{im}(\mathrm{inf}_{E \cap K(\Gamma) \to K(\Gamma)})$, thus there exists

$$\bar{\beta} \in H^2\big((E \cap K(\Gamma))/K\big) \quad \text{such that} \quad \mathrm{inf}_{E \cap K(\Gamma) \to K(\Gamma)} \bar{\beta} = \bar{\alpha}.$$

Then, by virtue of the exactness of the inflation-restriction sequence, we have

$$\mathrm{res}_{k \to E \cap K(\Gamma)} \mathrm{cor}_{K \to k} \bar{\alpha} = \mathrm{res}_{k \to E \cap K(\Gamma)} \mathrm{cor}_{K \to k} \mathrm{inf}_{E \cap K(\Gamma) \to K(\Gamma)} \bar{\beta}$$
$$= \mathrm{res}_{k \to E \cap K(\Gamma)} \mathrm{inf}_{E \cap K(\Gamma) \to K(\Gamma)} \mathrm{cor}_{K \to k} \bar{\beta}$$
$$= \mathrm{res}_{K \to E \cap K(\Gamma)} \mathrm{res}_{k \to K} \mathrm{inf}_{E \cap K(\Gamma) \to K(\Gamma)} \mathrm{cor}_{K \to k} \bar{\beta}$$
$$= \mathrm{res}_{K \to E \cap K(\Gamma)} \mathrm{inf}_{E \cap K(\Gamma) \to K(\Gamma)} \mathrm{res}_{k \to K} \mathrm{cor}_{K \to k} \bar{\beta}$$
$$= 0 \cdot \mathrm{res}_{k \to K} \mathrm{cor}_{K \to k} \bar{\beta} = 0.$$

It thus follows that $\mathrm{res}_{k \to E \cap K(\Gamma)} \mathrm{cor}_{K \to k} \alpha = 1$, as is desired. Analogous methods can be applied to show $\mathrm{Res}_{k \to K}|_{S(E/k)} : S(E/k) \to S(E/K)$ and $\mathrm{Cor}_{K \to k}|_{S(E/K)} : S(E/K) \to S(E/k)$ are homomorphisms. $\square$

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Department of Mathematics
HanNam University
Taejon 306-791, Korea
E-mail: emc@math.hannam.ac.kr