J. Korean Math. Soc. **42** (2005), No. 1, pp. 153–169

ON FUZZY STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. A fuzzy stochastic differential equation contains a fuzzy valued diffusion term which is defined by stochastic integral of a fuzzy process with respect to 1-dimensional Brownian motion. We prove the existence and uniqueness of the solution for fuzzy stochastic differential equation under suitable Lipschitz condition. To do this we prove and use the maximal inequality for fuzzy stochastic integrals. The results are illustrated by an example.

1. Introduction

The theory of fuzzy ordinary differential equations has been extensively developed in conjunction with fuzzy valued analysis [4, 9]. The techniques used have been closed related to set valued integrals [1] and set valued differential equations [7].

Random sets have proved valuable in a wide class of estimation and imaging problems, see [2] as a sample from a very large literature. A natural extension, from which many applications follow, was taken by Puri and Ralescu [12]. Further advances have been made in this area by Bàn [3] and Stojacoviĉ [13], where important stochastic concepts, such as set valued conditional expectation and martingales, have been exploited in a fuzzy context. Stochastic integrals, in the Itô or Stratonovich sense (see [11] for a lucid and concise treatment), have proved extraordinarily useful on the study and practical application of stochastic processes and differential equations to a wide variety of fields, including engineering and finance. Natural extensions to set valued situations, where further incertitude is introduced, have been made by Kim and Kim [10]. Very recently, Jung and Kim [8] have developed a more usable definition for which, like the single valued case, the maximal inequality holds under some assumptions for integrands. These advances also made it possible

Received October 10, 2003.
2000 Mathematics Subject Classification: Primary 60D05, 60H30.
Key words and phrases: fuzzy martingale, fuzzy stochastic differential equation.

to define a fuzzy stochastic integral, thus extending the class of uncertain processes which could be studied. This new tool opens the way to consider fuzzy stochastic differential equations.

This paper develops a theory of existence and uniqueness of the solution for such systems, and demonstrates how the technical machinery can be applied to uncertain dynamical problems. The techniques are illustrated by a worked example involving stochastic logistic growth. More precisely, in this paper we consider the following fuzzy stochastic differential equation

$$\begin{cases} dX(t) = F(t, X(t))dt + G(t, X(t))dB_t, \\ X(0) = x_0, \end{cases} \tag{1}$$

on a complete probability space $(\Omega, \mathcal{A}, P)$ with a filtration $(\mathcal{A}_t)_{t\geq 0}$, where $F : [0, \infty] \times \mathcal{F}_c(\mathbf{R}) \to \mathcal{F}_c(\mathbf{R})$ and $G : [0, \infty] \times \mathcal{F}_c(\mathbf{R}) \to \mathcal{F}_c(\mathbf{R})$. Here $\mathcal{F}_c(\mathbf{R})$ is the family of all fuzzy sets of which level sets are nonempty closed convex subsets of $\mathbf{R}$, the set of all real numbers, and $(B_t)_{t\geq 0}$ is a 1-dimensional Brownian motion. The question of existence of a solution of (1) is interpreted to be process $(X(t))_{t\geq 0}$ satisfying the stochastic integral equation

$$X(t) = x_0 + \int_0^t F(s, X(s))ds + \int_0^t G(s, X(s))dB_s \quad \text{a.s.} \tag{2}$$

Here, the second term on the right-hand side of (2) is the integral of which level sets are set valued integrals of level sets of $F(s, X(s))$ in the sense of Aumann [1] and the third term is the fuzzy stochastic integral which will be defined in Section 2. We will prove the uniqueness and existence of the solution of (1) under some Lipschitz conditions for F and G. Because of the lack of the martingale property for fuzzy stochastic integrals and of the distributive law for fuzzy sets, we may need some additional conditions to study fuzzy stochastic analysis unlike usual real valued case. So it is meaningful to find these additional conditions which is, of cause, trivial in real valued case (see Theorem 2.3 and 3.2).

The organization of this paper is as follows. In Section 2, we state some useful results of set valued and fuzzy stochastic integrals. In Section 3, we prove the existence and uniqueness of the solution of fuzzy stochastic differential equation (1). Section 4 gives a worked example of a fuzzy stochastic differential equation of which solution is given concretely.

2. Fuzzy stochastic integrals

Denote by $\mathcal{K}(\mathbf{R})$ the family of all nonempty closed subsets of $\mathbf{R}$ and let $\mathcal{K}_c(\mathbf{R})$ be the class of all such sets which are also convex. For any $A, B \in \mathcal{K}(\mathbf{R})$, we define

$$d_H(A, B) = \max \left\{ \sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A) \right\}$$

where $d(x, B) = \inf_{y\in B} |x - y|$ and

$$|||A||| = d_H(A, \{0\}) = \sup_{x\in A} |x|.$$

Let $L^p(\Omega, \mathcal{A}; \mathbf{R})$ be the space of all real valued random variables h such that $||h||_p^p = E[|h|^p] < \infty$, where $E[g]$ is the expectation of a random variable g. A $\mathcal{K}(\mathbf{R})$ valued random set F is called *L^p-bounded* if $|||F||| \in L^p(\Omega, \mathcal{A}; \mathbf{R})$. Let $L^p(\Omega, \mathcal{A}; \mathcal{K}(\mathbf{R}))$(resp. $L^p(\Omega, \mathcal{A}; \mathcal{K}_c(\mathbf{R}))$) be the set of all $\mathcal{K}(\mathbf{R})$ (resp. $\mathcal{K}_c(\mathbf{R})$) valued L^p-bounded random sets. If A and B are bounded, then $d_H(A, B)$ is the Hausdorff metric of A and B.

Let $\mathcal{F}(\mathbf{R})$ denote the family of all fuzzy sets $u : \mathbf{R} \to [0, 1]$ such that the level set(or α-cut) $[u]^\alpha = \{x \in \mathbf{R} : u(x) \geq \alpha\} \in \mathcal{K}(\mathbf{R})$, for $0 < \alpha \leq 1$, and $[u]^0 = \cup_{\alpha\in(0,1]}[u]^\alpha$ is bounded. For all $0 \leq \alpha \leq \beta \leq 1$

$$[u]^\beta \subset [u]^\alpha \subset [u]^0.$$

For two fuzzy sets $u, v \in \mathcal{F}(\mathbf{R})$, we denote $u \leq v$ if and only if $[u]^\alpha \subset [v]^\alpha$ for every $\alpha \in [0, 1]$. Let $\mathcal{F}_c(\mathbf{R})$ denote the family of all fuzzy sets in $\mathcal{F}(\mathbf{R})$ with level sets contained in $\mathcal{K}_c(\mathbf{R})$. Define a metric D on $\mathcal{F}(\mathbf{R})$ by

$$D(u, v) = \sup_{\alpha\in[0,1]} d_H([u]^\alpha, [v]^\alpha).$$

For $u_i \in \mathcal{F}(\mathbf{R})$, i=1,2,3,4, the followings are well known.

$$D(u_1 + u_3, u_2 + u_3) = D(u_1, u_2) \tag{3}$$

and

$$D(u_1 + u_3, u_2 + u_4) \leq D(u_1, u_2) + D(u_3 + u_4). \tag{4}$$

A *fuzzy random variable* is a measurable function $X : \Omega \to \mathcal{F}(\mathbf{R})$. A fuzzy random variable X is called *L^p-bounded* if there exists a function $h \in L^p(\Omega, \mathcal{A}; \mathbf{R})$ such that $|||[X(\omega)]^0||| \leq h(\omega)$ for a.a. $\omega \in \Omega$. Let $L^p(\Omega; \mathcal{F}(\mathbf{R}))$ be the set of all L^p-bounded fuzzy random variables and $L^p(\Omega; \mathcal{F}_c(\mathbf{R}))$ be the set of all L^p-bounded fuzzy random variables whose level sets belong to $\mathcal{K}_c(\mathbf{R})$. For $X_1, X_2 \in L^p(\Omega;\mathcal{F}(\mathbf{R}))$, they are considered to be identical if for all $\alpha \in (0,1]$, it holds $[X_1]^\alpha = [X_2]^\alpha$ a.s.. For any $\mathcal{A}$-measurable $\mathcal{K}(\mathbf{R})$ valued random set F, we define

$$S_F^p = \{f \in L^p(\Omega, \mathcal{A}; \mathbf{R}) : f(\omega) \in F(\omega) \text{ a.a. } \omega \in \Omega\},\ p = 1, 2, 3, \ldots.$$

The expectation of a fuzzy random variable X, denoted by $E[X]$ also, is a fuzzy set such that, for $\alpha \in (0,1]$,

$$[E[X]]^\alpha = \mathcal{E}[[X]^\alpha] = \{E[g] : g \in S^1_{[X]^\alpha}\}.$$

Let F be a $\mathcal{K}(\mathbf{R})$ valued random set satisfying $|||F||| \in L^1(\Omega, \mathcal{A}; \mathbf{R})$ and $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$. Hiai and Umegaki [6] defined the conditional expectation $\mathcal{E}[F|\mathcal{B}]$ of F given $\mathcal{B}$ by

$$S^1_{\mathcal{E}[F|\mathcal{B}]} = cl\{E[f|\mathcal{B}] : f \in S^1_F\}.$$

Stojaković [13] showed that for any $X \in L^1(\Omega;\mathcal{F}(\mathbf{R}))$ and σ-field $\mathcal{B} \subset \mathcal{A}$, there exists a unique fuzzy random variable $\Phi \in L^1(\Omega, \mathcal{B}; \mathcal{F}(\mathbf{R}))$ such that for $\alpha \in (0,1]$,

$$[\Phi]^\alpha = \mathcal{E}[[X]^\alpha|\mathcal{B}] \quad \text{a.s..}$$

The fuzzy random variable Φ is called the *fuzzy conditional expectation* of X given $\mathcal{B}$ and denote it by $E[X|\mathcal{B}]$. Let $L^p(\mathbf{R})$ be the set of all $\mathcal{A}_t$-adapted measurable $\mathbf{R}$ valued stochastic process $(h(t))_{t\geq 0}$ satisfying for every $t \geq 0$, $E[\int_0^t |h(s)|^p ds] < \infty$. A $\mathcal{K}(\mathbf{R})$ valued set valued process $(F(t))_{t\geq 0}$ is called $\mathcal{L}^2$-*bounded* if $(|||F(t)|||)_{t\geq 0} \in L^p(\mathbf{R})$. Let $\mathcal{L}^p(\mathcal{K}(\mathbf{R}))$ be the set of all $\mathcal{K}(\mathbf{R})$ valued $\mathcal{A}_t$-adapted measurable set valued processes.

We call $(Y_t)_{t\geq 0}$ a *fuzzy stochastic process* if each level set $[Y]_t^\alpha$ is a nonempty, closed and convex set valued random variable and each Y_t is a fuzzy random variable. A fuzzy stochastic process $(Y_t)_{t\geq 0}$ is called $\mathcal{A}_t$-*adapted* if for each $t \geq 0$, Y_t is $\mathcal{A}_t$-measurable, and *measurable* if Y is $\mathcal{B}_{[0,\infty)} \otimes \mathcal{A}$-measurable. A fuzzy stochastic process $(Y_t)_{t\geq 0}$ is called $\mathcal{L}^2$-*bounded* if there exists a process $(h(t))_{t\geq 0} \in L^2(\mathbf{R})$ such that $|||[Y_t(\omega)]^0||| \leq h(t,\omega)$ for a.a. (t,ω). Let $\mathcal{L}^2(\mathcal{F}_c(\mathbf{R}))$ be the set of $\mathcal{A}_t$-adapted measurable $\mathcal{L}^2$-bounded $\mathcal{F}_c(\mathbf{R})$ valued fuzzy processes. A fuzzy stochastic process $(Y_t)_{t\geq 0}$ is called a *fuzzy martingale* (resp. *submartingale, supermartingale*) with respect to $\mathcal{A}_t$ if Y_t is L^1-bounded, $\mathcal{A}_t$-measurable and for $t > s$, $E[Y_t|\mathcal{A}_s] = Y_s$ (resp. $\geq, \leq$) a.s.. By the definition of fuzzy conditional expectation we can see that $(Y_t)_{t\geq 0}$ is a fuzzy martingale if and only if $([Y]_t^\alpha)_{t\geq 0}$ is a set valued martingale for any $\alpha \in [0,1]$.

A set H of real valued random variables is called *decomposable* with respect to $\mathcal{A}$ if $f_1, f_2 \in H$ and $A \in \mathcal{A}$ imply $1_A f_1 + 1_{A^c} f_2 \in H$, where 1_A is the indicator function of A. For any set Γ in $L^p(\Omega, \mathcal{A}; \mathbf{R})$, we denote by $\overline{de}\Gamma$ the smallest closed set in $L^p(\Omega, \mathcal{A}; \mathbf{R})$ which contains Γ and is decomposable with respect to $\mathcal{A}$.

Let $(F(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{K}_c(\mathbf{R})$. Jung and Kim [8] defined a *set valued stochastic integral* $I(F)(t) = \int_0^t F(s)dB_s$ of $(F(t))_{t\geq 0}$ with respect to a 1-dimensional Brownian motion $(B_t)_{t\geq 0}$ by

$$S^2_{\int_0^t F(s)dB_s}(\mathcal{A}_t) = \overline{de}\left\{\int_0^t f(s)dB_s : (f(t))_{t\geq 0} \in \mathcal{S}^2((F(t))_{t\geq 0})\right\}$$

for all $t \geq 0$. Here $\mathcal{S}^2((F(t))_{t\geq 0})$ is the set of all stochastic processes $(f(t))_{t\geq 0} \in L^2(\mathbf{R})$ such that $f(t,\omega) \in F(t,\omega)$ for all $t \geq 0$ and a.a $\omega \in \Omega$. And they proved that $\int_0^t F(s)dB_s$ is a random set in $L^2(\Omega, \mathcal{A}_t; \mathcal{K}_c(\mathbf{R}))$ for every $t \geq 0$.

For some $(f_1(t))_{t\geq 0}, (f_2(t))_{t\geq 0} \in L^2(\mathbf{R})$, let $(F(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{K}_c(\mathbf{R}))$ be given by $F(t) = [\underline{f}(t), \overline{f}(t)]$, where $\underline{f}(t) = f_1(t)$ and $\overline{f}(t) = f_2(t)$ if $f_1(t) \leq f_2(t)$, and $\underline{f}(t) = f_2(t)$ and $\overline{f}(t) = f_1(t)$ if $f_2(t) \leq f_1(t)$. That is, $(F(t))_{t\geq 0}$ is a set valued process defined by the set between $(f_1(t))_{t\geq 0}$ and $(f_2(t))_{t\geq 0}$. In this case we will say that $(F(t))_{t\geq 0}$ is defined by

$$F(t) = \langle f_1; f_2\rangle(t). \tag{5}$$

Then we see that $I(F)(t) = \langle \int_0^\cdot f_1(s)dB_s; \int_0^\cdot f_2(s)dB_s\rangle(t)$. Note that $(I(F)(t))_{t\geq 0}$ does not satisfy the martingale property.

By the same method as Theorem 4.6 in [10], we can see that, for any $(Y(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{F}_c(\mathbf{R}))$, there exists a unique fuzzy random variable $J(Y)(t) \in L^2(\Omega, \mathcal{A}_t; \mathcal{F}_c(\mathbf{R}))$ such that for all $\alpha \in (0, 1]$, $[J(Y)]^\alpha(t,\omega) = (\int_0^t [Y]^\alpha(s)dB_s)(\omega)$ a.a. $\omega \in \Omega$. Moreover, $(J(Y)(t))_{t\geq 0}$ is an $\mathcal{A}_t$-adapted measurable fuzzy stochastic process.

DEFINITION 2.1. We call above $J(Y)(t)$ a *fuzzy stochastic integral* of $(Y(t))_{t\geq 0}$ with respect to $(B_t)_{t\geq 0}$ and denote it by $J(Y)(t) = \int_0^t Y(s)dB_s$.

Clearly $[J(Y)]^\alpha(t) = I([Y]^\alpha)(t)$. Since $(I([Y]^\alpha)(t))_{t\geq 0}$ is not a set valued martingale, $(J(Y)(t))_{t\geq 0}$ is not a fuzzy martingale. So it is difficult to deal with fuzzy stochastic integrals unlike usual real valued case.

EXAMPLE 2.2. Let $(c; d)_S$ denote the symmetric fuzzy number with the interval $[c, d]$ as its support (see [4]). For $(f_1(t))_{t\geq 0}, (f_2(t))_{t\geq 0} \in L^2(\mathbf{R})$ satisfying $f_1(t) \leq f_2(t)$ for all $t \geq 0$ a.s., define $(Y(t))_{t\geq 0} \in$

$\mathcal{L}^2(\mathcal{F}_c(\mathbf{R}))$ by $Y(t) = (f_1(t); f_2(t))_S$. We can calculus its level sets as follows.

$$\begin{aligned}[Y]^\alpha(t) &= \left[f_1(t) + \frac{\alpha}{2}(f_2(t) - f_1(t)), f_2(t) - \frac{\alpha}{2}(f_2(t) - f_1(t))\right] \\ &\equiv [\underline{y}(\alpha;t), \overline{y}(\alpha;t)], \quad \alpha \in [0,1]. \end{aligned} \tag{6}$$

Hence $J(Y)(t) = \int_0^t Y(s)dB_s$ is given by

$$\begin{aligned}[J(Y)]^\alpha(t) &= I([Y]^\alpha)(t) \\ &= \langle \int_0^\cdot \underline{y}(\alpha;s)dB_s; \int_0^\cdot \overline{y}(\alpha;s)dB_s\rangle(t).\end{aligned}$$

From now on, let T be any given real number. The following result is the maximal inequality for fuzzy stochastic integrals.

THEOREM 2.3. *Let $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ be in $\mathcal{L}^2(\mathcal{F}_c(\mathbf{R}))$ and satisfy that for $\alpha \in [0,1]$, $([X]^\alpha(t))_{t\geq 0}$ and $([Y]^\alpha(t))_{t\geq 0}$ are defined by $[X]^\alpha(t) = \langle f_1^\alpha; f_2^\alpha\rangle(t)$ and $[Y]^\alpha(t) = \langle g_1^\alpha; g_2^\alpha\rangle(t)$ for some $(f_i^\alpha(t))_{t\geq 0}$ and $(g_i^\alpha(t))_{t\geq 0}, i = 1,2$ with the the following inequality*

$$\left(\int_0^t (f_2^\alpha(s) - f_1^\alpha(s))dB_s\right)\left(\int_0^t (g_2^\alpha(s) - g_1^\alpha(s))dB_s\right) \geq 0$$

for all $t \in [0,T]$ a.s.. Then it holds that for all $t \in [0,T]$,

$$\begin{aligned}&E\left[\sup_{0\leq u\leq t} D^2\left(\int_0^u X(s)dB_s, \int_0^u Y(s)dB_s\right)\right] \\ \leq\ & 4E\left[D^2\left(\int_0^t X(s)dB_s, \int_0^t Y(s)dB_s\right)\right].\end{aligned}$$

Proof. Since for each $i = 1,2$, $\left(\int_0^t f_i^\alpha(s)dB_s\right)_{t\geq 0}$ is continuous $\mathcal{A}_t$-martingale, there is a constant $c_i > 0$ such that with probability one,

$$\left|\int_0^t f_i^\alpha(s)dB_s\right| \leq c_i, \quad t \in [0,T].$$

Hence $\left(-c_1 + \int_0^t f_1^\alpha(s)dB_s\right)_{t\in[0,T]}$ (resp. $\left(c_2 + \int_0^t f_2^\alpha(s)dB_s\right)_{t\in[0,T]}$) is non-positive (resp. nonnegative) martingale. Put

$$\begin{aligned}I_{c_1}^{c_2}(f_1^\alpha, f_2^\alpha)(t) &= \left[-c_1 + \int_0^t f_1^\alpha(s)dB_s, c_2 + \int_0^t f_2^\alpha(s)dB_s\right] \\ &\equiv [m_1^\alpha(t), m_2^\alpha(t)].\end{aligned}$$

Then we have

$$(7)\qquad I_{c_1}^{c_2}(f_1^\alpha, f_2^\alpha)(t) = \frac{(m_2^\alpha(t) - m_1^\alpha(t))}{2}[-1,1] + \frac{(m_1^\alpha(t) + m_2^\alpha(t))}{2}.$$

Since $\left(\frac{1}{2}\left(m_2^\alpha(t) - m_1^\alpha(t)\right)\right)_{t\in[0,T]}$ is a nonnegative $\mathcal{A}_t$-martingale, by Theorem 2.6 in [8], the first term on the right-hand side of (7) is a set valued $\mathcal{A}_t$-martingale. And $\left(\frac{1}{2}\left(m_1^\alpha(t) + m_2^\alpha(t)\right)\right)_{t\in[0,T]}$ is an $\mathcal{A}_t$-martingale. Thus $\left(I_{c_1}^{c_2}(f_1^\alpha, f_2^\alpha)(t)\right)_{t\in[0,T]}$ is a set valued $\mathcal{A}_t$-martingale. By the same argument, we can check that $\left(I_{d_1}^{d_2}(g_1^\alpha, g_2^\alpha)(t)\right)_{t\in[0,T]}$ is a set valued $\mathcal{A}_t$-martingale for some constants $d_1, d_2 \geq 0$. Let $k_i = \max(c_i, d_i), i = 1, 2$. Then $\left(I_{k_1}^{k_2}(f_1^\alpha, f_2^\alpha)(t)\right)_{t\in[0,T]}$ and $\left(I_{k_1}^{k_2}(g_1^\alpha, g_2^\alpha)(t)\right)_{t\in[0,T]}$ are set valued $\mathcal{A}_t$-martingales trivially. Using Theorem 2.6 in [5], a real valued process $\left(d_H\left(I_{k_1}^{k_2}(f_1^\alpha, f_2^\alpha)(t), I_{k_1}^{k_2}(g_1^\alpha, g_2^\alpha)(t)\right)\right)_{t\in[0,T]}$ is a submartingale. Hence, by the definition of D and our assumption, we have for $0 \leq s \leq t \leq T$,

$$\begin{aligned}
& E\left[D\left(J(X)(t), J(Y)(t)\right) \,\middle|\, \mathcal{A}_s\right] \\
= \;& E\left[\sup_{\alpha\in[0,1]} d_H\left(I([X]^\alpha)(t), I([Y]^\alpha)(t)\right) \,\middle|\, \mathcal{A}_s\right] \\
= \;& E\left[\sup_{\alpha\in[0,1]} d_H\left(I_{k_1}^{k_2}(f_1^\alpha, f_2^\alpha)(t), I_{k_1}^{k_2}(g_1^\alpha, g_2^\alpha)(t)\right) \,\middle|\, \mathcal{A}_s\right] \\
\geq \;& \sup_{\alpha\in[0,1]} E\left[d_H\left(I_{k_1}^{k_2}(f_1^\alpha, f_2^\alpha)(t), I_{k_1}^{k_2}(g_1^\alpha, g_2^\alpha)(t)\right) \,\middle|\, \mathcal{A}_s\right] \\
\geq \;& \sup_{\alpha\in[0,1]} d_H\left(I_{k_1}^{k_2}(f_1^\alpha, f_2^\alpha)(s), I_{k_1}^{k_2}(g_1^\alpha, g_2^\alpha)(s)\right) \\
= \;& \sup_{\alpha\in[0,1]} d_H\left(I([X]^\alpha)(s), I([Y]^\alpha)(s)\right) \\
= \;& D\left(J(X)(s), J(Y)(s)\right).
\end{aligned}$$

This shows that $\left(D\left(J(X)(t), J(Y)(t)\right)\right)_{t\in[0,T]}$ is a real valued submartingale. Using the Doob maximal inequality [14, Theorem 1.2.3], the proof is complete. □

REMARK 2.4 In [8], Jung and Kim have proved the maximal inequality for set valued stochastic integrals. To do this they assumed that the maximal and minimal selections of set valued stochastic integrals are bounded on **R**. But from the first part of the proof of Theorem 2.3, we can see that this assumption is not necessary. And, for example, in the case that for any given $c \geq 0$, $f_i^\alpha(t) = cg_i^\alpha(t), i = 1, 2$, we can see that the inequality in Theorem 2.3 is satisfied.

THEOREM 2.5. *Let* $(X(t))_{t\geq 0}, (Y(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{F}_c(\mathbf{R}))$. *Then, we have*

$$E\left[D^2\left(\int_0^t X(s)dB_s, \int_0^t Y(s)dB_s\right)\right] \leq E\left[\int_0^t D^2(X(s), Y(s))ds\right].$$

Proof. By the definition of D, we have for all $t \geq 0$

$$\begin{aligned} & E\left[D^2\left(\int_0^t X(s)dB_s, \int_0^t Y(s)dB_s\right)\right] \\ = \ & E\left[\sup_{\alpha\in[0,1]} d_H^2\left(\int_0^t [X]^\alpha(s)dB_s, \int_0^t [Y]^\alpha(s)dB_s\right)\right] \\ \leq \ & E\left[\sup_{\alpha\in[0,1]} \int_0^t d_H^2\left([X]^\alpha(s), [Y]^\alpha(s)\right) ds\right] \\ \leq \ & E\left[\int_0^t \sup_{\alpha\in[0,1]} d_H^2\left([X]^\alpha(s), [Y]^\alpha(s)\right) ds\right] \\ = \ & E\left[\int_0^t D^2\left(X(s), Y(s)\right) ds\right], \end{aligned}$$

where the first inequality is a result in [8]. The proof is complete. □

3. Main results

In this section we prove the existence and uniqueness of the solution of fuzzy stochastic differential equation (1). The solution of (1) is defined as follows.

DEFINITION 3.1. By a *solution* of fuzzy stochastic differential equation (1), we mean a $\mathcal{F}_c(\mathbf{R})$ valued fuzzy process $(X(t))_{t\geq 0}$ defined on $(\Omega, \mathcal{A}, P)$ with a reference family $(\mathcal{A}_t)_{t\geq 0}$ such that

(i) there exists a 1-dimensional Brownian motion $(B_t)_{t\geq 0}$ with $B(0) = 0$ a.s.,

(ii) $(X(t))_{t\geq 0}$ is $\mathcal{A}_t$-adapted and continuous in t a.s., i.e., with probability one, $D(X(t+h), X(t)) \to 0$ as $h \to 0$,

(iii) $(X(t))_{t\geq 0}$ and $(B_t)_{t\geq 0}$ satisfy (2).

And we say that the *pathwise uniqueness* of solutions for (1) holds if whenever $(X(t))_{t>0}$ and $(X'(t))_{t>0}$ are any two solutions defined on the same probability space with same reference family $(\mathcal{A}_t)_{t\geq 0}$ and the same 1-dimensional Brownian motion $(B_t)_{t\geq 0}$ such that $X(0) = X'(0)$ a.s., then $X(t) = X'(t)$ for all $t \geq 0$ a.s..

The following theorem is our main result.

THEOREM 3.2. *Assume that $F, G : [0,\infty) \times \mathcal{F}_c(\mathbf{R}) \longrightarrow \mathcal{F}_c(\mathbf{R})$ satisfy the following conditions;*

(i) *there exists a $K > 0$ such that*

$$D^2(F(t,x), F(t,y)) + D^2(G(t,x), G(t,y)) \leq KD^2(x,y),$$

$$|||[F(t,x)]^0|||^2 + |||[G(t,x)]^0|||^2 \leq K(1 + |||[x]^0|||^2)$$

for all $x, y \in \mathcal{F}_c(\mathbf{R})$ and $t \in [0,\infty)$,

(ii) *for any $(Y(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{F}_c(\mathbf{R}))$, a process $(G(t,Y(t)))_{t\geq 0}$ satisfies that for all $\alpha \in [0,1]$, $([G(\cdot, Y(\cdot))]^\alpha(t))_{t\geq 0}$ is defined by $[G(\cdot, Y(\cdot))]^\alpha(t) = \langle g_1^{\alpha,Y} ; g_2^{\alpha,Y}\rangle(t)$ for some $(g_i^{\alpha,Y}(t))_{t\geq 0}, i = 1,2$,*

(iii) *for any $(Y_1(t))_{t\geq 0}, (Y_2(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{F}_c(\mathbf{R}))$, it holds that*

$$\left(\int_0^t (g_2^{\alpha,Y_1}(s) - g_1^{\alpha,Y_1}(s))dB_s\right)\left(\int_0^t (g_2^{\alpha,Y_2}(s) - g_1^{\alpha,Y_2}(s))dB_s\right) \geq 0$$

for all $t \geq 0$ a.s..

Then a solution $(X(t))_{t\geq 0}$ of the fuzzy stochastic differential equation (1) exists and the pathwise uniqueness of solutions holds.

Proof. Let $T > 0$ be any given. Define $X_0(t) = x_0$ a.s. for all $t \in [0,T]$. Then by the condition (i), we have

$$E\left[\int_0^T |||[G(t,x_0)]^0|||^2 dt\right] \leq TK\left(1 + E\left[|||[x_0]^0|||^2\right]\right) < \infty$$

and hence $(G(t,x_0))_{t\geq 0} \in \mathcal{L}^2(\mathcal{F}_c(\mathbf{R}))$. By the definition of the fuzzy stochastic integral, $\int_0^t G(s,x_0)dB_s$ is defined. And thus we can define a continuous stochastic process

$$X_1(t) = x_0 + \int_0^t F(s,x_0)ds + \int_0^t G(s,x_0)dB_s$$

with $E[|||[X_1(t)]^0|||^2] < \infty$ for all $t \in [0,T]$. Now assume that continuous processes

$$X_i(t) = x_0 + \int_0^t F(s, X_{i-1}(s))ds + \int_0^t G(s, X_{i-1}(s))dB_s,\ i = 2,3,\ldots,n,$$

are defined and these satisfy $\sup_{0\le t\le T} E[|||[X_i(t)]^0|||^2] < \infty$. Then by the condition (i),

$$E\left[\int_0^t |||[G(s,X_n(s))]^0|||^2\right] \le TK\left(1+\sup_{0\le t\le T}|||[X_n(t)]^0|||^2\right) < \infty$$

and hence $(G(t,X_n(t)))_{t\ge 0}\in\mathcal{L}^2(\mathcal{F}_c(\mathbf{R}))$. This shows that a fuzzy stochastic integral $\int_0^t G(s,X_n(s))dB_s$ can is defined. And hence we can define a continuous stochastic process

$$X_{n+1}(t) = x_0 + \int_0^t F(s,X_n(s))ds + \int_0^t G(s,X_n(s))dB_s. \tag{8}$$

By mathematical induction, we obtain a sequence $\{(X_n(t))_{t\ge 0}\}$, $n = 1,2,\ldots$, of stochastic processes in $\mathcal{L}^2(\mathcal{F}_c(\mathbf{R}))$. By Theorem 2.3, Theorem 2.5, the equality (3), the inequality (4) and our assumptions, it holds

$$\begin{aligned}
& E\left[\sup_{0\le t\le T} D^2\left(X_n(t),X_{n+1}(t)\right)\right]\\
\le\ & 2E\left[\sup_{0\le t\le T} D^2\left(\int_0^t F(s,X_{n-1}(s))ds, \int_0^t F(s,X_n(s))ds\right)\right]\\
& +2E\left[\sup_{0\le t\le T} D^2\left(\int_0^t G(s,X_{n-1}(s))dB_s, \int_0^t G(s,X_n(s))dB_s\right)\right]\\
\le\ & 2TE\left[\int_0^T D^2\left(F(s,X_{n-1}(s)),F(s,X_n(s))\right)ds\right]\\
& +8E\left[\int_0^T D^2\left(G(s,X_{n-1}(s)),G(s,X_n(s))\right)ds\right]\\
\le\ & (2T+8)K\int_0^T E\left[\sup_{0\le s\le t} D^2(X_{n-1}(s),X_n(s))\right]dt\\
\le\ & K_T^n\int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}} E\left[\sup_{0\le s\le t_n} D^2(x_0,X_1(s))\right]dt_n\cdots dt_2dt_1,
\end{aligned}$$

where $K_T=(2T+8)K$. By the same argument as above, we have

$$E\left[\sup_{0\le t\le t_n} D^2(x_0,X_1(s))\right] \le K_T\left(1+|||\ [x_0]^0|||^2\right).$$

Thus we have

$$E\left[\sup_{0\le t\le T} D^2\left(X_n(t),X_{n+1}(t)\right)\right] \le \frac{K_T^{n+1}}{(n+1)!}\left(1+|||\ [x_0]^0|||^2\right).$$

By Chebyshev's inequality, we obtain

$$\begin{aligned} & P\left(\sup_{0\le t\le T} D\left(X_n(t), X_{n+1}(t)\right) \ge \frac{1}{2^{n+1}}\right) \\ \le\ & 4^{n+1}E\left[\sup_{0\le t\le T} D^2\left(X_n(t), X_{n+1}(t)\right)\right] \le C(T)\frac{K_T^{n+1}}{(n+1)!}, \end{aligned}$$

where $C(T) > 0$ is a constant depending only on x_0 and T. By the fact that $\mathcal{F}_c(\mathbf{R})$ is a closed subspace of a complete metric space $(\mathcal{F}(\mathbf{R}), D)$ (see [13]) and the Borel-Cantelli lemma, we see that $X_n(t)$ converges uniformly on $[0, T]$ with probability one. Since T was arbitrary, $X(t) = \lim_{n\to\infty} X_n(t)$ determines a continuous process which is clearly a solution of (1). Now to prove the uniqueness of the solution, let $(X(t))_{t\ge0}$ and $(X'(t))_{t\ge0}$ be any two solutions of (1). Then by the similar calculations as above, we have

$$E\left[D^2(X(t), X'(t))\right] \le 2K(1+T)\int_0^t E\left[D^2(X(s), X'(s))\right] ds$$

for all $t \in [0, T]$. By Gronwall's lemma, we obtain

$$E[D^2(X(t), X'(t))] = 0$$

for all $t \in [0, T]$. Hence, letting $T \to \infty$, we have $X(t) = X'(t)$ a.s. for all $t \ge 0$. Since $(X(t))_{t\ge0}$ and $(X'(t))_{t\ge0}$ are D-continuous in t a.s., we can conclude that $X(t) = X'(t)$ for all $t \ge 0$ a.s.. The proof is complete. □

4. An example

In this section we give a worked example of fuzzy stochastic differential equation (1) of which the solution is given concretely. For $(f_1(t))_{t\ge0}, (f_2(t))_{t\ge0} \in L^2(\mathbf{R})$ satisfying $f_1(t) \le f_2(t)$ for all $t \ge 0$ a.s., define $(Y(t))_{t\ge0} \in \mathcal{L}^2(\mathcal{F}_c(\mathbf{R}))$ by $Y(t) = (f_1(t); f_2(t))_S$ as in Example 2.2. In (1), take $F(t, x) = -x$ and $G(t, x) = Y(t)$. Then we get the following fuzzy stochastic differential equation.

$$\left\{\begin{array}{l} dX(t) = -X(t)dt + Y(t)dB_t, \\ X(0) = X_0, \end{array}\right. \tag{9}$$

where $[X_0]^\alpha = [\underline{c}(\alpha), \overline{c}(\alpha)]$ for any $\alpha \in [0, 1]$. Since coefficients in (9) satisfy the conditions in Theorem 3.2, there is a unique solution $(X(t))_{t>0}$

of (9). This means that for any $\alpha \in [0,1]$, $([X]^\alpha(t))_{t\geq 0}$ satisfies the following set valued stochastic differential equation.

$$\text{(10)}\qquad \begin{cases} d[X]^\alpha(t) = -[X]^\alpha(t)dt + [\underline{y}(\alpha;t), \overline{y}(\alpha;t)]dB_t, \\ [X]^\alpha(0) = [\underline{c}(\alpha), \overline{c}(\alpha)], \end{cases}$$

that is,

$$\text{(11)}\qquad \begin{aligned} [X]^\alpha(t) &= [\underline{c}(\alpha), \overline{c}(\alpha)] + \int_0^t -[X]^\alpha(s)ds \\ &\quad + \langle \int_0^\cdot \underline{y}(\alpha;s)dB_s; \int_0^\cdot \overline{y}(\alpha;s)dB_s\rangle(t), \end{aligned}$$

where $\underline{y}(\alpha;s)$ and $\overline{y}(\alpha;s)$ are given by (6) in Example 2.2. Denote $[X]^\alpha(t) = [\underline{X}(\alpha;t), \overline{X}(\alpha;t)]$. We will give the concrete representations of $\underline{X}(\alpha;t)$ and $\overline{X}(\alpha;t)$. We can, without loss of generality, assume that there is a $\delta > 0$ such that $\int_0^t \underline{y}(\alpha;s)dB_s < \int_0^t \overline{y}(\alpha;s)dB_s$ for all $t \in (0,\delta]$. Define two sequences $\{\tau_i\}_{i=1,2,\ldots}$ and $\{\sigma_i\}_{i=0,1,2,\ldots}$ of stopping times by

$$\begin{aligned} \sigma_0 &= 0, \\ \tau_1 &= \inf\left\{t > 0 : \int_0^t \underline{y}(\alpha;s)dB_s > \int_0^t \overline{y}(\alpha;s)dB_s\right\}, \\ \sigma_1 &= \inf\left\{t > \tau_1 : \int_0^t \underline{y}(\alpha;s)dB_s < \int_0^t \overline{y}(\alpha;s)dB_s\right\}, \\ &\ \vdots \\ \tau_i &= \inf\left\{t > \sigma_{i-1} : \int_0^t \underline{y}(\alpha;s)dB_s > \int_0^t \overline{y}(\alpha;s)dB_s\right\},\ i = 2,3,\ldots \\ \sigma_i &= \inf\left\{t > \tau_i : \int_0^t \underline{y}(\alpha;s)dB_s < \int_0^t \overline{y}(\alpha;s)dB_s\right\},\ i = 2,3,\ldots. \end{aligned}$$

First we consider the case $t \in [0,\tau_1]$. Since $\int_0^t \underline{y}(\alpha;s)dB_s < \int_0^t \overline{y}(\alpha;s)dB_s$ for all $t \in [0,\tau_1]$, (11) is equivalent to the following equation

$$\text{(12)}\ \begin{aligned} [X]^\alpha(t) &= \int_0^t -[X(s)]^\alpha ds \\ &\quad + \left[\underline{c}(\alpha) + \int_0^t \underline{y}(\alpha;s)dB_s, \overline{c}(\alpha) + \int_0^t \overline{y}(\alpha;s)dB_s\right]. \end{aligned}$$

Since $-[a,b]=[-b,-a]$ for all $a \le b \in \mathbf{R}$, $\underline{X}(\alpha;t)$ and $\overline{X}(\alpha;t)$ are the solutions of the following stochastic differential system.

$$\underline{X}(\alpha;t)=\underline{c}(\alpha)-\int_0^t \overline{X}(\alpha;s)ds+\int_0^t \underline{y}(\alpha;s)dB_s, \tag{13}$$

$$\overline{X}(\alpha;t)=\overline{c}(\alpha)-\int_0^t \underline{X}(\alpha;s)ds+\int_0^t \overline{y}(\alpha;s)dB_s. \tag{14}$$

Rewriting (13) and (14) in the vector valued stochastic differential equation, we have

$$\begin{cases} d\mathbf{X}(\alpha;t)=A\mathbf{X}(\alpha;t)dt+\mathbf{Y}(\alpha;t)dB_t, \\ \mathbf{X}(\alpha;0)=\mathbf{X}_0(\alpha) \end{cases} \tag{15}$$

where

$$\mathbf{X}(\alpha;t)=\begin{bmatrix} \underline{X}(\alpha;t) \\ \overline{X}(\alpha;t) \end{bmatrix}, \mathbf{Y}(\alpha;t)=\begin{bmatrix} \underline{y}(\alpha;t) \\ \overline{y}(\alpha;t) \end{bmatrix}, \mathbf{X}_0(\alpha)=\begin{bmatrix} \underline{c}(\alpha) \\ \overline{c}(\alpha) \end{bmatrix}$$

and

$$A=\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Since $A^{2n}=I$ and $A^{2n+1}=A$ for all nonnegative integer n, we see that

$$e^{tA}=I\cosh t+A\sinh t=\begin{bmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{bmatrix}.$$

Here I is the unit 2×2 matrix. The unique solution of stochastic differential equation (15) is given by, for $t \in [0,\tau_1]$,

$$\begin{aligned} \mathbf{X}(\alpha;t) &= e^{tA}\mathbf{X}_0(\alpha)+\int_0^t e^{(t-s)A}\mathbf{Y}(\alpha;s)dB_s \\ &= e^t\mathbf{X}_0(\alpha)+\int_0^t e^{(t-s)}\mathbf{Y}(\alpha;s)dB_s \\ &= e^t\begin{bmatrix} \underline{c}(\alpha)+\int_0^t e^{-s}\underline{y}(\alpha;s)dB_s \\ \overline{c}(\alpha)+\int_0^t e^{-s}\overline{y}(\alpha;s)dB_s \end{bmatrix}. \end{aligned}$$

The second equality holds from the fact that $\cosh t+\sinh t=e^t$. From the definition of $\mathbf{X}(\alpha;t)$, we have

$$\underline{X}(\alpha;t)=e^t\left(\underline{c}(\alpha)+\int_0^t e^{-s}\underline{y}(\alpha;s)dB_s\right) \tag{16}$$

and

$$\overline{X}(\alpha;t)=e^t\left(\overline{c}(\alpha)+\int_0^t e^{-s}\overline{y}(\alpha;s)dB_s\right). \tag{17}$$

We can see that

$$(18)\qquad \underline{c}(\alpha) + \int_0^t e^{-s}\underline{y}(\alpha;s)dB_s \leq \overline{c}(\alpha) + \int_0^t e^{-s}\overline{y}(\alpha;s)dB_s$$

for all $t \in [0,\tau_1]$ a.s.. In fact, to prove this, put

$$Z(t) = \overline{c}(\alpha) - \underline{c}(\alpha) + \int_0^t (\overline{y}(\alpha;s) - \underline{y}(\alpha;s))dB_s.$$

Then $Z(t) \geq 0$ for all $t \in [0,\tau_1]$ a.s. clearly. By Itô's formula, we have

$$\begin{aligned}
& \overline{c}(\alpha) + \int_0^t e^{-s}\overline{y}(\alpha;s)dB_s - \underline{c}(\alpha) - \int_0^t e^{-s}\underline{y}(\alpha;s)dB_s \\
= \ & \overline{c}(\alpha) - \underline{c}(\alpha) + \int_0^t e^{-s}(\overline{y}(\alpha;s) - \underline{y}(\alpha;s))dB_s \\
= \ & e^{-t}Z(t) + \int_0^t e^{-s}Z(s)ds \\
\geq \ & 0.
\end{aligned}$$

Thus, from (16), (17) and (18), we can see that $\overline{X}(\alpha;t) \geq \underline{X}(\alpha;t)$. Furthermore, the expectation and variance of $\overline{X}(\alpha;t)$ for $t \in [0,\tau_1]$ are calculated as follows.

$$E[\overline{X}(\alpha;t)] = \overline{c}(\alpha)e^t$$

and

$$\begin{aligned}
Var(\overline{X}(\alpha;t)) &= E[(\overline{X}(\alpha;t) - E[\overline{X}(\alpha;t)])^2] \\
&= E[(e^t\int_0^t e^{-s}\overline{y}(\alpha;s)dB_s)^2] \\
&= e^{2t}\int_0^t e^{-2s}E[\overline{y}(\alpha;s)]^2 ds.
\end{aligned}$$

Similarly we obtain

$$E[\underline{X}(\alpha;t)] = \underline{c}(\alpha)e^t$$

and

$$Var(\underline{X}(\alpha;t)) = e^{2t}\int_0^t e^{-2s}E[\underline{y}(\alpha;s)]^2 ds.$$

Next, we consider the case $t \in (\tau_1, \sigma_1]$. Since, for all $t \in (\tau_1, \sigma_1]$, $\int_{\tau_1}^t \overline{y}(\alpha; s)dB_s \le \int_{\tau_1}^t \underline{y}(\alpha; s)dB_s$ and $\overline{X}(\alpha;\tau_1) \ge \underline{X}(\alpha;\tau_1)$ we get the following equation

$$
(19) \qquad \begin{aligned}[X]^\alpha(t) &= \int_{\tau_1}^t -[X(s)]^\alpha ds + [\underline{X}(\alpha;\tau_1) \\ &\quad + \int_{\tau_1}^t \overline{y}(\alpha;s)dB_s, \overline{X}(\alpha;\tau_1) + \int_{\tau_1}^t \underline{y}(\alpha;s)dB_s].\end{aligned}
$$

By the same argument as the first part, we get the following vector valued stochastic differential equation

$$
(20) \qquad \begin{cases} d\mathbf{X}(\alpha;t) = A\mathbf{X}(\alpha;t)dt + \widetilde{\mathbf{Y}}(\alpha;t)dB_t, \\ \mathbf{X}(\alpha;\tau_1) = \mathbf{X}_{\tau_1}(\alpha), \ t \in (\tau_1, \sigma_1] \end{cases}
$$

where

$$
\mathbf{X}_{\tau_1}(\alpha) = \begin{bmatrix} \underline{X}(\alpha;\tau_1) \\ \overline{X}(\alpha;\tau_1) \end{bmatrix}, \widetilde{\mathbf{Y}}(\alpha;t) = \begin{bmatrix} \overline{y}(\alpha;t) \\ \underline{y}(\alpha;t) \end{bmatrix}.
$$

The solution of (20) is given by

$$
\begin{bmatrix} \underline{X}(\alpha;t) \\ \overline{X}(\alpha;t) \end{bmatrix} = \mathbf{X}(\alpha;t) = e^{(t-\tau_1)} \begin{bmatrix} \underline{X}(\alpha;\tau_1) + \int_{\tau_1}^t e^{-s}\overline{y}(\alpha;s)dB_s \\ \overline{X}(\alpha;\tau_1) + \int_{\tau_1}^t e^{-s}\underline{y}(\alpha;s)dB_s \end{bmatrix}.
$$

In this case the expectation and the variation of $\overline{X}(\alpha;t)$ are given by

$$
\begin{aligned} E[\overline{X}(\alpha;t)] &= e^{(t-\tau_1)}E[\overline{X}(\alpha;\tau_1)] \\ &= e^{(t-\tau_1)}e^{\tau_1}\overline{c}(\alpha) \\ &= \overline{c}(\alpha)e^t \end{aligned}
$$

and

$$
\begin{aligned} & Var(\overline{X}(\alpha;t)) \\ = \ & E[(\overline{X}(\alpha;t) - E[\overline{X}(\alpha;t)])^2] \\ = \ & E\left[e^{2t}\left(\int_0^{\tau_1} e^{-s}\underline{y}(\alpha;s)dB_s - e^{-\tau_1}\int_{\tau_1}^t e^{-s}\overline{y}(\alpha;s)dB_s\right)^2\right] \\ \le \ & e^{2t}\left(2\int_0^{\tau_1} e^{-2s}E[|\underline{y}(\alpha;s)|^2]ds + 2e^{-2\tau_1}\int_{\tau_1}^t e^{-2s}E[|\overline{y}(\alpha;s)|^2]ds\right) \\ \le \ & 2e^{2t}\int_0^t e^{-2s}E[|\underline{y}(\alpha;s)|^2 + |\overline{y}(\alpha;s)|^2]ds. \end{aligned}
$$

Repeating this procedure, when $[X]^\alpha(t) = [\underline{X}(\alpha;t), \overline{X}(\alpha;t)]$ is the solution of (10), $\overline{X}(\alpha;t)$ is given by, for each $t \in (\tau_k, \sigma_k]$,

$$
\begin{aligned}
&\overline{X}(\alpha;t) \\
= \;& e^{(t-\tau_k)}\left(\overline{X}(\alpha;\tau_k) + \int_{\tau_k}^{t} e^{-s}\underline{y}(\alpha;s)dB_s\right) \\
= \;& e^{t}\left(\overline{c}(\alpha) + \sum_{i=1}^{k} e^{-\sigma_{i-1}} \int_{\sigma_{i-1}}^{\tau_i} e^{-s}\overline{y}(\alpha;s)dB_s\right) \\
& + e^{t}\left(\sum_{i=1}^{k-1} e^{-\tau_i}\int_{\tau_i}^{\sigma_i} e^{-s}\underline{y}(\alpha;s)dB_s + e^{-\tau_k}\int_{\tau_k}^{t} e^{-s}\underline{y}(\alpha;s)dB_s\right)
\end{aligned}
$$

and, for each $t \in (\sigma_k, \tau_{k+1}]$,

$$
\begin{aligned}
&\overline{X}(\alpha;t) \\
= \;& e^{(t-\sigma_k)}\left(\overline{X}(\alpha;\sigma_k) + \int_{\sigma_k}^{t} e^{-s}\overline{y}(\alpha;s)dB_s\right) \\
= \;& e^{t}\left(\overline{c}(\alpha) + \sum_{i=1}^{k} e^{-\sigma_{i-1}} \int_{\sigma_{i-1}}^{\tau_i} e^{-s}\overline{y}(\alpha;s)dB_s\right) \\
& + e^{t}\left(\sum_{i=1}^{k} e^{-\tau_i}\int_{\tau_i}^{\sigma_i} e^{-s}\underline{y}(\alpha;s)dB_s + e^{-\sigma_k}\int_{\sigma_k}^{t} e^{-s}\overline{y}(\alpha;s)dB_s\right).
\end{aligned}
$$

From these we obtain $E[\overline{X}(\alpha;t)] = \overline{c}(\alpha)e^t$ for all $t \geq 0$ and

$$
\begin{aligned}
&Var(\overline{X}(\alpha;t)) \\
\leq \;& \begin{cases} 2ke^{2t}\int_0^t e^{-2s}E[||\underline{y}(\alpha;s)|^2 + |\overline{y}(\alpha;s)|^2]ds, \; t \in (\tau_k, \sigma_k] \\ (2k+1)e^{2t}\int_0^t e^{-2s}E[|\underline{y}(\alpha;s)|^2 + |\overline{y}(\alpha;s)|^2]ds, \; t \in (\sigma_k, \tau_{k+1}]. \end{cases}
\end{aligned}
$$

This shows that, for given $\overline{c}(\alpha)$ and the distribution of $(\underline{y}(\alpha;t))_{t\geq 0}$ and $(\overline{y}(\alpha;t))_{t\geq 0}$, the expectation and upper bound of variance of $\overline{X}(\alpha;t)$ can be computed. We can get the results related to $\underline{X}(\alpha;t)$ similarly.

Acknowledgement. Much of this research was done during a visit to Department of Mathematics, University of Queensland, QLD 4072, Australia. The author thanks them for supporting a generous environment.

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