J. Korean Math. Soc. **42** (2005), No. 1, pp. 111–127

COMPOSITION OPERATORS ON THE PRIVALOV SPACES OF THE UNIT BALL OF $\mathbb{C}^n$

SEI-ICHIRO UEKI

ABSTRACT. Let B and S be the unit ball and the unit sphere in $\mathbb{C}^n$, respectively. Let σ be the normalized Lebesgue measure on S. Define the *Privalov spaces* $N^p(B)$ $(1 < p < \infty)$ by

$$N^p(B) = \left\{ f \in H(B) : \sup_{0<r<1} \int_S \{\log(1 + |f(r\zeta)|)\}^p d\sigma(\zeta) < \infty \right\},$$

where $H(B)$ is the space of all holomorphic functions in B. Let φ be a holomorphic self-map of B. Let μ denote the pull-back measure $\sigma \circ (\varphi^*)^{-1}$. In this paper, we prove that the composition operator C_φ is *metrically bounded* on $N^p(B)$ if and only if $\mu(\mathcal{S}(\zeta,\delta)) \leq C\delta^n$ for some constant C and C_φ is *metrically compact* on $N^p(B)$ if and only if $\mu(\mathcal{S}(\zeta,\delta)) = o(\delta^n)$ as $\delta \downarrow 0$ uniformly in $\zeta \in S$. Our results are an analogous results for MacCluer's Carleson-measure criterion for the boundedness or compactness of C_φ on the Hardy spaces $H^p(B)$.

1. Introduction

Let $n \geq 1$ be a fixed integer. Let $B \equiv B_n$ and $S \equiv \partial B$ denote the unit ball and the unit sphere of the complex n-dimensional Euclidean space $\mathbb{C}^n$, respectively. Let ν and σ denote the normalized Lebesgue measure on B and S, respectively. For each $\alpha \in (-1, \infty)$, we set $c_\alpha = \Gamma(n+\alpha+1)/\{\Gamma(n+1)\Gamma(\alpha+1)\}$ and $d\nu_\alpha = c_\alpha(1-|z|^2)^\alpha d\nu(z)$ $(z \in B)$. Note that $\nu_\alpha(B) = 1$. Let $H(B)$ denote the space of all holomorphic functions in B. For each $p \in (0, \infty)$ and $\alpha \in (-1, \infty)$, the *Hardy space*

Received August 25, 2003.

2000 Mathematics Subject Classification: Primary 32A35, 32A37; Secondary 47B33.

Key words and phrases: Hardy spaces, Privalov spaces, composition operators, unit ball of $\mathbb{C}^n$.

$H^p(B)$ and the *weighted Bergman space* $A^p(\nu_\alpha)$ are as usual defined by

$$H^p(B) = \left\{ f \in H(B) \,:\, \|f\|_{H^p}^p \equiv \sup_{0<r<1} \int_S |f_r|^p d\sigma < \infty \right\},$$
$$A^p(\nu_\alpha) = \left\{ f \in H(B) \,:\, \|f\|_{A^p(\nu_\alpha)}^p \equiv \int_B |f|^p d\nu_\alpha < \infty \right\},$$

where $f_r(z) = f(rz)$ for $0 < r < 1$, $z \in \mathbb{C}^n$ with $rz \in B$. As in [12], the *Privalov spaces* $N^p(B)$ $(1 < p < \infty)$ is defined by

$$N^p(B) = \left\{ f \in H(B) \,:\, \|f\|_{N^p}^p \equiv \sup_{0<r<1} \int_S \{\log(1 + |f_r|)\}^p d\sigma < \infty \right\}.$$

For each $p \in [1, \infty)$ and $\alpha \in (-1, \infty)$, we define the *weighted Bergman-Privalov space* $(AN)^p(\nu_\alpha)$ by

$$(AN)^p(\nu_\alpha) = \left\{ f \in H(B) \,:\, \|f\|_{(AN)^p(\nu_\alpha)}^p \equiv \int_B \{\log(1 + |f|)\}^p d\nu_\alpha < \infty \right\}.$$

The properties of the spaces $N^p(B)$ were studied in [10] and [12]. The studies on the spaces $(AN)^p(\nu_\alpha)$ were in [7], [10] and [13].

If φ is a holomorphic self-map of B, then φ induces a linear operator C_φ on $H(B)$ by means of the equation $C_\varphi f = f \circ \varphi$. This C_φ is called the *composition operator* induced by φ.

Recently, C_φ on $N^p(B_1)$ have been studied by J. S. Choa and H. O. Kim [1, 2]. And C_φ on $(AN)^1(\nu)$ (in the case $n = 1$) have been studied by J. Xiao [13]. According to [2], we say that C_φ is *metrically bounded* on $N^p(B)$ if there exists a positive constant K such that $\|C_\varphi f\|_{N^p(B)} \leq K\|f\|_{N^p(B)}$ for all $f \in N^p(B)$. We call C_φ *metrically compact* if C_φ maps every closed ball $B_R = \{f \in N^p(B) : \|f\|_{N^p(B)} \leq R\}$ into a relatively compact set in $N^p(B)$. We also define the metrically boundedness and the metrically compactness of C_φ on $(AN)^p(\nu_\alpha)$ in a similar way.

In 1985, B. D. MacCluer gave the following measure-theoretic characterizatios of the holomorphic self-map φ that induce bounded or compact composition operatars on $H^p(B)$ $(0 < p < \infty)$.

THEOREM 1.1 (B. D. MacCluer [5]). *Let φ be a holomorphic self-map of B. Let μ be the measure on $\overline{B}$ defined by $\sigma \circ (\varphi^*)^{-1}$. Then for $0 < p < \infty$*

(a) *C_φ is bounded on $H^p(B)$ if and only if there exists a positive constant C such that*

$$\mu(\mathcal{S}(\zeta, \delta)) \leq C\delta^n \quad (\zeta \in S, \delta > 0).$$

(b) C_φ *is compact on* $H^p(B)$ *if and only if*

$$\limsup_{\delta\downarrow 0}\sup_{\zeta\in S}\frac{\mu(\mathcal{S}(\zeta,\delta))}{\delta^n}=0.$$

Here $\mathcal{S}(\zeta,\delta)$ *is the Carleson set in* $\overline{B}$.

As expected, an analogous results involving Carleson-measure conditions hold for the spaces $A^p(\nu_\alpha)$ $(0 < p < \infty, -1 < \alpha < \infty)$. See [3] pp.161 164.

The purpose of the present paper is to prove an analogous results for the metrically boudedness and the metrically compactness of C_φ on the spaces $N^p(B)$ $(1 < p < \infty)$. The proofs of our main theorems are essentially the same as those of MacCluer's theorems. Moreover, as a corollary of our main theorems we obtain that C_φ is metrically compact (respectively metrically bounded) on $N^p(B)$ if and only if C_φ is compact (respectively bounded) on $H^2(B)$. The last results is the higher dimensional cases $(n \geq 1)$ of the result by J. S. Choa and H. O. Kim [1]. Also, we obtain the same results for the spaces $(AN)^p(\nu_\alpha)$ $(1 \leq p < \infty, -1 < \alpha < \infty)$.

2. Preliminaries

In order to prove our main results we will use some notations and lemmas. For $\zeta \in S$ and $\delta > 0$, we introduce the *Carleson set* $\mathcal{S}(\zeta,\delta)$ in $\overline{B}$ which is defined by

$$\mathcal{S}(\zeta,\delta)=\left\{z\in\overline{B}:|1-\langle z,\zeta\rangle|<\delta\right\}.$$

Moreover, we put $B(\zeta,\delta)=\mathcal{S}(\zeta,\delta)\cap B$ and $S(\zeta,\delta)=\mathcal{S}(\zeta,\delta)\cap S$.

The proof of the following lemma is essentially the same as one of S. C. Power's theorem in [8].

LEMMA 2.1. *Let* $1 < p < \infty$. *Suppose that* μ *is a positive Borel measure on* B *and there exists a constant* $C > 0$ *such that*

$$\mu(B(\zeta,\delta))\leq C\sigma(S(\zeta,\delta))\quad(\zeta\in S,\delta>0).$$

Then there exists a constant $K > 0$ *such that*

$$\int_B\{\log(1+|f|)\}^p d\mu\leq K\|f\|^p_{N^p(B)},$$

for all $f \in N^p(B)$.

Proof. Fix $f \in N^p(B)$ and $t > 0$. By the same argument as the proof of Theorem in [8](pp.14-15), we can prove that there exists a constant $C' > 0$ such that

$$\begin{aligned}(2.1) \qquad &\mu(\{z \in B : \log(1+|f(z)|) \geq t\}) \\ \leq\ & C'\sigma(\{\zeta \in S : \log(1 + Mf(\zeta)) \geq t\}),\end{aligned}$$

where Mf is the *admissible maximal function* of f which is defined by

$$Mf(\zeta) = \sup\{|f(z)| : z \in D(\zeta)\},$$

and $D(\zeta)$ is the set of all $z \in \mathbb{C}^n$ such that $|1 - \langle z, \zeta\rangle| < 1 - |z|^2$ for $\zeta \in S$.

By (2.1) we have

$$\begin{aligned}\int_B \{\log(1+|f|)\}^p d\mu &= p\int_0^\infty \mu\{\log(1+|f|) > t\}t^{p-1}dt \\ &\leq C'p\int_0^\infty \sigma\{\log(1+Mf) \geq t\}t^{p-1}dt \\ &\leq C'p2^{p-1}\int_0^\infty \sigma\left\{\log(1+Mf) > \frac{t}{2}\right\}\left(\frac{t}{2}\right)^{p-1}dt \\ (2.2) \qquad &= C'2^p\int_S \{\log(1+Mf)\}^p d\sigma.\end{aligned}$$

Since $\log(1+e^t) : [-\infty, \infty) \to [0, \infty)$ is a nondecreasing convex function, not identically 0 and $f \in N^p(B)$, by Theorem 5.6.2 (b) in [9] there exists a $h \in L^p(\sigma)$ such that

$$(2.3) \qquad \log(1+|f(z)|) \leq P[h](z) \quad (z \in B),$$

$$(2.4) \qquad \|h\|_{L^p(\sigma)} = \|f\|_{N^p(B)},$$

where $P[h]$ is the (*invariant*) *Poisson integral* of h. Moreover, since $1 < p < \infty$ and $h \in L^p(\sigma)$, it follows from Theorem 5.4.10 in [9] that

$$(2.5) \qquad \int_S \{MP[h]\}^p d\sigma \leq A(p)\int_S |h|^p d\sigma,$$

for some constant $A(p) > 0$ depending only on p. By (2.2)~(2.5) we obtain

$$\begin{aligned}\int_B \{\log(1+|f|)\}^p d\mu &\le C'2^p \int_S \{\log(1+Mf)\}^p d\sigma \\ &= C'2^p \int_S \{M\log(1+|f|)\}^p d\sigma \\ &\le C'2^p \int_S \{MP[h]\}^p d\sigma \\ &\le C'2^p A(p)\|f\|^p_{N^p(B)}.\end{aligned}$$

This completes the proof. □

LEMMA 2.2 ([5] Lemma 1.3). *Suppose that μ is a positive Borel measure on S such that*

$$\mu(S(\zeta,\delta)) \le C\delta^n \quad (\zeta \in S, \delta > 0),$$

for some constant $C > 0$. Then $d\mu = g d\sigma$, where $g \in L^\infty(S)$ with $\|g\|_{L^\infty} \le C'$, where C' is the product of C and a constant depending only on the dimension n.

Proof. See [5], p. 238, Lemma 1.3. □

LEMMA 2.3. *Let $1 < p < \infty$. Suppose that μ is a positive Borel measure on $\overline{B}$ such that*

$$\mu(S(\zeta,\delta)) \le C\delta^n \quad (\zeta \in S, \delta > 0), \tag{2.6}$$

for some constant $C > 0$. Then there exists a constant $K > 0$ such that

$$\int_{\overline{B}} \{\log(1+|f^*|)\}^p d\mu \le K\|f\|^p_{N^p(B)}, \tag{2.7}$$

for all $f \in N^p(B)$. Here the notation f^* denotes the function defined on $\overline{B}$ by $f^* = f$ in B and $f^* = \lim_{r\uparrow 1} f_r$ a.e. $[\sigma]$ on S.

Proof. (cf. [5], p.239) Put $\mu_1 = \mu\,|_B$ and $\mu_2 = \mu\,|_S$. By (2.6) we have

$$\mu_1(B(\zeta,\delta)) \le C\delta^n, \tag{2.8}$$

$$\mu_2(S(\zeta,\delta)) \le C\delta^n. \tag{2.9}$$

On the other hand, it follows from Proposition 5.1.4 in [9] that $\sigma(S(\zeta,\delta)) \sim \delta^n$. That is, there exist positive constants A_1 and A_2 such that

$$A_1\delta^n \le \sigma(S(\zeta,\delta)) \le A_2\delta^n, \tag{2.10}$$

for all $\zeta \in S$ and $\delta > 0$. By (2.8), (2.10) and Lemma 2.1, there exists a constant $K' > 0$ such that

$$\int_B \{\log(1+|f|)\}^p d\mu_1 \le K' \|f\|^p_{N^p(B)}. \tag{2.11}$$

for all $f \in N^p(B)$. And it follows from (2.9), (2.10) and Lemma 2.2 that $d\mu_2 = g d\sigma$ for some $g \in L^\infty(S)$. Thus using (2.11), we have that for $f \in N^p(B)$

$$\begin{aligned}\int_{\overline{B}} \{\log(1+|f^*|)\}^p d\mu &= \int_B \{\log(1+|f|)\}^p d\mu_1 + \int_S \{\log(1+|f^*|)\}^p d\mu_2 \\ &\le K' \|f\|^p_{N^p(B)} + \|g\|_{L^\infty} \int_S \{\log(1+|f^*|)\}^p d\sigma.\end{aligned}$$

This proves (2.7). □

Remark. In Lemma 2.3, we see that the constant K of (2.7) can be chosen to be the product of C and a constant depending only on p and the dimension n.

Let φ_z ($z \in B$) be the biholomorphic involution of B described in [9], p.25. For $z \in B$ and $0 < r < 1$, we set $E(z,r) = \varphi_z(rB)$. According to [9], p.29, §2.2.7,

$$E(z,r) = \left\{ w \in B : \frac{|P_z w - c|^2}{(r\rho)^2} + \frac{|w - P_z w|^2}{r^2 \rho} < 1 \right\}, \tag{2.12}$$

where $P_z w = \frac{\langle w,z\rangle}{\langle z,z\rangle} z$, $c = \frac{(1-r^2)z}{1-(r|z|)^2}$ and $\rho = \frac{1-|z|^2}{1-(r|z|)^2}$.

LEMMA 2.4. *For any $z \in B$ and $0 < r < 1$, there exist $\zeta \in S$ and $\delta > 0$ with $E(z,r) \subset B(\zeta,\delta)$. Furthermore, $\delta \sim 1 - |z|^2$.*

Proof. Since we easily prove this lemma in the case $z = 0$, we consider only the case $z \ne 0$. Put $\zeta = \frac{1}{|z|} z \in S$. For $w \in E(z,r)$ we have

$$\begin{aligned}|1 - \langle w,\zeta\rangle| &= \left|1 - \frac{(1-r^2)|z|}{1-r^2|z|^2} + \frac{(1-r^2)|z|}{1-r^2|z|^2} - \langle w,\zeta\rangle\right| \\ &\le \frac{1-r^2|z|^2 - (1-r^2)|z|}{1-r^2|z|^2} + \left|\langle w,\zeta\rangle\zeta - \frac{(1-r^2)|z|\zeta}{1-r^2|z|^2}\right|. \qquad (2.13)\end{aligned}$$

By (2.12) we see that

$$|P_z w - c| < r\rho = \frac{(1-|z|^2)r}{1-r^2|z|^2},$$

$$P_z w = \langle w,\zeta\rangle\zeta, \quad c = \frac{(1-r^2)|z|\zeta}{1-r^2|z|^2}.$$

Hence, we obtain

$$\left|\langle w,\zeta\rangle\zeta - \frac{(1-r^2)|z|\zeta}{1-r^2|z|^2}\right| = |P_z w - c|$$

$$< \frac{(1-|z|^2)r}{1-r^2|z|^2}. \tag{2.14}$$

By (2.13) and (2.14) we have

$$\begin{aligned} |1-\langle w,\zeta\rangle| &< \frac{1-r^2|z|^2-(1-r^2)|z|}{1-r^2|z|^2} + \frac{(1-|z|^2)r}{1-r^2|z|^2} \\ &= \frac{(1+r)(1-|z|)}{1-r|z|}. \end{aligned}$$

That is, $w \in B(\zeta,\delta)$ where $\delta = \frac{(1+r)(1-|z|)}{1-r|z|}$.

Furthermore, we obtain

$$1-|z|^2 = \frac{(1+|z|)(1-r|z|)}{1+r}\delta.$$

This completes the proof. □

Lemma 2.5. *Let* $-1<\alpha<\infty$. *Suppose that* μ *is a positive Borel measure on* B *with*

$$\mu(B(\zeta,\delta)) \le C\nu_\alpha(B(\zeta,\delta)) \quad (\zeta\in S, \delta>0), \tag{2.15}$$

for some constant $C>0$. *Then there is a positive constant* K *such that*

$$\int_B g\,d\mu \le K\int_B g\,d\nu_\alpha, \tag{2.16}$$

for all nonnegative $\mathcal{M}$*-subharmonic function* g *on* B.

Proof. Let $z\in B$ and $\frac{1}{2}<r<1$ be fixed. By Lemma 2.4, there exist $\zeta\in S$ and $\delta>0$ such that

$$E(z,r)\subset B(\zeta,\delta), \quad \delta\sim 1-|z|^2. \tag{2.17}$$

Since $\nu_\alpha(B(\zeta,\delta))\sim\delta^{n+1+\alpha}$ (see [6], Lemma 6.10), by (2.15) and (2.17) we obtain

$$\mu(E(z,r)) \le C'(1-|z|^2)^{n+1+\alpha}, \tag{2.18}$$

for some constant $C'>0$. Now, let g be a nonnegative $\mathcal{M}$-subharmonic function on B. By [11], p.33, (4.3) and (4.4), for $z\in B$

$$g(z) \le 3^n \int_{E(z,\frac{1}{2})} g\,d\lambda, \tag{2.19}$$

where $d\lambda(z) = (1-|z|^2)^{-n-1}d\nu(z)$ $(z \in B)$. Hence (2.19) and Fubini's theorem give

$$\begin{aligned}\int_B g(z)d\mu(z) &\leq 3^n \int_B d\mu(z) \int_{E(z,\frac{1}{2})} g(w)d\lambda(w) \\ &= 3^n \int_B g(w)d\lambda(w) \int_B \chi_{E(z,\frac{1}{2})}(w)d\mu(z). \qquad (2.20)\end{aligned}$$

On the other hand, we can easily see that $\chi_{E(z,\frac{1}{2})}(w) \leq \chi_{E(w,r)}(z)$. Thus using (2.18) and (2.20), we have

$$\begin{aligned}\int_B g(z)d\mu(z) &\leq 3^n \int_B g(w)d\lambda(w) \int_B \chi_{E(w,\frac{1}{2})}(z)d\mu(z) \\ &= 3^n \int_B g(w)\mu(E(w,r))d\lambda(w) \\ &\leq 3^n C' \int_B g(w)(1-|w|^2)^{n+1+\alpha}\frac{d\nu(w)}{(1-|w|^2)^{n+1}} \\ &= 3^n C' \int_B g(w)(1-|w|^2)^{\alpha}d\nu(w).\end{aligned}$$

This proves (2.16). □

Remark. By a careful computation, we see that the constant K of (2.16) is taken to be the product of C and a positive constant depending on α and the dimension n.

3. Composition operators on $N^p(B)$

In this section, let p be in $(1,\infty)$ and φ be a holomorphic self-map of B. Let φ^* denote the radial limit of the mapping φ considered as a map of $S \to \overline{B}$. We define a Borel measure μ on $\overline{B}$ by $\mu(E) = \sigma(\varphi^{*-1}(E))$ for all Borel sets E of $\overline{B}$.

THEOREM 3.1. *C_φ is metrically bounded on $N^p(B)$ if and only if there exists a positive constant C such that*

$$\mu(\mathcal{S}(\zeta,\delta)) \leq C\delta^n,$$

for all $\zeta \in S$ and $\delta > 0$.

Proof. Suppose that for some constant $C > 0$ we have $\mu(\mathcal{S}(\zeta,\delta)) \leq C\delta^n$. By Lemma 2.3, there exists a constant $K > 0$ such that

$$\int_B \{\log(1+|f^*|)\}^p d\mu \leq K\|f\|^p_{N^p(B)}, \qquad (3.1)$$

for all $f \in N^p(B)$. Let f be in $N^p(B)$. Since the ball algebra $A(B)$ is dense in $N^p(B)$, then there exists a sequence $\{f_j\}$ in $A(B)$ such that $\lim_{j\to\infty} \|f_j - f\|_{N^p(B)} = 0$. Noting that $f_j \in A(B)$ implies that $C_\varphi f_j = f_j \circ \varphi \in N^p(B)$ and $(f_j \circ \varphi)^*(\zeta) = f_j(\varphi^*(\zeta))$ a.e. $[\sigma]$ $\zeta \in S$, by (3.1) we have

$$\begin{aligned}
\|f_j \circ \varphi\|^p_{N^p(B)} &= \int_S \{\log(1 + |(f_j \circ \varphi)^*|)\}^p d\sigma \\
&= \int_S \{\log(1 + |(f_j \circ \varphi^*)|)\}^p d\sigma \\
&= \int_{\overline{B}} \{\log(1 + |f_j|)\}^p d\mu \le K\|f_j\|^p_{N^p(B)}, \qquad (3.2)
\end{aligned}$$

for all $j \in \mathbb{N}$. Since $\lim_{j\to\infty} \|f_j - f\|_{N^p(B)} = 0$, it follows from (3.2) that $\{f_j \circ \varphi\}$ is a Cauchy sequence in $N^p(B)$. The completeness of $N^p(B)$ gives $f \circ \varphi \in N^p(B)$ and $\|f \circ \varphi\|^p_{N^p(B)} \le K\|f\|^p_{N^p(B)}$. This proves that C_φ is metrically bounded on $N^p(B)$.

Conversely, suppose that C_φ is metrically bounded on $N^p(B)$. Then there exists a constant $K > 0$ such that

$$\|C_\varphi f\|_{N^p(B)} \le K\|f\|_{N^p(B)}, \qquad (3.3)$$

for all $f \in N^p(B)$. For $\zeta \in S$ and $0 < \delta < 1$, put $w = (1 - \delta)\zeta \in B$. And define

$$f_w(z) = \exp\left\{\frac{1 - |w|^2}{(1 - \langle z, w\rangle)^2}\right\}^{\frac{n}{p}},$$

for $z \in B$. Using the elementary inequality $\log(1 + x) \le \log 2 + \log^+ x$ $(x \ge 0)$, we can easily see that $f_w \in A(B)$ and $\|f_w\|^p_{N^p(B)} \le 2^{p-1}\{(\log 2)^p + 1\}$. Hence by using the fact that $(f_w \circ \varphi)^* = f_w \circ \varphi^*$ a.e. $[\sigma]$ on S and (3.3), we obtain

$$\begin{aligned}
K^p 2^{p-1}\{(\log 2)^p + 1\} &\ge \int_S \{\log(1 + |f_w \circ \varphi^*|)\}^p d\sigma \\
&= \int_{\overline{B}} \{\log(1 + |f_w|)\}^p d\mu \\
&\ge \int_{\overline{B}} \{\log^+ |f_w|\}^p d\mu \\
&= \int_{\overline{B}} \left\{\log^+ \exp\left[\mathrm{Re}\left\{\frac{1 - |w|^2}{(1 - \langle z, w\rangle)^2}\right\}^{\frac{n}{p}}\right]\right\}^p d\mu(z) \\
&= \int_{\overline{B}} \left\{\left[\mathrm{Re}^+\left\{\frac{1 - |w|^2}{(1 - \langle z, w\rangle)^2}\right\}^{\frac{n}{p}}\right]\right\}^p d\mu(z), \qquad (3.4)
\end{aligned}$$

where $\mathrm{Re}^+(v) = \max\{\mathrm{Re}(v), 0\}$ for $v \in \mathbb{C}$.

On the other hand, by using the continuity of the function $F(v) = \mathrm{Re}(1+v)^{-\frac{2n}{p}}$ ($v \in \mathbb{C}$) at the origin in $\mathbb{C}$, we can choose $t_0 > 0$ such that

$$\mathrm{Re}\left\{1 + \frac{|w|(1-\langle z,\zeta\rangle)}{1-|w|}\right\}^{-\frac{2n}{p}} > \frac{1}{2}, \tag{3.5}$$

for any $z \in \mathcal{S}(\zeta, t_0\delta)$. By (3.5), for $z \in \mathcal{S}(\zeta, t_0\delta)$

$$\begin{aligned}
\mathrm{Re}\left\{\frac{1-|w|^2}{(1-\langle z,w\rangle)^2}\right\}^{\frac{n}{p}} &= \left\{\frac{1-|w|^2}{(1-|w|)^2}\right\}^{\frac{n}{p}} \times \mathrm{Re}\left\{\frac{1-|w|}{(1-\langle z,w\rangle)}\right\}^{\frac{2n}{p}} \\
&= \left(\frac{1+|w|}{1-|w|}\right)^{\frac{n}{p}} \times \mathrm{Re}\left\{1 + \frac{|w|(1-\langle z,\zeta\rangle)}{1-|w|}\right\}^{-\frac{2n}{p}} \\
&> \left(\frac{2-h}{h}\right)^{\frac{n}{p}} \times \frac{1}{2} > \frac{1}{2\delta^{\frac{n}{p}}}. \qquad (3.6)
\end{aligned}$$

By (3.4) and (3.6), we have

$$\begin{aligned}
K^p 2^{p-1}\{(\log 2)^p + 1\} &\geq \int_B \left\{\left[\mathrm{Re}^+\left\{\frac{1-|w|^2}{(1-\langle z,w\rangle)^2}\right\}^{\frac{n}{p}}\right]\right\}^p d\mu(z) \\
&\geq \int_{\mathcal{S}(\zeta,t_0\delta)} \left\{\left[\mathrm{Re}^+\left\{\frac{1-|w|^2}{(1-\langle z,w\rangle)^2}\right\}^{\frac{n}{p}}\right]\right\}^p d\mu(z) \\
&\geq \int_{\mathcal{S}(\zeta,t_0\delta)} \frac{1}{2^p\delta^n} d\mu(z) = \frac{1}{2^p\delta^n}\mu(\mathcal{S}(\zeta, t_0\delta)).
\end{aligned}$$

That is,

$$\mu(\mathcal{S}(\zeta, t_0\delta)) \leq K^p 2^{p-1}\{(\log 2)^p + 1\} 2^p \delta^n,$$

for any $\zeta \in S$ and $0 < \delta < 1$. This proves the desired conclusion. □

By an application Theorem 3.1 and Lemma 2.2, we obtain the following corollary.

COROLLARY 3.1. *If C_φ is metrically bounded on $N^p(B)$, then φ^* cannot carry a set of positive σ-measure in S into a set of σ-measure 0 in S.*

Proof. We can prove this corollary in the same way that is used to prove the corresponding one for the Hardy space $H^p(B)$. See [5], Corollary 1.4. □

The next lemma is also an analogous result for corresponding one for $H^p(B)$, due to B. D. MacCluer.

LEMMA 3.1. *Suppose that C_φ is metrically bounded on $N^p(B)$ and let $f \in N^p(B)$. Then $(f \circ \varphi)^* = f^* \circ \varphi^*$ a.e. $[\sigma]$ on S. Here the notation f^* is used in the same sense of Lemma 2.3.*

Proof. By Corollary 3.1 and the same argument as the proof of Lemma 1.6 in [5], we can prove this lemma. □

The proof of Theorem 3.2 relies on the following characterization of the metrically compactness of C_φ on $N^p(B)$ expressed in terms of sequential convergence. By using some growth estimate satisfied by functions in $N^p(B)$ (cf. [12] §3), we can prove the next lemma in the same way that is used to prove the corresponding one for the spaces $H^p(B)$ (See [4], Proposition 1.10). We omit the detail.

LEMMA 3.2. *C_φ is metrically compact on $N^p(B)$ if and only if for every sequence $\{f_j\}$ in $N^p(B)$ which satisfies $\sup_{j\in\mathbb{N}} \|f_j\|_{N^p(B)} < \infty$ and converges to 0 uniformly on compact subsets of B, we have $f_j \circ \varphi$ converges to 0 in $N^p(B)$.*

THEOREM 3.2. *C_φ is metrically compact on $N^p(B)$ if and only if μ satisfies*

$$\lim_{\delta\downarrow 0} \sup_{\zeta\in S} \frac{\mu(\mathcal{S}(\zeta,\delta))}{\delta^n} = 0. \tag{3.7}$$

Proof. Suppose that C_φ is metrically compact on $N^p(B)$. Assume, to reach a contradiction, that $\mu(\mathcal{S}(\zeta,\delta)) \neq o(\delta^n)$ as $\delta \downarrow 0$ uniformly in $\zeta \in S$. Then there are $\{\zeta_j\} \subset S$, $\{\delta_j\} \in (0,1)$ with $\delta_j \downarrow 0$ $(j \to \infty)$ and $\varepsilon_0 > 0$ such that

$$\mu(\mathcal{S}(\zeta_j,\delta_j)) \geq \varepsilon_0 \delta_j^n, \tag{3.8}$$

for each $j \in \mathbb{N}$. Put $a_j = (1-\delta_j)\zeta_j$ and define

$$f_j(z) = (1-|a_j|)\exp\left\{\frac{1-|a_j|^2}{(1-\langle z,a_j\rangle)^2}\right\}^{\frac{n}{p}},$$

for $j \in \mathbb{N}$ and $z \in B$. As in the proof of Theorem 3.1, we see that $f_j \in A(B)$ and $\|f_j\|^p_{N^p(B)} \leq 2^{p-1}\{(\log 2)^p + 1\}$. Moreover, we can easily see that f_j converges to 0 uniformly on compact subsets of B. Hence, it follows from Lemma 3.2 that $f_j \circ \varphi$ converges to 0 in $N^p(B)$.

On the other hand, the same argument in the second part of the proof of Theorem 3.1 shows that there exists a $t_0 \in (0,1)$ such that for $z \in \mathcal{S}(\zeta_j, t_0\delta_j)$ and $j \in \mathbb{N}$

$$\exp\left[\operatorname{Re}\left\{\frac{1-|a_j|^2}{(1-\langle z,a_j\rangle)^2}\right\}^{\frac{n}{p}}\right] > \exp(2\delta_j^{\frac{n}{p}})^{-1}. \tag{3.9}$$

Thus using (3.9), we have

$$\log^+|f_j(z)| = \log^+\left[(1-|a_j|)\exp\operatorname{Re}\left\{\frac{1-|a_j|^2}{(1-\langle z,a_j\rangle)^2}\right\}^{\frac{n}{p}}\right]$$

$$\geq \log^+\left\{\delta_j \exp(2\delta_j^{\frac{n}{p}})^{-1}\right\}, \tag{3.10}$$

for $z \in \mathcal{S}(\zeta_j, t_0\delta_j)$ and $j \in \mathbb{N}$. Using Fatou's lemma and (3.10), we obtain that for each $j \in \mathbb{N}$

$$\left[\log^+\left\{\delta_j \exp(2\delta_j^{\frac{n}{p}})^{-1}\right\}\right]^p \mu(\mathcal{S}(\zeta_j, t_0\delta_j)) \tag{3.11}$$

$$\leq \int_{\mathcal{S}(\zeta_j,t_0\delta_j)} \{\log^+|f_j|\}^p d\mu$$

$$\leq \int_{\overline{B}} \{\log(1+|f_j|)\}^p d\mu$$

$$= \int_S \{\log(1+|(f_j\circ\varphi)^*|)\}^p d\sigma$$

$$\leq \liminf_{r\uparrow 1} \int_S \{\log(1+|(f_j\circ\varphi)_r|)\}^p d\sigma$$

$$= \|f_j\circ\varphi\|^p_{N^p(B)}. \tag{3.12}$$

Since $\lim_{j\to\infty}\|f_j\circ\varphi\|_{N^p(B)} = 0$ and

$$\lim_{j\to\infty} \delta_j^n \left[\log^+\left\{\delta_j \exp(2\delta_j^{\frac{n}{p}})^{-1}\right\}\right]^p = \frac{1}{2^p},$$

it follows from (3.11) that

$$\lim_{j\to\infty} \frac{\mu(\mathcal{S}(\zeta_j, t_0\delta_j))}{\delta_j^n} = 0.$$

Using this, we have

$$\lim_{j\to\infty} \frac{\mu(\mathcal{S}(\zeta_j, \delta_j))}{\delta_j^n} = 0.$$

This contradicts (3.8). Therefore we obtain (3.7).

In order to prove the other direction, we assume that (3.7) holds. Fix $\varepsilon > 0$. By the same argument as the proof of Theorem 1.1 (ii) in [5], we see that there exists a positive constant C_1 depending only on the dimension n such that

$$\widetilde{\mu}(\mathcal{S}(\zeta,\delta)) \leq C_1\varepsilon\delta^n \quad (\zeta \in S, \delta > 0), \tag{3.13}$$

where $\widetilde{\mu} = \mu\,|_{\overline{B}\setminus(1-\delta_0)\overline{B}}$ for some $0 < \delta_0 < 1$.

Now, suppose $\{f_j\}$ is a sequence in $N^p(B)$ which satisfies $\|f_j\|^p_{N^p(B)} \leq M$ ($j \in \mathbb{N}$) and converges to 0 uniformly on compact subsets of B. By (3.7) and Theorem 3.1, we see that C_φ is metrically bounded on $N^p(B)$. Hence, it follows from Lemma 3.1 that

$$(f_j \circ \varphi)^* = f_j^* \circ \varphi^* \quad \text{a.e. } [\sigma] \text{ on } S,$$

for each $j \in \mathbb{N}$. We obtain that for $j \in \mathbb{N}$

$$\begin{aligned}
\|f_j \circ \varphi\|^p_{N^p(B)} &= \int_S \{\log(1 + |(f_j \circ \varphi)^*|)\}^p d\sigma \\
&= \int_S \{\log(1 + |f_j^* \circ \varphi^*|)\}^p d\sigma \\
&= \int_{\overline{B}} \{\log(1 + |f_j^*|)\}^p d\mu \\
&= \int_{\overline{B}} \{\log(1 + |f_j^*|)\}^p d\widetilde{\mu} + \int_{(1-\delta_0)\overline{B}} \{\log(1 + |f_j|)\}^p d\mu. \qquad (3.14)
\end{aligned}$$

By (3.12) and Lemma 2.3, there exists a constant $C_2 > 0$ depending only on p and the dimension n such that

$$\begin{aligned}
\int_{\overline{B}} \{\log(1 + |f_j^*|)\}^p d\widetilde{\mu} &\leq C_2 C_1 \varepsilon \|f_j\|^p_{N^p(B)} \\
&\leq M C_2 C_1 \varepsilon, \qquad (3.15)
\end{aligned}$$

for each $j \in \mathbb{N}$. Since f_j converges to 0 uniformly on compact subsets on B, we have

$$\lim_{j \to \infty} \int_{(1-\delta_0)\overline{B}} \{\log(1 + |f_j|)\}^p d\mu = 0. \qquad (3.16)$$

By (3.13), (3.14) and (3.15), we see that $\lim_{j \to \infty} \|f_j \circ \varphi\|_{N^p(B)} = 0$. Hence, it follows from Lemma 3.2 that C_φ is metrically compact on $N^p(B)$. □

As a corollary of Theorem 3.1 and 3.2, we obtain the following corollary. This results is the higher dimensional cases of the result by J. S. Choa and H. O. Kim [1].

COROLLARY 3.2. (a) *C_φ is bounded on $H^2(B)$ if and only if C_φ is metrically bounded on $N^p(B)$.*

(b) *C_φ is compact on $H^2(B)$ if and only if C_φ is metrically compact on $N^p(B)$.*

4. Composition operators on $(AN)^p(\nu_\alpha)$

In this section, let $1 \le p < \infty$ and $-1 < \alpha < \infty$. Let φ be a holomorphic self-map of B. We define a Borel measure $\mu(E) = \nu_\alpha(\varphi^{-1}(E))$ for all Borel sets E of B.

THEOREM 4.1. *C_φ is metrically bounded on $(AN)^p(\nu_\alpha)$ if and only if there exists a positive constant C such that*

$$\mu(B(\zeta,\delta)) \le C\delta^{n+1+\alpha}, \tag{4.1}$$

for all $\zeta \in S$ and $\delta > 0$.

Proof. We first prove that (4.1) is a sufficient condition for C_φ is metrically bounded on $(AN)^p(\nu_\alpha)$. By (4.1) and the fact $\nu_\alpha(B(\zeta,\delta)) \sim \delta^{n+1+\alpha}$, there exists a constant $C' > 0$ such that

$$\mu(B(\zeta,\delta)) \le C'\nu_\alpha(B(\zeta,\delta)), \tag{4.2}$$

for all $\zeta \in S$ and $\delta > 0$.

Fix $f \in (AN)^p(\nu_\alpha)$. Since $\{\log(1+|f|)\}^p$ is a nonnegative $\mathcal{M}$-subharmonic function on B, it follows from (4.2) and Lemma 2.5 that

$$\int_B \{\log(1+|f|)\}^p d\mu \le K\int_B \{\log(1+|f|)\}^p d\nu_\alpha, \tag{4.3}$$

for some constant $K > 0$ depending on C', α and the dimension n. (4.3) proves that C_φ is metrically bounded on $(AN)^p(\nu_\alpha)$.

For the other direction, we pick $\zeta \in S$ and $0 < \delta < 1$. Take the function

$$f_w(z) = \exp\left\{\frac{1-|w|^2}{(1-\langle z,w\rangle)^2}\right\}^{\frac{n+1+\alpha}{p}} \quad (z \in B),$$

where $w = (1-\delta)\zeta \in B$. We can easily see that $f_w \in A(B) \subset (AN)^p(\nu_\alpha)$. Moreover, by Proposition 1.4.10 in [9] we see that there exists a positive constant C such that

$$\begin{aligned}&\int_B \frac{(1-|w|^2)^{n+1+\alpha}(1-|z|^2)^\alpha}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}d\nu(z)\\ &\le C(1-|w|^2)^{n+1+\alpha}(1-|w|^2)^{-(n+1+\alpha)} = C.\end{aligned}$$

Hence, we obtain that

$$
\begin{aligned}
(4.4)\qquad & \|f_w\|^p_{(AN)^p(\nu_\alpha)} \\
&= \int_B \{\log(1+|f_w|)\}^p d\nu_\alpha \\
&\le 2^{p-1}\left[\{\log 2\}^p + \int_B \{\log^+|f_w|\}^p d\nu_\alpha\right] \\
&\le 2^{p-1}\{\log 2\}^p + 2^{p-1}c_\alpha \int_B \frac{(1-|w|^2)^{n+1+\alpha}(1-|z|^2)^\alpha}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} d\nu(z) \\
(4.5)\qquad &\le 2^{p-1}[\{\log 2\}^p + c_\alpha C].
\end{aligned}
$$

By (4.4) and the completely analogous argument that is used to prove Theorem 3.1, we can also prove that (4.1) is a necessary condition for C_φ is metrically bounded on $(AN)^p(\nu_\alpha)$. We omit the detail. □

For the metrically compactness of C_φ on $(AN)^p(\nu_\alpha)$, an analogous result of Lemma 3.2 holds.

LEMMA 4.1. *C_φ is metrically compact on $(AN)^p(\nu_\alpha)$ if and only if for every sequence $\{f_j\}$ in $(AN)^p(\nu_\alpha)$ which satisfies $\sup_{j\in\mathbb{N}}\|f_j\|_{(AN)^p(\nu_\alpha)} < \infty$ and converges to 0 uniformly on compact subsets of B, we have $f_j\circ\varphi$ converges to 0 in $(AN)^p(\nu_\alpha)$.*

Proof. By using the growth estimate satisfied by functions in $(AN)^p(\nu_\alpha)$ ([7], Lemma 1) and modifying the proof of Proposition 2.3 in [13], this lemma can be proven. □

THEOREM 4.2. *C_φ is metrically compact on $(AN)^p(\nu_\alpha)$ if and only if μ satisfies*

$$
\lim_{\delta\downarrow 0}\sup_{\zeta\in S}\frac{\mu(B(\zeta,\delta))}{\delta^{n+1+\alpha}} = 0. \tag{4.6}
$$

Proof. The proof is essentially the same as one of Theorem 3.2. For sufficiency, we choose test functions

$$
f_j(z) = (1-|a_j|)\exp\left\{\frac{1-|a_j|^2}{(1-\langle z,a_j\rangle)^2}\right\}^{\frac{n+1+\alpha}{p}} \qquad (z\in B,\ j\in\mathbb{N}),
$$

where we choose a_j in the same way that is used to prove Theorem 3.2. As in the proof of Theorem 4.1, we see that $f_j \in A(B)$ and $\|f_j\|^p_{(AN)^p(\nu_\alpha)} \le 2^{p-1}\{(\log 2)^p + c_\alpha C\}$ for some constant $C > 0$. By Lemma 4.1 and the same argument that is used to prove Theorem 3.2, we see that (4.5) is a sufficient condition for the metrically compactness of C_φ on $(AN)^p(\nu_\alpha)$.

Now, we prove that (4.5) is a necessary condition for the metrically compactness of C_φ on $(AN)^p(\nu_\alpha)$. We set $D(\zeta,\delta) = \{z \in B : 1-\delta < |z|,\ z/|z| \in S(\zeta,\delta)\}$. We easily see that

$$B(\zeta,\delta/2) \subset D(\zeta,\delta) \subset B(\zeta,2\delta), \tag{4.7}$$

for $\zeta \in S$ and $\delta > 0$. By (4.5) and (4.6), we have $\mu(D(\zeta,\delta))/\delta^{n+1+\alpha} \to 0$ as $\delta \downarrow 0$ uniformly in $\zeta \in S$. Hence, as in the proof of Theorem 1.1 (ii) in [5], we can prove that there exists a constant $C_1 > 0$ depending only on the dimension n such that

$$\widetilde{\mu}(B(\zeta,\delta)) \leq C_1 \varepsilon \delta^{n+1+\alpha} \quad (\zeta \in S,\ \delta > 0), \tag{4.8}$$

where $\varepsilon > 0$ be fixed and $\widetilde{\mu} = \mu\,|_{B\setminus(1-\delta_0)\overline{B}}$ for some $0 < \delta_0 < 1$.

Suppose that $\{f_j\}$ is a sequence in $(AN)^p(\nu_\alpha)$ which satisfies

$$\|f_j\|^p_{(AN)^p(\nu_\alpha)} \leq M\ (j \in \mathbb{N})$$

and converges to 0 uniformly on compact subsets of B. For each $j \in \mathbb{N}$ we have

$$\begin{aligned} \|f_j \circ \varphi\|^p_{(AN)^p(\nu_\alpha)} &= \int_B \{\log(1+|f_j\circ\varphi|)\}^p d\nu_\alpha \\ &= \int_B \{\log(1+|f_j|)\}^p d\mu \\ &= \int_B \{\log(1+|f_j|)\}^p d\widetilde{\mu} + \int_{(1-\delta_0)\overline{B}} \{\log(1+|f_j|)\}^p d\mu. \end{aligned} \tag{4.9}$$

By (4.7) and Lemma 2.5, there exists a positive constant C_2 depending only on α and n such that

$$\int_B \{\log(1+|f_j|)\}^p d\widetilde{\mu} \leq C_2 C_1 \varepsilon \|f_j\|^p_{(AN)^p(\nu_\alpha)}, \tag{4.10}$$

for each $j \in \mathbb{N}$. Moreover, it follows from f_j converges to uniformly 0 on $(1-\delta_0)\overline{B}$ that

$$\lim_{j\to\infty} \int_{(1-\delta_0)\overline{B}} \{\log(1+|f_j|)\}^p d\mu = 0. \tag{4.11}$$

By (4.8), (4.9) and (4.10), we obtain that $f_j \circ \varphi$ converges to 0 in $(AN)^p(\nu_\alpha)$. By Lemma 4.1, we see that C_φ is metrically compact on $(AN)^p(\nu_\alpha)$. □

As a corollary of Theorem 4.1 and 4.2, we obtain the following results.

COROLLARY 4.1. (a) *C_φ is bounded on $A^2(\nu_\alpha)$ if and only if C_φ is metrically bounded on $(AN)^p(\nu_\alpha)$.*

(b) *C_φ is compact on $A^2(\nu_\alpha)$ if and only if C_φ is metrically compact on $(AN)^p(\nu_\alpha)$.*

References

[1] J. S. Choa and H. O. Kim, *Composition operators on some F-algebras of holomorphic functions*, Nihonkai Math. J. **7** (1996), 29–39.

[2] ———, *Composition operators between Nevanlinna-type spaces*, J. Math. Anal. Appl. **257** (2001), 378–402.

[3] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, 1994.

[4] B. D. MacCluer, *Spectra of compact composition operators on $H^p(B_N)$*, Analysis **4** (1984), 87–103.

[5] ———, *Compact composition operators on $H^p(B_N)$*, Michigan Math. J. **32** (1985), 237–248.

[6] B. D. MacCluer and J. H. Shapiro, *Angular derivatives and compact composition operators on the Hardy and Bergman spaces*, Canad. J. Math. **38** (1986), 878–906.

[7] Y. Matsugu and S. Ueki, *Isometries of weighted Bergman-Privalov spaces on the unit ball of $\mathbb{C}^n$*, J. Math. Soc. Japan **54** (2002), 341–347.

[8] S. C. Power, *Hörmander's Carleson theorem for the ball*, Glasg. Math. J. **26** (1985), 13–17.

[9] W. Rudin, *Function Theory on the Unit Ball of $\mathbb{C}^n$*, Springer-Verlag, Berlin, New York, Heiderberg, 1980.

[10] M. Stoll, *Mean growth and Taylor coefficients of some topological algebras of analytic functions*, Ann. Polon. Math. **35** (1977), 139–158.

[11] ———, Invariant Potential Theory in the Unit Ball of $\mathbb{C}^n$, Cambridge Univ. Press, 1994.

[12] A. V. Subbotin, *Functional properties of Privalov spaces of holomorphic functions in several variables*, Math. Notes **65** (1999), 230–237.

[13] J. Xiao, *Compact composition operators on the Area-Nevanlinna class*, Expo. Math. **17** (1999), 255–264.

Graduate School of Science and Technology
Shinshu University
Matsumoto 390-8621
Japan
E-mail: sueki@math.shinshu-u.ac.jp