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A Note on Relationship between T-sum and T-product on LR Fuzzy Numbers

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Abstract

In this note, we show that Theorem 2.1[Kybernetika, 28(1992) 45-49], a result of a functional relationship between the membership function of LR fuzzy numbers of T-sum and T-product, remains valid for convex additive generator and concave shape functions L and R with simple proof. We also consider the case for 0-symmetric R fuzzy numbers.

Keywords : Extension principle, Fuzzy numbers, t-norm

1. Preliminaries

As defined in [1], by a fuzzy number we mean a fuzzy subset ξ of the real line with a continuous, compactly supported, unimodal membership function such that there exists a unique real number m satisfying $\xi(m)=\sup_x\xi(x)=1$. A fuzzy set ξ is said to be positive if $\xi=0$ for all $x<0$. A function $T:[0,1]\times[0,1]\rightarrow[0,1]$ is said to be a triangular norm (t-norm for short) if and only if T is symmetric, associative, non-decreasing in each argument, and $T(x,1)=x$ for all $x\in[0,1]$. Now suppose that a sequence of fuzzy numbers $\xi_1,\xi_2,\cdots,\xi_n,\cdots$ and a t-norm T are given. The T-product $\xi_1\cdots\xi_n$ and the T-sum $\xi_1+\cdots+\xi_n$ are the fuzzy numbers defined by

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$$(\xi_1 \cdots \xi_n)(z) := \sup_{x_1 \cdots x_n = z} T(\xi_1(x_1), \cdots, \xi_n(x_n))$$

and

$$(\xi_1 + \cdots + \xi_n)(z) := \sup_{x_1 + \cdots + x_n = z} T(\xi_1(x_1), \cdots, \xi_n(x_n))$$

respectively.

Recall that a t-norm T is called Archimedean if and only if T is continuous and $T(x,x) < x$ for all $x \in (0,1)$. A well-known theorem asserts that for each Archimedean t-norm there exists a continuous, decreasing function $f: [0, 1] \to [0, \infty]$ with $f(1) = 0$ such that

$$T(x_1, \cdots, x_n) = f^{[-1]}(f(x_1) + \cdots + f(x_n))$$

for all $x_i \in (0,1)$, $1 \le i \le n$. Here $f^{[-1]}: [0, \infty] \to [0, 1]$ is defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{for} \quad y \in [0, f(0)], \\ 0 & \text{if} \quad y > f(0). \end{cases}$$

The function f is called the additive generator of T. Since f is continuous and decreasing, $f^{[-1]}$ is also continuous and non--increasing, we have

$$\begin{aligned} (\xi_1 \cdots \xi_n)(z) &= \sup_{x_1 \cdots x_n = z} f^{[-1]}\Big(\sum_{i=1}^{n} f(\xi_i(x_i))\Big) \\ &= f^{[-1]}\Big(\inf_{x_1 \cdots x_n = z}\Big(\sum_{i=1}^{n} f(\xi_i(x_i))\Big)\Big). \end{aligned} \tag{1}$$

An LR fuzzy number $\tilde{a} = (a, \alpha, \beta)_{LR}$ is a function from the reals into the interval $[0,1]$ satisfying

$$\tilde{a}(t) = \begin{cases} R\left(\dfrac{t-a}{\beta}\right) & \text{for} \quad a \le t \le a + \beta, \\ L\left(\dfrac{a-t}{\alpha}\right) & \text{for} \quad a - \alpha \le t \le a, \\ 0 & else, \end{cases}$$

where L and R are strictly decreasing, continuous function from $[0,1]$ to $[0,1]$ satisfying $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. In particular, if $\alpha = 0$, then $\tilde{a}=(a,0,\beta)_{LR}=(a,\beta)_R$ is R fuzzy number. A fuzzy number $\tilde{a}$ is called positive if its membership function is such that $\tilde{a}(t)=0$ for any $t<0$.

The following theorem(Fullér and Keresztfalvi [4]) gave a functional relationship between the membership function of T-sum and T-product of LR fuzzy numbers.

Theorem 1[4]. Let T be an Archimedean t-norm with additive generator f and let $\xi=\xi_i=(a,\ \alpha,\ \beta)_{LR}$ be positive fuzzy numbers of LR type. If L and R are twice differentiable, concave functions and f is twice differentiable, strictly convex function , then

$$(\xi_1+\cdots+\xi_n)(nz)=(\xi_1\cdots\xi_n)(z^n)=f^{[-1]}(nf(\xi(z))).$$

In this note, we prove above theorem under weaker conditions that convex additive generator f and concave shape functions L and R.

2. The results

We need the following known result of Hong and Hwang[5].

Lemma 1[5]. Let T be an Archimedean t-norm with additive generator f and let $\xi=\xi_i=(a,\ \alpha,\ \beta)_{LR}$ be fuzzy numbers of LR type. If L and R are concave functions and f is convex function , then

$$(\xi_1+\cdots+\xi_n)(nz)=f^{[-1]}(nf(\xi(z))).$$

We now prove the main result which generalizes Theorem 1.

Theorem 2. Let T be an Archimedean t-norm with additive generator f and let $\xi=\xi_i=(a,\ \alpha,\ \beta)_{LR}$ be positive fuzzy numbers of LR type. If L and R are concave functions and f is convex function, then

$$(\xi_1+\cdots+\xi_n)(nz)=(\xi_1\cdots\xi_n)(z^n)=f^{[-1]}(nf(\xi(z))).$$

Proof. Let $z\geq 0$ be arbitrary fixed. From Lemma 1, it suffices to prove that

$$(\xi_1 \cdots \xi_n)(z) = f^{[-1]}(nf(\xi(z^{\frac{1}{n}}))).$$

As mentioned in (1),

$$(\xi_1 \cdots \xi_n)(z) = f^{[-1]}\left(\inf_{x_1 \cdots x_n = z}\left(\sum_{i=1}^{n} f(\xi_i(x_i))\right)\right).$$

By setting $w_i = \ln x_i$, we have

$$f^{[-1]}\left(\inf_{x_1 \cdots x_n = z}\left(\sum_{i=1}^{n} f(\xi_i(x_i))\right)\right) = f^{[-1]}\left(\inf_{w_1 + \cdots + w_n = \ln z}\left(\sum_{i=1}^{n} f(\xi_i(\exp(w_i)))\right)\right).$$

By the convex decreasing property of f and the concavity of ξ, we obtain

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} f(\xi_i(\exp(w_i))) &\geq f\left(\xi_i\left(\frac{1}{n}\sum_{i=1}^{n}(\exp(w_i))\right)\right) \\ &\geq f\left(\xi_i\left(\exp(\frac{1}{n}\sum_{i=1}^{n} w_i)\right)\right) \\ &= f\left(\xi\left(z^{\frac{1}{n}}\right)\right).\end{aligned}$$

By taking $w_i = \frac{1}{n}\ln z$, $i = 1, \cdots, n$, we have

$$\inf_{x_1 \cdots x_n = z}\left(\sum_{i=1}^{n} f(\xi_i(x_i))\right) = \inf_{w_1 + \cdots + w_n = \ln z}\left(\sum_{i=1}^{n} f(\xi_i(\exp(w_i)))\right) = nf\left(\xi(z^{\frac{1}{n}})\right),$$

which completes the proof.

For the case of non-positive fuzzy numbers, we consider the following known result for 0-symmetric fuzzy number.

Lemma 2[5]. Let T be an Archimedean t-norm with additive generator f and let $\tilde{0}_s = (0, \alpha, \alpha)_R$ be a symmetric fuzzy number. If R is concave and f is convex, then the membership function of T-product $\tilde{0}_s \cdots \tilde{0}_s$ is given by

$$\tilde{0}_s \cdots \tilde{0}_s(z) = \begin{cases} f^{-1}(nf(R(z^{\frac{1}{n}}))) & \text{if } \quad |z| \in [-\alpha^n \alpha^n], \\ 0 & \textit{otherwise} \quad . \end{cases}$$

Using this result, the following result is immediate.

Theorem 3. Let T be an Archimedean t-norm with additive generator f and let $\xi = \xi_i = (0, \alpha, \alpha)_R$ be 0-symmetric fuzzy numbers of R type. If R is concave function and f is convex function , then

$$(\xi_1 + \cdots + \xi_n)(nz) = (\xi_1 \cdots \xi_n)(z^n) = f^{[-1]}(nf(\xi(z))).$$

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