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NECESSARY CONDITIONS FOR OPTIMAL CONTROL PROBLEM UNDER STATE CONSTRAINTS

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ABSTRACT. Necessary conditions for a deterministic optimal control problem which involves states constraints are derived in the form of a maximum principle. The conditions are similar to those of F.H. Clarke, R.B. Vinter and G. Pappas who assume that the problem's data are Lipschitz. On the other hand, our data are not continuously differentiable but only differentiable. Fermat's rule and Rockafellar's duality theory of convex analysis are the basic techniques in this paper.

1. Introduction

In [5] and [11], Clarke, Vinter and Pappas derive the necessary conditions associated with the optimal control problem which involves the following state constrints

$$g(t, x(t)) \leq 0 \ \ \forall t \in [t_0, T]$$

when the function $g(t, \cdot)$ is Lipschtz. (See also [8] for the problem which represents state constraints in the inclusion form $x(t) \in X(t)$). In their approach, they reduce the optimal control problem to the Bolza problem of calculus of variations and use Ekeland's variational principle and a limiting process. Clarke's nonsmooth analysis (see [5]) is the main tool in which the derivatives are replaced by generalized gradient or generalized Jacobian. As a result, the adjoint equation is represented by the form of inclusions.

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We consider the following Mayer problem with free end points

$$\begin{array}{ll} \text{minimize} & \psi(x(T)) \\ \text{subject to} & x'(t) = f(t, x(t), u(t)) \text{ a.e. } t \in [t_0, T] \\ & u(t) \in U(t) \text{ a.e. } t \in [t_0, T] \\ & g(t, x(t)) \leq 0 \ \forall t \in [t_0, T]. \end{array}$$

In contrast with the above authors, we derive the necessary conditions with assumption that $g(t, \cdot)$ and $f(t, \cdot, u)$ are only differentiable not continuously differentiable. Under this assumption, our results can not be obtained directly from those of the above authors. In other words, Clarke's generalized gradient (respectively Jacobian) is not reduced directly to the classical gradient (respectively Jacobian). We will obtain the adjoint equation instead of the adjoint inclusion which can augment the number of controls verifying the maximum principle.

The underlying idea is to take a direct approach (without reduce the original optimal control problem to the Bolza problem of calculus of variations) by using Fermat's rule: at the point of minimum the derivative of the function is nonnegative in the directions which are tangent to constraints.

2. Statement of problem and preliminaries

Consider a complete separable metric space $\mathcal{Z}$, two real numbers $t_0 < T$ and the following functions:

$$\begin{array}{l} \psi : \mathbf{R}^n \to \mathbf{R}, \\ f : [t_0, T] \times \mathbf{R}^n \times \mathcal{Z} \to \mathbf{R}^n, \\ g : [t_0, T] \times \mathbf{R}^n \to \mathbf{R}. \end{array}$$

Let $U : [t_0, T] \rightsquigarrow \mathcal{Z}$ be a set-valued map. We associate to the above data the following control system:

$$\text{(1)} \qquad \begin{cases} x'(t) = f(t, x(t), u(t)), \ u(t) \in U(t) \ \text{ a.e. in } [t_0, T] \\ g(t, x(t)) \leq 0 \ \ \forall t \in [t_0, T]. \end{cases}$$

We say that a function $x : [t_0, T] \to \mathbf{R}^n$ is a solution of (1) if x is absolutely continuous(AC) and verifies (1).

We study here the following optimal control problem:

$$\min \left\{ \psi(x(T)) \mid x \text{ is a solution of } (1) \right\}.$$

Let us introduce the following notations:

$$S^g_{[\tau,t]}(y) = \left\{ x \in AC(\tau,t;\mathbf{R}^n) \;\middle|\; \begin{array}{l} x \text{ verifies (1) in } [\tau,t], \\ x(\tau) = y \end{array} \right\},$$

$R^g(t) = \{x(t) \mid x \in S^g_{[t_0,t]}(x_0) \text{ for some } x_0 \text{ such that } g(t_0,x_0) \leq 0\}$.

Throughout the whole paper we suppose that

i) g is differentiable

ii) ψ is differentiable

iii) $f(t,\cdot,u)$ is differentiable

iv) $f(\cdot,x,u)$ is measurable

v) $f(t,x,\cdot)$ is continuous

vi) there exists $k \in L^\infty(t_0,T;\mathbf{R}_+)$ such that for all $t \in [t_0,T]$, for all $(x,y) \in \mathbf{R}^n \times \mathbf{R}^n$,

$$\|f(t,x,u) - f(t,y,u)\| \leq k(t)\|x-y\|$$

vii) there exists $m \in L^2(t_0,T;\mathbf{R}_+)$ such that for all $t \in [t_0,T]$,

$$\sup_{v \in f(t,x,U(t))} \|v\| \leq m(t)(1+\|x\|)$$

viii) $U(\cdot)$ is measurable

ix) for all $t \in [t_0,T]$, $U(t)$ is nonempty and compact

x) for all (t,x), $f(t,x,U(t))$ is convex.

We first recall the definition of polar cone. Let K be a subset of a Banach space X. The positive polar cone of K is defined by

$$K^+ = \{p \in X^* \mid \forall u \in K, \;\; \langle p,u\rangle \geq 0\}$$

where X^* is the dual space of X. The negative polar cone of K is defined by

$$K^- = \{p \in X^* \mid \forall u \in K, \;\; \langle p,u\rangle \leq 0\}.$$

The contingent cone $T_K(x)$ to K at x is defined by:

$$T_K(x) = \{v \in X \mid \liminf_{h\to 0^+} \frac{\operatorname{dist}(x+hv,K)}{h} = 0\}.$$

In the following, we fix a trajectory-control pair $(\bar{x},\bar{u})$. Let us introduce the following sets.

$$A = \left\{ w \in W^{1,2}(t_0,T;\mathbf{R}^n) \;\middle|\; \begin{array}{l} w'(t) = \frac{\partial f}{\partial x}(t,\bar{x}(t),\bar{u}(t))w(t) + v(t), \\ v(t) \in T_{f(t,\bar{x}(t),U(t))}(\bar{x}'(t)) \;\text{ a.e.} \end{array} \right\},$$

$$\tilde{A} = \left\{ w \in A \;\middle|\; w \in W^{1,\infty}(t_0,T;\mathbf{R}^n) \right\},$$

$$S = \left\{ t \in [t_0,T] \;\middle|\; g(t,\bar{x}(t)) = 0 \right\},$$

$$S_i = \left\{ t \in [t_0, T] \;\middle|\; t \in S + \frac{1}{i}[-1,1] \right\},$$

$$D_i = \left\{ w \in C(t_0, T; \mathbf{R}^n) \;\middle|\; \langle \frac{\partial g}{\partial x}(t, \bar{x}(t)), w(t) \rangle \le 0 \quad \forall t \in S_i \right\},$$

$$\tilde{D} = \left\{ w \in C(t_0, T; \mathbf{R}^n) \;\middle|\; \langle \frac{\partial g}{\partial x}(t, \bar{x}(t)), w(t) \rangle \le 0 \quad \forall t \in S \right\},$$

where $W^{1,2}(t_0, T; \mathbf{R}^n)$ (respectively, $W^{1,\infty}(t_0, T; \mathbf{R}^n)$) is the space of functions $w \in L^2(t_0, T; \mathbf{R}^n)$ (respectively, $L^\infty(t_0, T; \mathbf{R}^n)$) such that $w' \in L^2(t_0, T; \mathbf{R}^n)$ (respectively, $L^\infty(t_0, T; \mathbf{R}^n)$).

PROPOSITION 2.1. *Suppose $\frac{\partial g}{\partial x}(\cdot, \bar{x}(\cdot))$ is continuous and there exists $\bar{w} \in \tilde{A}$ and $\rho > 0$ such that for all $t \in [t_0, T]$,*

$$\langle \frac{\partial g}{\partial x}(t, \bar{x}(t)), \bar{w}(t) \rangle \le -\rho.$$

Then

$$\gamma(\tilde{A} \cap D_i) \subset T_{R^g(T)}(\bar{x}(T)) \quad \forall i$$

where γ is a linear continuous function defined by:

$$\begin{array}{rcl} \gamma : C(t_0, T; \mathbf{R}^n) & \to & \mathbf{R}^n \\ w & \mapsto & w(T). \end{array}$$

Proof. Let $w \in \tilde{A} \cap D_i$ and $w_\lambda = \lambda w + (1-\lambda)\bar{w}$ where $\lambda \in]0,1[$. Because $\tilde{A}$ and D_i are convex and $\bar{w} \in \tilde{A} \cap D_i$, $w_\lambda \in \tilde{A} \cap D_i$. Fix λ. Set

$$F(t,x) = f(t, x, U(t))$$

and note that

$$\frac{\partial f}{\partial x}(t, \bar{x}(t), \bar{u}(t)) w_\lambda(t) \in D^b F(t, \bar{x}(t), \bar{x}'(t))(w_\lambda(t)) \text{ a.e.}$$

(for the definition of adjacent derivative $D^b F(t, \bar{x}(t), \bar{x}'(t))$, see [2]). Therefore we have

$$\begin{array}{rcl} w'_\lambda(t) & \in & D^b F(t, \bar{x}(t), \bar{x}'(t))(w_\lambda(t)) + T_{F(t,\bar{x}(t))}(\bar{x}'(t)) \\ & = & D^b F(t, \bar{x}(t), \bar{x}'(t))(w_\lambda(t)) + D^b F(t, \bar{x}(t), \bar{x}'(t))(0) \\ & = & D^b F(t, \bar{x}(t), \bar{x}'(t))(w_\lambda(t)) \end{array}$$

by Proposition 5.2.6 of [2].

In the proof of Theorem 10.5.1. of [2], we can see that there exists a constant $M \ge 0$ and the solutions x_h^λ of the system such that

$$\| x_h^{\lambda\prime} - \bar{x}' - h w'_\lambda \|_{L^1(t_0, T; \mathbf{R}^n)} \le M \cdot o(h),$$

and

$$\frac{x_h^\lambda - \bar{x}}{h} \longrightarrow w_\lambda \text{ uniformly.}$$

On the other hand, we have

$$\begin{aligned}
& \|x_h^\lambda(t) - \bar{x}(t)\| \\
\leq\ & \|x_h^\lambda(t) - \bar{x}(t) - hw_\lambda(t)\| + h\|w_\lambda(t)\| \\
\leq\ & \|x_h^\lambda(t_0) - \bar{x}(t_0) - hw_\lambda(t_0)\| + h\|w_\lambda(t_0)\| \\
& + \int_{t_0}^t \|x_h^{\lambda\prime}(s) - \bar{x}'(s) - hw'_\lambda(s)\| ds + h \int_{t_0}^t \|w'_\lambda(s)\| ds \\
(2) \qquad \leq\ & o(h) + h\|w_\lambda(t_0)\| + M \cdot o(h) + h\|w'_\lambda\|_{L^\infty(t_0,T;\mathbf{R}^n)}(T - t_0).
\end{aligned}$$

We want to prove that for all $h > 0$ small enough,

$$x_h^\lambda \in S^g_{[t_0,T]}(x_h^\lambda(t_0)).$$

1^{st} case: $t \in [t_0, T] \cap S_i$. In this case,

$$\langle \frac{\partial g}{\partial x}(t, \bar{x}(t)), w(t) \rangle \leq 0$$

because $w \in D_i$. Therefore we have

$$\begin{aligned}
& g(t, x_h^\lambda(t)) \\
=\ & g(t, \bar{x}(t)) + \Big\langle \frac{\partial g}{\partial x}(t, \bar{x}(t)), x_h^\lambda(t) - \bar{x}(t) \Big\rangle + o(\|x_h^\lambda(t) - \bar{x}(t)\|) \\
=\ & g(t, \bar{x}(t)) + \Big\langle \frac{\partial g}{\partial x}(t, \bar{x}(t)), x_h^\lambda(t) - \bar{x}(t) - hw_\lambda(t) \Big\rangle \\
& + h\Big\langle \frac{\partial g}{\partial x}(t, \bar{x}(t)), w_\lambda(t) \Big\rangle + o(\|x_h^\lambda(t) - \bar{x}(t)\|) \\
\leq\ & N(o(h) + M \cdot o(h)) - h\rho(1 - \lambda) + o(h) \\
=\ & h(N(\epsilon(h) + M \cdot \epsilon(h)) - \rho(1 - \lambda) + \epsilon(h))
\end{aligned}$$

where $\epsilon(h) \longrightarrow 0$ when h converges to 0 and

$$N = \sup_{t \in [t_0, T]} \|\frac{\partial g}{\partial x}(t, \bar{x}(t))\|.$$

2^{nd} case: $t \in [t_0, T] \backslash S_i$. Set

$$K_i = \sup_{t \in [t_0, T] \backslash S_i} g(t, \bar{x}(t)) < 0.$$

Then

$$\begin{aligned} & g(t,x_h^\lambda(t)) \\ = \;& g(t,\bar{x}(t)) + \left\langle \frac{\partial g}{\partial x}(t,\bar{x}(t)), x_h^\lambda(t)-\bar{x}(t) \right\rangle + o(\|x_h^\lambda(t)-\bar{x}(t)\|) \\ \leq \;& K_i + N(o(h) + M\cdot o(h) + h\|w_\lambda(t_0) + h\|w_\lambda'\|_{L^\infty}(T-t_0)) + o(h). \end{aligned}$$

From the above two cases, for all $t\in[t_0,T]$, for all $h>0$ sufficiently small,

$$g(t,x_h^\lambda(t)) \leq 0,$$

i.e.,

$$x_h^\lambda \in S^g_{[t_0,T]}(x_h^\lambda(t_0)).$$

This and the fact that $\frac{x_h^\lambda - \bar{x}}{h} \longrightarrow w_\lambda$ uniformly imply that

$$\begin{aligned} w_\lambda(T) &= \lambda w(T) + (1-\lambda)\bar{w}(T) \\ &\in T_{R^{g}(T)}(\bar{x}(T)). \end{aligned}$$

Therefore by taking limit when λ converges to 1, we have

$$w(T) \in T_{R^g(T)}(\bar{x}(T))$$

because $T_{R^g(T)}(\bar{x}(T))$ is closed. □

LEMMA 2.2. *$\tilde{A}$ is dense in A for the topology of uniform convergence.*

Proof. Let $w\in A$. If we prove that there exists a sequence $w_i\in\tilde{A}$ such that $w_i \longrightarrow w$, the proof ends. Let $v(s)\in T_{f(s,\bar{x}(s),U(s))}(\bar{x}'(s))$ be such that

$$w'(s) = \frac{\partial f}{\partial x}(s,\bar{x}(s),\bar{u}(s))w(s) + v(s).$$

Set

$$v_i(s) = \begin{cases} 0 & \text{if } \|v(s)\| > i, \\ v(s) & \text{otherwise.} \end{cases}$$

Then

$$v_i(s) \in T_{f(s,\bar{x}(s),U(s))}(\bar{x}'(s)).$$

Consider the solution w_i of the system

$$\begin{cases} w_i'(s) = \frac{\partial f}{\partial x}(s,\bar{x}(s),\bar{u}(s))w_i(s) + v_i(s), \quad s\in[t_0,T] \\ w_i(t_0) = w(t_0). \end{cases}$$

Note that

$$w_i \in \tilde{A}$$

and

$$\|v_i - v\|_{L^1} \longrightarrow 0.$$

By Gronwall's Lemma, we can show that w_i converges uniformly to w. □

LEMMA 2.3. *Set*

$$R(t) = \begin{cases} \left\{ x \in \mathbf{R}^n \;\middle|\; \frac{\partial g}{\partial x}(t, \bar{x}(t)) \cdot x \le \alpha \right\}, & t \in S \\ \mathbf{R}^n, & t \in [t_0, T] \backslash S \end{cases}$$

where $\alpha \in \mathbf{R}$ is fixed. If $\frac{\partial g}{\partial x}(\cdot, \bar{x}(\cdot))$ is continuous and $\frac{\partial g}{\partial x}(t, \bar{x}(t)) \neq 0$ for all $t \in S$, then the set-valued map R is lower semicontinuous with nonempty values.

Proof. Since

$$\alpha \frac{\frac{\partial g}{\partial x}(t, \bar{x}(t))}{\|\frac{\partial g}{\partial x}(t, \bar{x}(t))\|^2} \in R(t) \quad \forall t \in S,$$

$R(t) \neq \emptyset$ for all $t \in S$. Furthermore

$$R(t) = \mathbf{R}^n \quad \forall t \in [t_0, T] \backslash S.$$

Fix (τ, x) such that $x \in R(\tau)$. If we prove that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$(x + \epsilon B) \cap R(t) \neq \emptyset \quad \forall |\tau - t| < \delta$$

where B is open ball with center 0 and radius 1, then R is l.s.c. at τ. If $\tau \in [t_0, T] \backslash S$, there is nothing to prove. Fix $\tau \in S$ and $\epsilon > 0$. Set

$$M = \sup_{t \in S} \frac{1}{\|\frac{\partial g}{\partial x}(t, \bar{x}(t))\|}$$

and

$$x_\delta = x - \delta \frac{\frac{\partial g}{\partial x}(\tau, \bar{x}(\tau))}{\|\frac{\partial g}{\partial x}(\tau, \bar{x}(\tau))\|^2 M}.$$

Then

$$x_\delta \in x + \delta B.$$

Since $\frac{\partial g}{\partial x}(\cdot,\bar{x}(\cdot))$ is continuous, if $t\in S$ is sufficiently close to τ, then

$$\begin{aligned}\frac{\partial g}{\partial x}(t,\bar{x}(t))x_\delta &= \frac{\partial g}{\partial x}(\tau,\bar{x}(\tau))x_\delta + \Big(\frac{\partial g}{\partial x}(t,\bar{x}(t)) - \frac{\partial g}{\partial x}(\tau,\bar{x}(\tau))\Big)x_\delta \\ &= \frac{\partial g}{\partial x}(\tau,\bar{x}(\tau))\Big(x-\delta\frac{\frac{\partial g}{\partial x}(\tau,\bar{x}(\tau))}{\|\frac{\partial g}{\partial x}(\tau,\bar{x}(\tau))\|^2M}\Big) \\ &\quad +\Big(\frac{\partial g}{\partial x}(t,\bar{x}(t)) - \frac{\partial g}{\partial x}(\tau,\bar{x}(\tau))\Big)(x_\delta - x) \\ &\quad +\Big(\frac{\partial g}{\partial x}(t,\bar{x}(t)) - \frac{\partial g}{\partial x}(\tau,\bar{x}(\tau))\Big)x \\ &\le \alpha - \frac{\delta}{M} + \delta^2 + \delta^2 \\ &= \alpha + \delta(2\delta - \frac{1}{M}).\end{aligned}$$

Therefore if δ is sufficiently small, then

$$\frac{\partial g}{\partial x}(t,\bar{x}(t))x_\delta \le \alpha$$

i.e., $x_\delta \in R(t)$. □

LEMMA 2.4. *Suppose that* $\frac{\partial g}{\partial x}(\cdot,\bar{x}(\cdot))$ *is continuous and* $\frac{\partial g}{\partial x}(t,\bar{x}(t))\neq 0$ *for all* $t\in S$. *Then*

$$\tilde{D}^- = \left\{ \zeta\in C(t_0,T;\mathbf{R}^n)^* \;\middle|\; \begin{array}{l} \exists\theta\in C(t_0,T;\mathbf{R})^* \textit{ such that} \\ \theta \textit{ is positive, } \zeta \textit{ is absolutely} \\ \textit{continuous with respect to } \theta \textit{ and the Radon-} \\ \textit{Nykodym derivative } \frac{d\zeta}{d\theta}(t) \textit{ is contained} \\ \textit{in} \begin{cases} \mathbf{R}_+\frac{\partial g}{\partial x}(t,\bar{x}(t)) & \theta-a.e. \textit{ in } S \\ \{0\} & \theta-a.e. \textit{ in } [t_0,T]\backslash S \end{cases} \end{array} \right\}.$$

Proof. Set

$$Q(t) = \begin{cases} \{x \mid \langle\frac{\partial g}{\partial x}(t,\bar{x}(t)),x\rangle \le 0\} & \text{if } t\in S \\ \mathbf{R}^n & \text{if } t\notin S. \end{cases}$$

Q is l.s.c. by Lemma 2.3, $Q(t)$ is nonempty, closed and convex for all $t\in[t_0,T]$ and

$$\tilde{D} = \{w\in C(t_0,T;\mathbf{R}^n) \mid w(t)\in Q(t) \;\; \forall t\in[t_0,T]\}.$$

Then

$$\tilde{D}^- = \{\zeta\in C(t_0,T;\mathbf{R}^n)^* \mid \langle\zeta,w\rangle\le 0 \;\; \forall w\in\tilde{D}\}$$

and

$$Q(t)^- = \begin{cases} \mathbf{R}_+\frac{\partial g}{\partial x}(t,\bar{x}(t)) & \text{si } t\in S \\ \{0\} & \text{si } t\notin S. \end{cases}$$

By [10, Corollary 6A], we have the result. □

LEMMA 2.5. *If $\frac{\partial g}{\partial x}(t, \bar{x}(t)) \neq 0$ for all $t \in S$ and $\frac{\partial g}{\partial x}(\cdot, \bar{x}(\cdot))$ is continuous, then*

$$\mathrm{Int}(\tilde{D}) \neq \emptyset$$

for the topology of uniform convergence.

Proof. Set for all $t \in [t_0, T]$,

$$P(t) = \begin{cases} \{x | \langle \frac{\partial g}{\partial x}(t, \bar{x}(t)), x \rangle \leq -1\} & \text{if } t \in S \\ \mathbf{R}^n & \text{if } t \notin S. \end{cases}$$

P is l.s.c. by Lemma 2.3. Furthermore, for all $t \in [t_0, T]$, $P(t)$ is nonempty, convex and closed. By Theorem [1, p.82], there exists $w \in C(t_0, T; \mathbf{R}^n)$ such that

$$w(t) \in P(t) \quad \forall t \in [t_0, T],$$

i.e.,

$$\frac{\partial g}{\partial x}(t, \bar{x}(t)) \cdot w(t) \leq -1 \quad \forall t \in S.$$

Hence

$$\mathrm{Int}(\tilde{D}) \neq \emptyset.$$

□

Set

$$X = W^{1,2}(t_0, T; \mathbf{R}^n),$$
$$Y = L^2(t_0, T; \mathbf{R}^n) \times L^2(t_0, T; \mathbf{R}^n).$$

Recall that X and Y are Hilbert spaces and consider the linear continuous operator from X to Y:

$$(1 \times \mathcal{D}) : X \to Y$$

where $\mathcal{D}$ denotes the differential operator.

LEMMA 2.6. *Set*

$$\begin{aligned} L = \{(x, y) \in\ & L^2(t_0, T; \mathbf{R}^n) \times L^2(t_0, T; \mathbf{R}^n) \mid \\ & y(s) \in \frac{\partial f}{\partial x}(s, \bar{x}(s), \bar{u}(s))x(s) + T_{f(s,\bar{x}(s),U(s))}(\bar{x}'(s)) \\ & a.e.\ in\ [t_0, T]\}. \end{aligned}$$

Then

$$X^* \supset A^+ = (1 \times \mathcal{D})^*(L^+) \tag{3}$$

where X^* denotes the dual space of X and $(1 \times \mathcal{D})^*$ denotes the adjoint of $1 \times \mathcal{D}$.

Proof. See [7]. □

LEMMA 2.7. *If $\frac{\partial g}{\partial x}(t,\bar{x}(t)) \neq 0$ for all $t \in S$ and $\frac{\partial g}{\partial x}(\cdot,\bar{x}(\cdot))$ is continuous, then $\cup_{i=1}^{\infty} D_i$ is dense in $\tilde{D}$ for the topology of uniform convergence. Hence $\overline{\cup_{i=1}^{\infty} D_i} = \tilde{D}$ because $\tilde{D}$ is closed.*

Proof. Let $w \in \tilde{D}$. Set

$$M_i = \max\left\{\sup_{t\in S_i}\langle\frac{\partial g}{\partial x}(t,\bar{x}(t)), w(t)\rangle, 0\right\}$$

and

$$w_i(t) = w(t) - \frac{\frac{\partial g}{\partial x}(t,\bar{x}(t))}{\inf_{t\in S_i}\|\frac{\partial g}{\partial x}(t,\bar{x}(t))\|^2}M_i \quad \forall t \in [t_0, T].$$

Note that for all i sufficiently large, $\frac{\partial g}{\partial x}(t,\bar{x}(t)) \neq 0$ in S_i. We see that $M_i \longrightarrow 0$ and $w_i \in D_i$ because for all $t \in S_i$,

$$\begin{aligned}\langle\frac{\partial g}{\partial x}(t,\bar{x}(t)), w_i(t)\rangle &\leq \langle\frac{\partial g}{\partial x}(t,\bar{x}(t)), w(t)\rangle - M_i \\ &\leq 0.\end{aligned}$$

To end the proof, it is enough to observe that $w_i \longrightarrow w$ uniformly on $[t_0, T]$. □

LEMMA 2.8. *Under the assumptions of Proposition 2.1, $\tilde{A} \cap (\cup D_i)$ is dense in $\tilde{A} \cap \tilde{D}$.*

Proof. Let $w \in \tilde{A} \cap \tilde{D}$. Set

$$w_i = (1-\lambda_i)w + \lambda_i\bar{w}$$

where λ_i converges to 0^+. Note that $w_i \in \tilde{A}$ and $w_i \to w$. If we prove that $w_i \in \cup_{i=1}^{\infty} D_i$, then the proof ends.

Since $\langle\frac{\partial g}{\partial x}(t,\bar{x}(t)), w(t)\rangle \leq 0$ for all $t \in S$ and $\frac{\partial g}{\partial x}(\cdot,\bar{x}(\cdot))$ and $w(\cdot)$ are continuous, for all $\epsilon \leq \frac{\lambda_i}{1-\lambda_i}\rho$, there exists j such that

$$\langle\frac{\partial g}{\partial x}(t,\bar{x}(t)), w(t)\rangle \leq \epsilon \quad \forall t \in S_j.$$

Therefore, for all $t \in S_j$,

$$\begin{aligned}&\langle\frac{\partial g}{\partial x}(t,\bar{x}(t)), w_i(t)\rangle \\ =\ & (1-\lambda_i)\langle\frac{\partial g}{\partial x}(t,\bar{x}(t)), w(t)\rangle + \lambda_i\langle\frac{\partial g}{\partial x}(t,\bar{x}(t)), \bar{w}(t)\rangle \\ \leq\ & (1-\lambda_i)\epsilon - \lambda_i\rho \\ <\ & 0.\end{aligned}$$

Therefore $w_i \in D_j$. □

3. Main results

In this section, we assume that $(\bar{x}, \bar{u})$ is optimal,

$$\frac{\partial g}{\partial x}(\cdot, \bar{x}(\cdot)) \text{ is continuous,}$$

and there exists $\bar{w} \in \tilde{A}$ and $\rho > 0$ such that

$$\left\langle \frac{\partial g}{\partial x}(t, \bar{x}(t)), \bar{w}(t) \right\rangle \leq -\rho \quad \forall t \in [t_0, T].$$

Note that $\tilde{A} \neq \emptyset$ because $0 \in \tilde{A}$ and that $\mathrm{Int}(\tilde{D}) \neq \emptyset$ by Lemma 2.5.

The next proposition is the main idea to obtain the necessary conditions for optimality.

PROPOSITION 3.1. **(Fermat's Rule)** *If z is optimal, then*

$$\nabla\psi(z(T)) \in \Big(T_{R^g(T)}(z(T))\Big)^+.$$

Proof. Let $v \in T_{R^g(T)}(z(T))$. Then there exist sequences $h_i \longrightarrow 0^+$ and $v_i \longrightarrow v$ such that

$$z(T) + h_i v_i \in R^g(T) \quad \forall i.$$

Since z is optimal, we have

$$\psi(z(T) + h_i v_i) \geq \psi(z(T))$$

and thereby

$$\langle \nabla\psi(z(T)), v\rangle \geq 0.$$

Since v is arbitrary, we have

$$\nabla\psi(z(T)) \in \Big(T_{R^g(T)}(z(T))\Big)^+.$$

□

Now, we can prove the main theorem:

THEOREM 3.2. *Suppose that $(\bar{x}, \bar{u})$ is optimal. Then there exists $c \in \{0, 1\}$, a positive Radon measure μ, a measurable function ν and an absolutely continuous function p such that*

i)

$$-p'(t) = \frac{\partial f}{\partial x}(t, \bar{x}(t), \bar{u}(t))^* \Big(p(t) + \int_{[t_0,t]} \nu(s)d\mu(s)\Big) \quad a.e.$$

ii)

$$\max_{u\in U(t)}\Big\langle p(t)+\int_{[t_0,t]}\nu(s)d\mu(s), f(t,\bar{x}(t),u)\Big\rangle$$
$$=\Big\langle p(t)+\int_{[t_0,t]}\nu(s)d\mu(s), f(t,\bar{x}(t),\bar{u}(t))\Big\rangle \quad a.e.$$

iii)

$$\nu(t)\in\begin{cases}\mathbf{R}_+\frac{\partial g}{\partial x}(t,\bar{x}(t)) & \mu-a.e.\ \ in\ S\\ \{0\} & \mu-a.e.\ \ in\ [t_0,T]\backslash S\end{cases}$$

iv)

$$p(T)+\int_{[t_0,T]}\nu(s)d\mu(s)=-c\nabla\psi(\bar{x}(T))$$

v)

$$c+\|\mu\|>0$$

vi)

$$supp(\mu)\subset S.$$

Proof. The proof consists of two parts.

Case 1: $0\in Int(\tilde{D}-\tilde{A})$ in $C(t_0,T;\mathbf{R}^n)$

By Propositions 3.1, 2.1 and Lemma 2.8,

$$\begin{aligned}\nabla\psi(\bar{x}(T)) &\subset \Big(T_{R^{g}(T)}(\bar{x}(T))\Big)^+\\ &\subset \Big(\gamma\Big(\tilde{A}\cap(\cup_{i=1}^{\infty}D_i)\Big)\Big)^+\\ &= \Big(\gamma(\tilde{A}\cap\tilde{D})\Big)^+,\end{aligned}$$

i.e., for all $w\in\tilde{A}\cap\tilde{D}\subset C(t_0,T;\mathbf{R}^n)$,

$$\begin{aligned}\langle\nabla\psi(\bar{x}(T)),\gamma(w)\rangle &= \langle\gamma^*\nabla\psi(\bar{x}(T)),w\rangle\\ &\geq 0.\end{aligned}$$

It implies that

$$\gamma^*\nabla\psi(\bar{x}(T))\in(\tilde{A}\cap\tilde{D})^+. \tag{4}$$

Consider the set-valued map $V: C(t_0,T;\mathbf{R}^n)\rightsquigarrow C(t_0,T;\mathbf{R}^n)$ defined by:

$$V(x)=\begin{cases}\{x\} & \text{if } x\in\tilde{D}\\ \emptyset & \text{if } x\notin\tilde{D}.\end{cases}$$

Then V is a closed convex process, i.e., the graph of V is a closed convex cone and $\mathrm{Dom}(V) = \tilde{D}$. Since $0 \in Int(\tilde{D} - \tilde{A})$, we have $\mathrm{Dom}(V) - \tilde{A} = C(t_0, T; \mathbf{R}^n)$. Hence by Theorem 2.5.7 of [2],

$$(5) \qquad (\tilde{A} \cap \tilde{D})^+ = \Big(V(\tilde{A} \cap \tilde{D})\Big)^+ = \Big(V(\tilde{A})\Big)^+ = V^{*-1}(\tilde{A}^+)$$

where $V^* : C(t_0, T; \mathbf{R}^n)^* \rightsquigarrow C(t_0, T; \mathbf{R}^n)^*$ is defined by:

$$r \in V^*(q) \Leftrightarrow \forall x \in C(t_0, T; \mathbf{R}^n)\ \forall y \in V(x), \quad \langle r, x \rangle \leq \langle q, y \rangle.$$

We have

$$(6) \qquad V^{*-1}(\tilde{A}^+) = \tilde{A}^+ + \tilde{D}^+.$$

Indeed,

$$\begin{aligned} & q \in V^{*-1}(\tilde{A}^+) \\ \Leftrightarrow \quad & \exists r \in \tilde{A}^+ \text{ such that } q \in V^{*-1}(r) \\ \Leftrightarrow \quad & \exists r \in \tilde{A}^+ \text{ such that } r \in V^*(q) \\ \Leftrightarrow \quad & \exists r \in \tilde{A}^+ \text{ such that } \forall x \in C(t_0, T; \mathbf{R}^n)\ \forall y \in V(x), \quad \langle r, x \rangle \leq \langle q, y \rangle \\ \Leftrightarrow \quad & \exists r \in \tilde{A}^+ \text{ such that } \forall x \in \tilde{D}, \quad \langle r - q, x \rangle \leq 0 \\ \Leftrightarrow \quad & \exists r \in \tilde{A}^+ \text{ such that } r - q \in \tilde{D}^- \\ \Leftrightarrow \quad & q \in \tilde{A}^+ + \tilde{D}^+. \end{aligned}$$

Hence by (4), (5), (6) and Lemma 2.2,

$$(7) \qquad \gamma^* \nabla \psi(\bar{x}(T)) \in A^+ + \tilde{D}^+ \subset C(t_0, T; \mathbf{R}^n)^*.$$

Let $\xi \in A^+ \subset C(t_0, T; \mathbf{R}^n)^*$ and $\tilde{\mu} \in \tilde{D}^+ \subset C(t_0, T; \mathbf{R}^n)^*$ be such that

$$\gamma^* \nabla \psi(\bar{x}(T)) = \xi + \tilde{\mu}.$$

Note that

$$C(t_0, T; \mathbf{R}^n)^* \subset X^*.$$

Therefore we have

$$(8) \qquad \langle \gamma^* \nabla \psi(\bar{x}(T)), w \rangle = \langle \xi + \tilde{\mu}, w \rangle \quad \forall w \in X.$$

Lemma 2.6 implies that there exists $(r, q) \in L^+$ such that

$$(9) \qquad \xi = (1 \times \mathcal{D})^*(r, q)$$

because

$$\xi \in C(t_0, T; \mathbf{R}^n)^* \subset X^*.$$

(8) and (9) imply that for all $w \in X$,

$$\langle \gamma^* \nabla \psi(\bar{x}(T)), w \rangle = \langle (1 \times \mathcal{D})^*(r, q) + \tilde{\mu}, w \rangle.$$

Thus for all $w \in X$,

$$\begin{aligned} & \int_{t_0}^{T} r(t)w(t)dt + \int_{t_0}^{T} q(t)w'(t)dt + \int_{t_0}^{T} w(t)d\tilde{\mu}(t) \\ (10) \qquad & = \langle \nabla\psi(\bar{x}(T)), w(T)\rangle. \end{aligned}$$

On the other hand, by integrating by parts, we have

$$\int_{t_0}^{T} r(t)w(t)dt = -\int_{t_0}^{T} w'(t)\int_{t_0}^{t} r(s)dsdt + \left\langle w(t), \int_{t_0}^{t} r(s)ds \right\rangle \Big|_{t_0}^{T}.$$

Set

$$W_0^{1,2}(t_0, T; \mathbf{R}^n) = \{w \in W^{1,2}(t_0, T; \mathbf{R}^n) | w(t_0) = w(T) = 0\}.$$

If we define β_1 and β_2 by: for all $t_0 \leq t \leq T$,

$$\beta_1(t) = \tilde{\mu}^+([t_0, t])$$

and

$$\beta_2(t) = \tilde{\mu}^-([t_0, t])$$

(where $\tilde{\mu}^+$ is the positive part of $\tilde{\mu}$ and $\tilde{\mu}^-$ is the negative part of $\tilde{\mu}$) then for all $w \in W_0^{1,2}(t_0, T; \mathbf{R}^n)$,

$$\begin{aligned} & \int_{t_0}^{T} w(t)d\tilde{\mu}(t) \\ & = \int_{(t_0,T)} w(t)d\tilde{\mu}^+(t) - \int_{(t_0,T)} w(t)d\tilde{\mu}^-(t) \\ & = \int_{(t_0,T)} w(t)d\beta_1(t) - \int_{(t_0,T)} w(t)d\beta_2(t) \\ (11) \qquad & = -\int_{(t_0,T)} (\beta_1(t) - \beta_2(t))dw(t) \\ (12) \qquad & = -\int_{t_0}^{T} (\beta_1(t) - \beta_2(t))w'(t)dt \\ & = -\int_{t_0}^{T} \Big(\int_{[t_0,t]} d\tilde{\mu}^+(s) - \int_{[t_0,t]} d\tilde{\mu}^-(s)\Big) w'(t)dt \\ & = -\int_{t_0}^{T} w'(t)\int_{[t_0,t]} d\tilde{\mu}(s)dt \end{aligned}$$

(where we have (11) by [6, p.154] and (12) by [9, p.311]).

Therefore (10) becomes, for all $w \in W_0^{1,2}(t_0, T; \mathbf{R}^n)$,

$$\int_{t_0}^{T} w'(t)\Big(q(t) - \int_{t_0}^{t} r(s)ds - \int_{[t_0,t]} d\tilde{\mu}(s)\Big)dt = 0.$$

By Dubois-Reymond Lemma ([4, p.42]), there exists a constant $c_0 \in \mathbf{R}^n$ such that

$$q(t) = c_0 + \int_{t_0}^{t} r(s)ds + \int_{[t_0,t]} d\tilde{\mu}(s).$$

Set

$$p(t) = -c_0 - \int_{t_0}^{t} r(s)ds. \tag{13}$$

Then

$$p'(t) = -r(t)$$

and

$$q(t) = -p(t) + \int_{[t_0,t]} d\tilde{\mu}(s).$$

On the other hand, by integrating by parts ([6, p.154]), for all $w \in W^{1,2}(t_0.T; \mathbf{R}^n)$ such that $w(t_0) = 0$, we have

$$\begin{aligned}
&\int_{t_0}^{T} w(t)d\tilde{\mu}(t)\\
= \ &\int_{t_0}^{T} w(t)d\tilde{\mu}^+(t) - \int_{t_0}^{T} w(t)d\tilde{\mu}^-(t)\\
= \ &\int_{(t_0,T)} w(t)d\tilde{\mu}^+(t) + \langle w(T), \tilde{\mu}^+(\{T\})\rangle\\
&- \int_{(t_0,T)} w(t)d\tilde{\mu}^-(t) - \langle w(T), \tilde{\mu}^-(\{T\})\rangle\\
= \ &\int_{(t_0,T)} w(t)d\beta_1(t) + \langle w(T), \tilde{\mu}^+(\{T\})\rangle\\
&- \int_{(t_0,T)} w(t)d\beta_2(t) - \langle w(T), \tilde{\mu}^-(\{T\})\rangle\\
= \ &-\int_{(t_0,T)} \beta_1(t)dw(t) + \langle w(T), \beta_1(T-) + \tilde{\mu}^+(\{T\})\rangle\\
&+ \int_{(t_0,T)} \beta_2(t)dw(t) - \langle w(T), \beta_2(T-) + \tilde{\mu}^-(\{T\})\rangle
\end{aligned}$$

$$= -\int_{t_0}^{T} w'(t)\int_{[t_0,t]} d\tilde{\mu}(s)dt + \Big\langle w(T), \int_{[t_0,T]} d\tilde{\mu}(t)\Big\rangle.$$

By (10), we have

$$\begin{aligned} &\Big\langle \int_{t_0}^{T} r(t)dt, w(T)\Big\rangle + \int_{t_0}^{T} w'(t)\Big(q(t) - \int_{t_0}^{t} r(s)ds - \int_{[t_0,t]} d\tilde{\mu}(s)\Big)dt \\ &-\langle \nabla\psi(\bar{x}(T)), w(T)\rangle + \Big\langle \int_{[t_0,T]} d\tilde{\mu}(t), w(T)\Big\rangle \\ = &\Big\langle \int_{t_0}^{T} r(t)dt + c_0 - \nabla\psi(\bar{x}(T)) + \int_{[t_0,T]} d\tilde{\mu}(t), w(T)\Big\rangle \\ = &\ 0. \end{aligned}$$

Therefore we have

$$\int_{t_0}^{T} r(t)dt = \nabla\psi(\bar{x}(T)) - c_0 - \int_{[t_0,T]} d\tilde{\mu}(t). \tag{14}$$

By (13), (14), Lemma 2.4 and the fact that $\tilde{\mu} \in \tilde{D}^+$ (i.e., $-\tilde{\mu} \in \tilde{D}^-$), there exists a positive Radon measure $\tilde{\mu}_1$ and a measurable function ν such that

$$\int_{[t_0,t]} d\tilde{\mu}(s) = -\int_{[t_0,t]} \nu(s)d\tilde{\mu}_1(s),$$

$$p(T) + \int_{[t_0,T]} \nu(t)d\tilde{\mu}_1(t) = -\nabla\psi(\bar{x}(T))$$

and

$$\nu(t) \in \begin{cases} \mathbf{R}_{+}\frac{\partial g}{\partial x}(t,\bar{x}(t)) & \tilde{\mu}_1 - \text{a.e. in } S \\ \{0\} & \tilde{\mu}_1 - \text{a.e. in } [t_0,T]\backslash S. \end{cases}$$

Define a positive Radon measure $\mu : \mathcal{B}([t_0,T]) \to \mathbf{R}^+$ by setting

$$\forall E \in \mathcal{B}([t_0,T]), \quad \mu(E) = \tilde{\mu}_1(S \cap E)$$

where $\mathcal{B}([t_0,T])$ is the σ-algebra generated by the open subsets of $[t_0,T]$. Then

$$\int_{[t_0,T]} \nu(t)d\tilde{\mu}_1(t) = \int_{[t_0,T]} \nu(t)d\mu(t)$$

and

$$\text{supp}(\mu) \subset S.$$

Now, recall that $(r,q) \in L^+$. For all $v \in L^2$ which verifies

$$v(t) \in T_{f(t,\bar{x}(t),U(t))}(\bar{x}'(t)),$$

we have

$$(0,v) \in L.$$

Therefore

$$\langle (r,q),(0,v)\rangle = \int_{t_0}^{T} \langle q(t), v(t)\rangle dt \geq 0. \tag{15}$$

From the measurable selection theorem and the fact that $f(s,\bar{x}(s),U(s)) - \bar{x}'(s) \subset T_{f(s,\bar{x}(s),U(s))}(\bar{x}'(s))$, we deduce that

$$\sup\{\langle -q(s),\xi\rangle | \xi \in f(s,\bar{x}(s),U(s)) - f(s,\bar{x}(s),\bar{u}(s))\} \leq 0 \quad \text{a.e.}$$

On the other hand,

$$f(s,\bar{x}(s),\bar{u}(s)) \in f(s,\bar{x}(s),U(s)),$$

hence

$$\max\{\langle -q(s),\xi\rangle | \xi \in f(s,\bar{x}(s),U(s)) - f(s,\bar{x}(s),\bar{u}(s))\} = 0 \quad \text{a.e.,}$$

therefore we obtain the maximum principle:

$$\langle -q(s), f(s,\bar{x}(s),\bar{u}(s))\rangle = \max_{u\in U(s)} \langle -q(s), f(s,\bar{x}(s),u)\rangle \quad \text{a.e.}$$

i.e.,

$$\begin{aligned}&\Big\langle p(s) + \int_{[t_0,s]} \nu(t)d\mu(t), f(s,\bar{x}(s),\bar{u}(s))\Big\rangle \\ = \;& \max_{u\in U(s)} \Big\langle p(s) + \int_{[t_0,s]} \nu(t)d\mu(t), f(s,\bar{x}(s),u)\Big\rangle \quad \text{a.e.}\end{aligned}$$

Now, since

$$0 \in T_{f(s,\bar{x}(s),U(s))}(\bar{x}'(s)) \;\; \forall s \in [t_0,T],$$

we have

$$\Big(w, \frac{\partial f}{\partial x}(\cdot,\bar{x}(\cdot),\bar{u}(\cdot))w\Big) \in L \quad \forall w \in L^2.$$

Since $(r,q) \in L^+$, we have for all $w \in L^2$,

$$\begin{aligned}&\Big\langle (r,q),(w,\frac{\partial f}{\partial x}(\cdot,\bar{x}(\cdot),\bar{u}(\cdot))w)\Big\rangle \\ =\;& \Big\langle r + \frac{\partial f}{\partial x}(\cdot,\bar{x}(\cdot),\bar{u}(\cdot))^* q, w\Big\rangle \\ \geq\;& 0\end{aligned}$$

therefore

$$r(t) = -\frac{\partial f}{\partial x}(t,\bar{x}(t),\bar{u}(t))^* q(t) \;\; \text{a.e.,}$$

i.e.,

$$-p'(t) = \frac{\partial f}{\partial x}(t,\bar{x}(t),\bar{u}(t))^* \Big(p(t) + \int_{[t_0,t]} \nu(s)d\mu(s)\Big) \quad \text{a.e.}$$

Case 2: $0 \notin Int(\tilde{D} - \tilde{A})$ in $C(t_0, T; \mathbf{R}^n)$

We know that $\tilde{A} \neq \emptyset$ and $\mathrm{Int}(\tilde{D}) \neq \emptyset$. In this case $\mathrm{Int}(\tilde{D}) \cap \tilde{A} = \emptyset$. By Separation Theorem (see [6, p.417]), there exists $\hat{\mu} \in C(t_0, T; \mathbf{R}^n)^*$ which is not equal to 0 such that

$$\sup_{d \in \mathrm{Int}(\tilde{D})} \langle \hat{\mu}, d \rangle \leq \inf_{a \in \tilde{A}} \langle \hat{\mu}, a \rangle \leq 0$$

because $0 \in \tilde{A}$. On the other hand, $\tilde{D} = \overline{\mathrm{Int}(\tilde{D})}$. Therefore

$$\sup_{d \in \tilde{D}} \langle \hat{\mu}, d \rangle \leq \inf_{a \in \tilde{A}} \langle \hat{\mu}, a \rangle \leq 0. \tag{16}$$

Hence

$$\hat{\mu} \in \tilde{D}^-.$$

Furthermore, by (16),

$$\hat{\mu} \in \tilde{A}^+ = A^+.$$

Since $\hat{\mu} \in \tilde{D}^- \cap \tilde{A}^+ \subset C(t_0, T; \mathbf{R}^n)^* \subset X^*$, there exists $(\hat{r}, \hat{q}) \in L^+$ such that for all $w \in X$,

$$\begin{aligned} \int_{t_0}^{T} w(t) d\hat{\mu}(t) &= \langle \hat{\mu}, w \rangle \\ &= \Big\langle (1 \times \mathcal{D})^*(\hat{r}, \hat{q}), w \Big\rangle \\ &= \int_{t_0}^{T} \hat{r}(t) w(t) dt + \int_{t_0}^{T} \hat{q}(t) w'(t) dt. \end{aligned} \tag{17}$$

We can see that (17) is of the same form as (10) with the term $\langle \nabla\psi(\bar{x}(T)), w(T) \rangle$ replaced by zero. Hence, by replacing $\tilde{\mu}$ by $-\hat{\mu}$ and (r, q) by $(\hat{r}, \hat{q})$, we can obtain all the same conclusions as in the Case 1. □

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