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HOLOMORPHIC EMBEDDINGS OF STEIN SPACES IN INFINITE-DIMENSIONAL PROJECTIVE SPACES

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Abstract. Let X be a reduced Stein space and L a holomorphic line bundle on X. L is spanned by its global sections and the associated holomorphic map $h_L : X \to \mathbf{P}(H^0(X,L)^*)$ is an embedding. Choose any locally convex vector topology τ on $H^0(X,L)^*$ stronger than the weak-topology. Here we prove that $h_L(X)$ is sequentially closed in $\mathbf{P}(H^0(X,L)^*)$ and arithmetically Cohen - Macaulay, i.e. for all integers $k \geq 1$ the restriction map ρ_k : $H^0(\mathbf{P}(H^0(X,L)^*), \mathcal{O}_{\mathbf{P}(H^0(X,L)^*)}(k)) \to H^0(h_L(X), \mathcal{O}_{h_L(X)}(k)) \cong H^0(X, L^{\otimes k})$ is surjective.

1. Introduction

For any complex vector space V, let $\mathbf{P}(V)$ be the projective space of all one-dimensional linear subspaces of V. If V is an infinite-dimensional topological vector space, then $\mathbf{P}(V)$ is an infinite-dimensional complex manifold. Let X be a finite-dimensional Hausdorff complex space countable at infinity and $\mathcal{F}$ a coherent analytic sheaf on X. The vector space $H^0(X.\mathcal{F})$ has a natural topology for which it is a nuclear Fréchet space (see [2], Th. 9 at p. 240, for a proof that $H^0(X,\mathcal{F})$ is Fréchet and Montel, [1], Th. 6 at p. 168, for a proof that $H^0(X,\mathcal{F})$ is Fréchet). If $\mathcal{F}$ is a line bundle and $H^0(X,\mathcal{F})$ spans $\mathcal{F}$, then there is a holomorphic map $h_{\mathcal{F}} : X \to \mathbf{P}(H^0(X,\mathcal{F})^*)$, where $H^0(X,\mathcal{F})^*$ denotes the topological dual of $H^0(X,\mathcal{F})$. Since $H^0(X,\mathcal{F})$ is nuclear and complete, it is semi-reflexive ([3], p. 144). Hence $\mathbf{P}(H^0(X,\mathcal{F})^*)$ determines $\mathbf{P}(H^0(X,\mathcal{F}))$. In this paper we prove the following results.

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THEOREM 1. *Let X be a reduced Stein space and L a holomorphic line bundle on X. L is spanned by its global sections and the associated holomorphic map $h_L : X \to \mathbf{P}(H^0(X,L)^*)$ is an embedding. Choose any locally convex vector topology τ on $H^0(X,L)^*$ stronger that the weak-topology. Then $h_L(X)$ is sequentially closed in $\mathbf{P}(H^0(X,L)^*)$.*

THEOREM 2. *Let X be a finite-dimensional Stein space and L a holomorphic line bundle on X. L is spanned by its global sections and the associated holomorphic map $h_L : X \to \mathbf{P}(H^0(X,L)^*)$ is an embedding. Choose any locally convex vector topology τ on $H^0(X,L)^*$ stronger than the weak-topology. Then $h_L(X)$ is arithmetically Cohen - Macaulay, i.e. for all integers $k \geq 1$ the restriction map ρ_k : $H^0(\mathbf{P}(H^0(X,L)^*), \mathcal{O}_{\mathbf{P}(H^0(X,L)^*)}(k)) \to H^0(h_L(X), \mathcal{O}_{h_L(X)}(k)) \cong H^0(X, L^{\otimes k})$ is surjective.*

For a generalization of Theorem 1 to embeddings in Grassmannians, see Theorem 3. For a generalization of Theorem 2 to non-complete embeddings, see Theorem 5. We also prove that the embedding $h_L(X)$ is extremely twisted (e.g. it has no inflectional point) (see Theorem 4 and Remark 2).

2. The proofs and other related results

LEMMA 1. *Let X be a Stein space and L a holomorphic line bundle on X. L is spanned by its global sections and the associated holomorphic map $h_L : X \to \mathbf{P}(H^0(X,L)^*)$ is an embedding*

Proof. L is spanned by its global sections by Cartan's Theorem A ([2], th. 13 at p. 243). Fix $P \in X$ and let $\mathcal{I}_{P,X}$ denote the ideal sheaf of P in X. The coherent ideal sheaf $(\mathcal{I}_{P,X})^2$ defines a non-reduced zero-dimensional analytic subspace Z of X such that $Z_{red} = \{P\}$. Z is called the first infinitesimal neighborhood of P in X. The restriction map $\rho : H^0(X,L) \to H^0(Z, L|Z)$ is surjective because $H^1(X, \mathcal{I}_Z \otimes L) = 0$ by Cartan's Theorem B ([2], Th. 14 at p. 243). The surjectivity of ρ is equivalent to the fact that h_L is a local embedding at P. Now fix $Q \in X$ such that $Q \neq P$. Since $H^1(X, \mathcal{I}_{\{P,Q\}} \otimes L) = 0$ by Cartan's Theorem B ([2], Th. 14 at p. 243), we have $h_L(P) \neq h_L(Q)$. Hence h_L is injective, too. □

Proof of Theorem 1. By Lemma 1 L is spanned by its global sections and h_L is an embedding. Assume the existence of a sequence $\{h_L(P_n)\}_{n\geq 0}$ in $h_L(X)$ converging to $Q \in \mathbf{P}(H^0(X,L)^*) \backslash h_L(X)$ and choose $f \in H^0(X, L)^* \setminus \{0\}$ representing Q. Since the support, Z, $\{P_n\}_{n \geq 0}$ is a discrete and infinite set in X and X is Stein, there is a holomorphic map $g : X \to \mathbf{C}$ such that $\lim_{n \to +\infty} |g(P_n)| = +\infty$. Thus $H^1(X, \mathcal{I}_Z \otimes L) = 0$ by Cartan's Theorem B ([2], Th. 14 at p. 243). Thus the restriction map $H^0(X, L) \to H^0(Z, L|Z)$ is surjective. Hence there is $s \in H^0(X, L)$ such that $s(P_n) \neq 0$ for every n. Since both s and gs are holomorphic sections of L, $f(gs)$ and $f(s)$ are complex numbers. Since $\{h_L(P_n)\}_{n \geq 0}$ converges to Q, we must have $f(s) = 0$. Let A be the set of all $\alpha \in H^0(X, L)$ such that α does not vanish at all except finitely many points of Z. Taking a subsequence of the sequence $\{P_n\}_{n \geq 0}$ we see that $f(\alpha) = 0$ for every $\alpha \in A$. We claim that A is dense in $H^0(X, L)$. Indeed, take any $\alpha \in A$ and $\beta \in H^0(X, L) \setminus A$. For every $z \in \mathbf{C} \setminus \{0\}$ we have $\beta + z\alpha \in A$, proving the claim. Since $f(\mu) = 0$ for every $\mu \in A$ by the first part of the proof, we have $f = 0$, contradiction. □

REMARK 1. Let X be a complex space and L a line bundle on X spanned by its global sections. Since $H^0(X, L)$ is semi-reflexive ([3], p. 144), the set of all closed hyperplanes of $\mathbf{P}(H^0(X, L)^*)$ may be identified with $\mathbf{P}(H^0(X, L))$. For any closed hyperplane H of $\mathbf{P}(H^0(X, L)^*)$, say corresponding to $u \in \mathbf{P}(H^0(X, L))$, the set $h_L^{-1}(H \cap h_L(X))$ is the zero-locus of any $f \in H^0(X, L) \setminus \{0\}$ representing u. Hence $H \cap h_L(X) = \emptyset$ for some closed hyperplane H of $\mathbf{P}(H^0(X, L)^*)$ if and only if L has a nowhere vanishing section, i.e. if and only if $L \cong \mathcal{O}_X$.

LEMMA 2. *Let X be an n-dimensional Stein space and L a holomorphic line bundle on X. Then there is a linear subspace W of $H^0(X, L)$ such that W spans L at each point of X and* $\dim(W) \leq n + 1$.

Proof. We will prove the same assertion even when X is not reduced because it is easier to prove by induction on n this more general statement. If $n = 0$, then L is trivial and the constant function 1 spans L. Assume $n \geq 1$ and that the result is true for all $(n-1)$-dimensional Stein spaces (even the unreduced ones). There is $s \in H^0(X, L)$ such that the zero-locus, D, of s is of dimension at most $n - 1$. The section s spans L at each point of $X \setminus D$. In particular s spans L at each point of X if $D = \emptyset$. Hence we may assume $D \neq \emptyset$. Since D is closed in X, D is a possibly non-reduced Stein space of dimension at most $n - 1$. By the inductive assumption there are $a_1, \ldots, a_n \in H^0(D, L|D)$ spanning $L|D$ at each point of D. By Cartan's Theorem B ([2], Th. 14 at p. 243) there is $s_i \in H^0(X, L)$ such that $s_i|D = a_i$. Since $a_1, \ldots, a_n$ span $L|D$, $s_1, \ldots, s_n$ span L at each point of D. Hence $s, s_1, \ldots, s_n$ span L at each point of X, concluding the proof. □

Proof of Theorem 2. The last isomorphism is true because h_L is an embedding. The map ρ_1 is surjective because h_L is the embedding associated to the complete linear system $H^0(X, L)$. Hence ρ_k is surjective for some integer $k > 0$ if and only if the map $\sigma_k : H^0(X, L) \otimes \cdots \otimes H^0(X, L) \to H^0(X, L^{\otimes k})$ is surjective; here the domain of σ_k is the algebraic tensor product of k copies of $H^0(X, L)$. By Lemma 2 there is a finite-dimensional linear subspace W of L such that W spans L at each point of X. Hence the evaluation map $\alpha : \mathcal{O}_X \otimes W \to L$ is surjective. We have an exact sequence

$$0 \to \mathrm{Ker}(\alpha) \to \mathcal{O}_X \otimes W \to L \to 0 \tag{1}$$

and hence $\mathrm{Ker}(\alpha)$ is a coherent analytic sheaf on X. Thus $H^1(X, L \otimes \mathrm{Ker}(\alpha)) = 0$ by Cartan's Theorem B. Thus by tensoring (1) with L we obtain the surjectivity of the multiplication map $W \otimes H^0(X, L) \to H^0(X, L^{\otimes 2})$. Thus σ_2 is surjective. Now assume $k \geq 3$ and that σ_{k-1} is surjective. Since $H^1(X, L^{\otimes k} \otimes \mathrm{Ker}(\alpha)) = 0$ (Cartan's Theorem B), by tensoring (1) with $L^{\otimes(k-1)}$ we obtain the surjectivity of the multiplication map $W \otimes H^0(X, L^{\otimes(k-1)}) \to H^0(X, L^{\otimes k})$. Since σ_{k-1} is surjective, we obtain the surjectivity of σ_k, concluding the proof by induction on k. □

THEOREM 3. *Let V be a topological vector space and X an n-dimensional Stein space embedded in $\mathbf{P}(V)$. We do not require that the embedding is a closed embedding. For all integers $k \geq 1$, let $\rho_k : H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(k)) \to H^0(X, \mathcal{O}_X(k))$ denote the restriction map. Assume that $\mathrm{Im}(\rho_1)$ is a finite-codimensional linear subspace of $H^0(X, \mathcal{O}_X(1))$ and call x its codimension. Then $\dim(Coker(\rho_k)) \leq x(n+1)^{k-1}$ for every $k \geq 2$.*

Proof. By Lemma 2 there is an $(n+1)$-dimensional linear subspace W of $H^0(X, \mathcal{O}_X(1))$ spanning $\mathcal{O}_X(1)$. Use the exact sequence (1) as in the proof of Theorem 2 to show that $\dim(Coker(\rho_k)) \leq \dim(W) \cdot \dim(Coker(\rho_{k-1}))$ for every $k \geq 2$. Hence we conclude by induction on k. □

THEOREM 4. *Let X be a finite-dimensional Stein space and L a holomorphic line bundle on X. Let $h_L : X \to \mathbf{P}(H^0(X, L)^*)$ be the embedding associated to L. Then for every zero-dimensional subscheme Z of X supported by finitely many points of X the linear span $\langle h_L(Z) \rangle$ of $h_L(Z)$ has dimension $h^0(X, \mathcal{O}_Z) - 1$.*

Proof. The equality in the statement means that $H^0(X, \mathcal{I}_Z \otimes L)$ has codimension $h^0(X\mathcal{O}_Z)$ in $H^0(X, L)$ and this is true because $L|Z \cong \mathcal{O}_Z$

by the finiteness of Z_{red} and $H^1(X, \mathcal{I}_Z \otimes L) = 0$ by Cartan's Theorem B ([2], Th. 14 at p. 243). $\square$

REMARK 2. The last statement of Theorem 4 means that the embedding h_L of X is extremely twisted. For instance, no three points of $h_L(X)$ are collinear and for every length three subscheme Z of X supported by one point of X the scheme $h_L(Z)$ spans a plane. The last assertion of Theorem 4 means that no point of $h_L(X)$ is an inflectional point.

REMARK 3. Let X be any Stein space with at least three points. It is very easy to construct embeddings of X into a projective space $\mathbf{P}(V)$ for which the statement of Theorem 4 is false; of course, for any such embedding the restriction map $\rho_1 : H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1)) \to H^0(X, \mathcal{O}_X(1))$ cannot be surjective. Here we construct such examples with $\mathrm{Coker}(\rho_1)$ finite-dimensional. We will also take as $\mathcal{O}_X(1)$ any holomorphic line bundle on X. Fix a holomorphic line bundle L on X and three distinct points P, Q and R of X. The linear space $H^0(X, \mathcal{I}_{\{P,Q,R\}} \otimes L)$ spans L outside $\{P, Q, R\}$ (see the proof of Lemma 1). By Cartan's Theorem B the linear space $H^0(X, \mathcal{I}_{\{P,Q,R\}} \otimes L)$ has codimension three in $H^0(X, L)$. Choose $s_1, s_2 \in H^0(X, L)$ not vanishing on $\{P, Q, R\}$ and separating P, Q and R. Let W be the closed codimension one linear subspace of $H^0(X, L)$ spanned by s_1, s_2 and $H^0(X, \mathcal{I}_{P,Q,R} \otimes L)$. It is easy to check that for general s_1, s_2 the linear space W induces an embedding $h_{L,W} : X \to \mathbf{P}(W)$. In this embedding the three points P, Q, R are mapped onto three collinear points. Now assume $\dim(X) > 0$ and fix a smooth non-isolated point P of X. We will construct an embedding $h_{L,V} : X \to \mathbf{P}(V)$ of X such that $h_{L,V}(P)$ is an inflectional point of $h_{L,V}(X)$. Let Z_i, $1 \le i \le 3$, be the zero-dimensional analytic subspace of X with $(\mathcal{I}_{P,X})^i$ as its ideal sheaf. Hence $(Z_i)_{red} = \{P\}$ and $\mathrm{length}(Z_i) = \binom{n+i-1}{n}$, where n is the dimension of X at P. By Cartan's Theorem B the restriction maps $\alpha_i : H^0(X, L) \to H^0(Z_i, L|Z_i)$ are surjective. Let V be the linear subspace of $H^0(X, L)$ spanned by $H^0(X, \mathcal{I}_{Z_3} \otimes L)$, a section of $H^0(X, L)$ not vanishing at P and n sections of $H^0(X, L)$ vanishing at P and inducing local coordinates around P. Since $H^0(X, \mathcal{I}_{Z_3} \otimes L)$ induces an embedding of $X \backslash \{P\}$, it is easy to check that $h_{L,V}$ is an embedding. By construction $h_{L,V}(Z_3) = h_{L,V}(Z_2)$ and hence $h_{L,V}(P)$ is an inflectional point of $h_{L,V}(X)$.

3. The Grassmannian

Fix an integer $r \geq 1$, a complex space X, a rank r holomorphic vector bundle E on X and a closed linear subspace W of $H^0(X, E)$ such that W spans E at each point of X. The pair (E, W) induces a holomorphic map $h_{E,W} : X \to G(r, W)$, where $G(r, W)$ is the Grassmannian of all closed codimension r linear subspaces of W. If $W = H^0(X, E)$, write h_E instead of $h_{E,W}$. $G(r, W)$ is a smooth complex manifold. There is a natural rank r holomorphic vector bundle Q on $G(r, W)$, often called the tautological quotient bundle of $G(r, W)$. The vector bundle Q is spanned by its global sections and $H^0(G(r, W), Q) \cong W$.

The proof of Lemma 1 gives the following result.

LEMMA 3. *Let X be a Stein space and E a rank r holomorphic vector bundle on X. Then E is spanned by its global sections and the holomorphic map $h_E : X \to G(r, H^0(X, W))$ is an embedding.*

Since $h_E^*(Q|h_E(X)) \cong E$, the proof of Theorem 2 gives the following result.

THEOREM 5. *Let X be a Stein space and E a rank r holomorphic vector bundle on X. For every integer $k \geq 1$ the map $H^0(G(r, H^0(X, E)), S^k(Q)) \to H^0(X, S^k(E))$ is surjective.*

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