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Bayesian Model Selection in Analysis of Reciprocals

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Abstract

Tweedie (1957a) proposed a method for the analysis of residuals from an inverse Gaussian population paralleling the analysis of variance in normal theory. He called it the analysis of reciprocals. In this paper, we propose a Bayesian model selection procedure based on the fractional Bayes factor for the analysis of reciprocals. Using the proposed model selection procedures, we compare with the classical tests.

Keywords : Analysis of Reciprocals, Fractional Bayes Factor, Inverse Gaussian Distribution, Reference Prior

1. INTRODUCTION

Because of the versatility and flexibility in modelling right-skewed data, the inverse Gaussian distribution has potential useful applications in a wide variety of fields such as biology, economics, reliability theory, life testing and social sciences as discussed in Chhikara and Folks (1978, 1989) and Seshadri (1999). Tweedie (1957a, 1957b) established many important statistical properties of the inverse Gaussian distribution and discussed the similarity between statistical methods based on the inverse Gaussian distribution and those based on the normal theory.

Let X be an inverse Gaussian distribution with density function

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$$f(x \mid \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi}} x^{-3/2} \exp\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\}, \quad x>0, \tag{1}$$

where $\mu>0$ and $\lambda>0$. The parameter μ is the mean of the distribution and λ is a scale parameter. The inverse Gaussian distribution will be denoted by the $\mathrm{IG}(\mu,\lambda)$. In this model, we assume that there are n_i items from the ith population each of which is distributed as $\mathrm{IG}(\mu_i,\lambda_i)$ where $i=1,\cdots,k$. The classical analysis of reciprocals (Tweedie, 1957a) consists of testing whether the means μ_i are all equal when all populations have the same λ. From the frequentist viewpoints this problem poses the difficulty that an exact F test exists only under the condition that the λ_i's are all equal.

Our proposal here is to formulate the classical analysis of reciprocals as a model selection problem for which we propose a fully Bayesian procedure.

Models (or Hypotheses) H_1, $H_2, \cdots$, H_q are under consideration, with the data $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)$ having probability density function $f_i(\boldsymbol{x} \mid \theta_i)$ under model $H_i, i=1,2,\cdots,q$. The parameter vectors θ_i are unknown. Let $\pi_i(\theta_i)$ be the prior distribution of model H_i, and let p_i be the prior probabilities of model H_i, $i=1,2,\cdots,q$. Then the posterior probability that the model H_i is true is

$$P(H_i \mid \boldsymbol{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji}\right)^{-1}, \tag{2}$$

where B_{ji} is the Bayes factor of model H_j to model H_i defined by

$$B_{ji} = \frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})} = \frac{\int f_j(\boldsymbol{x} \mid \theta_j)\pi_j(\theta_j)d\theta_j}{\int f_i(\boldsymbol{x} \mid \theta_i)\pi_i(\theta_i)d\theta_i}. \tag{3}$$

The B_{ji} interpreted as the comparative support of the data for the model j to i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$. Usually, one can use the noninformative prior, often improper, such as uniform prior, Jeffreys prior and reference prior. Denote it as π_i^N. The use of improper priors $\pi_i^N(\cdot)$ in (3) causes the B_{ji} to contain unspecified constants.

Spiegalhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the fractional Bayes factor. To remove the arbitrariness in Bayes factor, he used to a portion of the likelihood with a so-called the fraction b. These two approaches mentioned above have shown to be quite useful in many statistical areas.

In this paper, we consider the Bayesian model selection problem for analysis of reciprocals. The outline of the remaining sections is as follows. In Section 2, using the reference priors, we provide the Bayesian model selection procedure based on the fractional Bayes factor for the analysis of reciprocals, and provide the test procedure for homogeneity of the λ's. In Section 3, some examples and conclusions of our Bayesian test procedure are given.

2. BAYES FACTOR FOR ANALYSIS OF RECIPROCALS

2.1 Preliminaries

It has known that the use of improper priors $\pi_i^N(\cdot)$ in (3) causes the B_{ji} to contain unspecified constants. To solve this problem, O'Hagan (1995) proposed the fractional Bayes factor for Bayesian testing and model selection problem as follow.

When the $\pi_i^N(\theta_i)$ is noninformative prior under H_i, equation (3) becomes

$$B_{ji}^N = \frac{\int f_j(x \mid \theta_j)\pi_j^N(\theta_j)d\theta_j}{\int f_i(x \mid \theta_i)\pi_i^N(\theta_i)d\theta_i}.$$

Then the fractional Bayes factor of model H_j versus model H_i is

$$B_{ji}^F = B_{ji}^N \cdot \frac{\int f_i^b(x \mid \theta_i)\pi_i^N(\theta_i)d\theta_i}{\int f_j^b(x \mid \theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^N \cdot \frac{m_i^b(x)}{m_j^b(x)},$$

and $f_i(x \mid \theta_i)$ is the likelihood function and b specifies a fraction of the likelihood which is to be used as a prior density. He proposed three ways for the choice of the fraction b. One frequently suggested choice is $b = m/n$, where m is the size of the minimal training sample, assuming this is well defined. (see O'Hagan, 1995, 1997 and the discussion by Berger and Mortera of O'Hagan, 1995).

2.2 Fractional Bayes Factor for Analysis of Reciprocals

Given samples of sizes n_i from $\mathrm{IG}(\mu_i, \lambda)$, $i = 1, \cdots, k$, we consider the testing of the following hypotheses:

$$H_1: \mu_1 = \cdots = \mu_k \equiv \mu \quad \text{v.s.} \quad H_2: \mu_1 \neq \cdots \neq \mu_k.$$

Our interest is to develop a Bayesian test based on the fractional Bayes factors for H_1 v.s. H_2 under the noninformative priors.

Under the hypothesis H_1, the reference prior for μ and λ is

$$\pi_1(\mu, \lambda) = \lambda^{-1} \mu^{-3/2}.$$

The likelihood function under H_1 is

$$L(\mu, \lambda \mid \mathbf{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-3/2}\right] \lambda^{n/2} \exp\left\{-\frac{\lambda}{2}\left[\sum_{j=1}^{n_1} s_i + \frac{n_i(\overline{x}_i - \mu)^2}{\mu^2 \overline{x}_i}\right]\right\},$$

where $n = n_1 + \cdots + n_k$, $\overline{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$ and $s_i = \sum_{j=1}^{n_i} [(1/x_{ij}) - (1/\overline{x}_i)], i = 1, \cdots, k$. Then the element of fractional Bayes factor under H_1 is given by

$$\begin{aligned} m_1^b(\mathbf{x}) &= \int_0^\infty \int_0^\infty L^b(\mu, \lambda \mid \mathbf{x}) \pi_1(\mu, \lambda) d\mu d\lambda \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^{nb} \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-3b/2}\right] \left(\frac{b}{2}\right)^{-\frac{nb}{2}} \Gamma\left(\frac{nb}{2}\right) S_1(\mathbf{x}; b), \end{aligned}$$

where

$$S_1(\mathbf{x}; b) = \int_0^\infty \theta^{-1/2} \left\{\sum_{i=1}^{k} [s_i + n_i \overline{x}_i (\theta - \overline{x}_i^{-1})^2]\right\}^{-\frac{nb}{2}} d\theta.$$

For the H_2, the reference prior is

$$\pi_2(\mu_1, \cdots, \mu_k, \lambda) = \mu_1^{-3/2} \cdots \mu_k^{-3/2} \lambda^{-1}.$$

Kang (2004) developed the above reference prior $\pi_2(\mu_1, \cdots, \mu_k, \lambda)$. The likelihood function under H_2 is

$$\begin{aligned} &L(\mu_1, \cdots, \mu_k, \lambda \mid \mathbf{x}) \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-3/2}\right] \lambda^{n/2} \exp\left\{-\sum_{i=1}^{k} \frac{\lambda}{2}\left[s_i + \frac{n_i(\overline{x}_i - \mu_i)^2}{\mu_i^2 \overline{x}_i}\right]\right\}. \end{aligned}$$

Thus the element of fractional Bayes factor under H_2 is given as follows.

$$m_2^b(x) = \int_0^\infty \cdots \int_0^\infty L^b(\mu_1, \cdots, \mu_k, \lambda \mid x)\pi_2(\mu_1, \cdots, \mu_k, \lambda)d\mu_1 \cdots d\mu_k d\lambda$$
$$= (\frac{1}{\sqrt{2\pi}})^{nb}[\prod_{i=1}^{k}\prod_{j=1}^{n_i} x_{ij}^{-3b/2}](\frac{b}{2})^{-\frac{nb}{2}}\Gamma(\frac{nb}{2})S_2(x;b),$$

where

$$S_2(x;b) = \int_0^\infty \cdots \int_0^\infty [\prod_{i=1}^{k}\theta_i^{-1/2}]\{\sum_{i=1}^{k}[s_i + n_i\overline{x}_i(\theta_i - \overline{x}_i^{-1})^2]\}^{-\frac{nb}{2}} d\theta_1 \cdots d\theta_k.$$

Therefore the B_{21}^N from $m_1^b(x)$ and $m_2^b(x)$ with $b=1$ is given by

$$B_{21}^N = \frac{S_2(x;1)}{S_1(x;1)}.$$

And also

$$\frac{m_1^b(x)}{m_2^b(x)} = \frac{S_1(x;b)}{S_2(x;b)}.$$

Thus the fractional Bayes factor of H_2 versus H_1 is given by

$$B_{21}^F = \frac{S_2(x;1)S_1(x;b)}{S_2(x;b)S_1(x;1)}. \quad (4)$$

Note that the calculation of the fractional Bayes factor of H_2 versus H_1 requires a numerical integration.

2.3 Fractional Bayes Factor for Homogeneity of the Scale Parameters

The Bayes factor in Section 2.2 was derived under the assumption that all the λ's are equal. Hence, it is of interest to test whether the same λ condition can be accepted.

Given samples of sizes n_i from $\mathrm{IG}(\mu_i, \lambda_i)$, $i=1,\cdots,k,$ we consider the testing of the following hypotheses:

$$H_1: \lambda_1 = \cdots = \lambda_k \equiv \lambda \quad \text{v.s.} \quad H_2: \lambda_1 \neq \cdots \neq \lambda_k.$$

Under the hypothesis H_1, the reference prior for $\mu_1, \cdots, \mu_k$ and λ is

$$\pi_1(\mu_1, \cdots, \mu_k, \lambda) = \lambda^{-1}\mu_1^{-3/2}\cdots\mu_k^{-3/2}.$$

The likelihood function under H_1 is

$$
\begin{aligned}
& L(\mu_1, \cdots, \mu_k, \lambda \mid \boldsymbol{x}) \\
& = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-3/2}\right] \lambda^{n/2} \exp\left\{- \sum_{i=1}^{k} \frac{\lambda}{2} \left[s_i + \frac{n_i(\overline{x}_i - \mu_i)^2}{\mu_i^2 \overline{x}_i} \right]\right\}.
\end{aligned}
$$

where $n = n_1 + \cdots + n_k$, $\overline{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$ and $s_i = \sum_{j=1}^{n_i}[(1/x_{ij}) - (1/\overline{x}_i)]$, $i = 1, \cdots, k$. Then the element of fractional Bayes factor under H_1 is given by

$$
\begin{aligned}
m_1^b(\boldsymbol{x}) &= \int_0^\infty \cdots \int_0^\infty L^b(\mu_1, \cdots, \mu_k, \lambda \mid \boldsymbol{x}) \pi_1(\mu_1, \cdots, \mu_k, \lambda) d\mu_1 \cdots d\mu_k d\lambda \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^{nb} \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-3b/2}\right] \left(\frac{b}{2}\right)^{-\frac{nb}{2}} \Gamma\left(\frac{nb}{2}\right) T_1(\boldsymbol{x}; b),
\end{aligned}
$$

where

$$
T_1(\boldsymbol{x}; b) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{i=1}^{k} \theta_i^{-\frac{1}{2}}\right] \left\{\sum_{i=1}^{k} [s_i + n_i \overline{x}_i (\theta_i - \overline{x}_i^{-1})^2]\right\}^{-\frac{nb}{2}} d\theta_1 \cdots d\theta_k.
$$

For the H_2, the reference prior is

$$
\pi_2(\mu_1, \cdots, \mu_k, \lambda_1, \cdots, \lambda_k) = \mu_1^{-3/2} \cdots \mu_k^{-3/2} \lambda_1^{-1} \cdots \lambda_l^{-1}.
$$

The likelihood function under H_2 is

$$
\begin{aligned}
& L(\mu_1, \cdots, \mu_k, \lambda_1, \cdots, \lambda_k \mid \boldsymbol{x}) \\
& = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-3/2}\right] \left[\prod_{i=1}^{k} \lambda_i^{n_i/2}\right] \exp\left\{- \sum_{i=1}^{k} \frac{\lambda_i}{2} \left[s_i + \frac{n_i(\overline{x}_i - \mu_i)^2}{\mu_i^2 \overline{x}_i} \right]\right\}.
\end{aligned}
$$

Thus the element of fractional Bayes factor under H_2 is given as follows.

$$
\begin{aligned}
m_2^b(\boldsymbol{x}) &= \int_0^\infty \cdots \int_0^\infty L^b(\mu_1, \cdots, \mu_k, \lambda_1, \cdots, \lambda_k \mid \boldsymbol{x}) \\
& \quad \times \pi_2(\mu_1, \cdots, \mu_k, \lambda_1, \cdots, \lambda_k) d\mu_1 \cdots d\mu_k d\lambda_1 \cdots d\lambda_k \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^{nb} \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-3b/2}\right] \left(\frac{b}{2}\right)^{-\frac{nb}{2}} \left\{\prod_{i=1}^{k} \Gamma\left[\frac{n_i b}{2}\right]\right\} T_2(\boldsymbol{x}; b),
\end{aligned}
$$

where

$$T_2(x;b)=\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^k\{\theta_i^{-1/2}[s_i+n_i\overline{x}_i(\theta_i-\overline{x}_i^{-1})^2]^{-\frac{n_ib}{2}}\}d\theta_1\cdots d\theta_k.$$

Therefore, the B_{21}^N from $m_1^b(x)$ and $m_2^b(x)$ with $b=1$ is given by

$$B_{21}^N=\frac{T_2(x;1)\prod_{i=1}^k\Gamma[\frac{n_i}{2}]}{T_1(x;1)\Gamma[\frac{n}{2}]}.$$

And also

$$\frac{m_1^b(x)}{m_2^b(x)}=\frac{T_1(x;b)\Gamma[\frac{nb}{2}]}{T_2(x;b)\prod_{i=1}^k\Gamma[\frac{n_ib}{2}]}.$$

Thus the fractional Bayes factor of H_2 versus H_1 is given by

$$B_{21}^F=\frac{\Gamma[\frac{nb}{2}]\prod_{i=1}^k\Gamma[\frac{n_i}{2}]}{\Gamma[\frac{n}{2}]\prod_{i=1}^k\Gamma[\frac{n_ib}{2}]}\frac{T_2(x;1)T_1(x;b)}{T_2(x;b)T_1(x;1)}. \quad (5)$$

Note that the calculation of the fractional Bayes factor of H_2 versus H_1 requires a numerical integration.

3. NUMERICAL STUDIES

In this section, we give some examples to show the usefulness of our test procedures by real data sets.

Example 1 : Testing equality of scale parameters

The data given in Table 1 is the results of an experiment designed to compare the performance of high-speed turbine bearings made out of five different compounds. In the experiment 10 bearings of each type were tested and the failure times in units of millions of cycles were recorded (McCool, 1979; Chhikara and Folks, 1989).

Let $V_i=\sum_{j=1}^{n_i}[(1/X_{ij})-(1/\overline{X}_i)]$ and $V=\sum_{i=1}^k V_i$. Under $H_1:\lambda_1=\cdots=\lambda_k$, the classical test statistic is given by

$$\Lambda = \frac{M}{C},$$

where $f_i = n_i - 1$, $f = \sum_{i=1}^{k}(n_i - 1)$, $C = 1 + \frac{1}{3(k-1)}\left[\sum_{i=1}^{k}(1/f_i) - (1/\sum_{i=1}^{k} f_i)\right]$ and $M = f\log(V/f) - \sum_{i=1}^{k} f_i \log(V_i/f_i)$. The test statistic Λ is distributed approximately as χ^2 with $(k-1)$ degrees of freedom.

Table 1. Failure Times of Bearing Specimens

I	3.03, 5.53, 5.60, 9.30, 9.92, 12.51, 12.95, 15.21, 16.04, 16.84
II	3.19, 4.26, 4.47, 4.53, 4.67, 4.69, 5.78, 6.79, 9.37, 12.75
III	3.46, 5.22, 5.69, 6.54, 9.16, 9.40, 10.19, 10.71, 12.58, 13.41
IV	5.88, 6.74, 6.90, 6.98, 7.21, 8.14, 8.59, 9.80, 12.28, 25.46
V	6.43, 9.97, 10.39, 13.55, 14.45, 14.72, 16.81, 18.39, 20.84, 21.51

The p-values based on the χ^2 statistics, the value of fractional Bayes factors of H_2 versus H_1 and the posterior probabilities for H_1 are given in Table 2. We computed the posterior probabilities for model H_1 corresponding to values of Bayes factors when the prior probabilities are equal. From the results of Table 2, we may conclude that the homoscedastic model is clearly favoured.

Table 2. p-values, Bayes Factor Values and Posterior Probabilities

H_1	p-*value*	B_{21}^F	$P(H_1 \mid x)$
$\lambda_1 = \lambda_3 = \lambda_4$	0.7446	0.1615	0.8610
$\lambda_1 = \lambda_2 = \lambda_5$	0.1953	0.5386	0.6499
$\lambda_1 = \lambda_2 = \lambda_3$	0.8621	0.1392	0.8778

Example 2 : Testing the equality of means

The three rows in Table 2 show that the equality of the scale parameters are accepted. In this situation it is desired to test whether the population means are all equal.

The p-values under the F statistics of the analysis of reciprocals, the value of fractional Bayes factors of H_2 versus H_1 and the posterior probabilities for H_1 are given in Table 3. We computed the posterior probabilities for model H_1 corresponding to values of Bayes factors when the prior probabilities are equal. From the results of Table 3, for the model H_1: $\mu_1 = \mu_3 = \mu_4$ and the model

$H_1: \mu_1 = \mu_2 = \mu_5$, the simpler model and the complex model are clearly favoured, respectively. The Bayes factors and the p-values give the same message. But for the model $H_1: \mu_1 = \mu_2 = \mu_3$, the classical test favors the complex model and the Bayes factor favors simpler model.

In this paper, we developed a Bayesian model selection procedures for the analysis of reciprocals. Under the reference priors, the fractional Bayes factors of O'Hagan (1996) are computed. Through the examples, we can conclude that the Bayes factors and the classical tests perform reasonably.

Table 3. p-values, Bayes Factor Values and Posterior Probabilities

H_1	*p-value*	B_{21}^F	$P(H_1 \mid x)$
$\mu_1 = \mu_3 = \mu_4$	0.6389	0.0816	0.9246
$\mu_1 = \mu_2 = \mu_5$	0.0017	25.4755	0.0378
$\mu_1 = \mu_2 = \mu_3$	0.0480	0.9831	0.5043

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