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Estimation for the Weibull Distribution Based on Censored Samples

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Abstract

We consider the problem of estimating the scale and shape parameters in the Weibull distribution based on censored samples. We propose the approximate maximum likelihood estimators (AMLEs) of the scale and shape parameters in the Weibull distribution based on Type-II censored samples. We compare the proposed estimators in the sense of the mean squared error (MSE).

Keywords : Approximate maximum likelihood estimator, Extreme value distribution, Type-II censored sample, Weibull distribution

1. Introduction

The probability density function (pdf) of the random variable X having the Weibull distribution is given by

$$f(x)=\frac{\delta}{\theta^{\delta}}x^{\delta-1}\exp\left\{-\left(\frac{x}{\theta}\right)^{\delta}\right\},\quad x>0,\ \theta>0,\ \delta>0. \tag{1.1}$$

The estimation of the parameters in the censored samples, has been investigated by many authors such as Balakrishnan (1989), Balakrishnan and Cohen (1991), and Fei and Kong (1995). Balakrishnan and Varadan (1990) obtained the AMLEs of the location and scale parameters in the extreme value distribution with censoring. Balakrishnan, Gupta, and Panchapakesan (1995) obtained the AMLEs of the location and scale parameters in the extreme value distribution based on multiply

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Type-II censored samples.

Kang (1996, 2003) obtained the AMLE for the scale parameter of the double exponential distribution based on Type-II censored samples and the AMLEs of the location and scale parameters of the exponential distribution based on multiple Type-II censored samples.

Recently, Kang, Lee, and Han (2004) introduced the AMLE of the scale parameter in the Weibull distribution based on multiply Type-II censored samples.

In this paper, we propose the AMLEs of the scale and shape parameters in the Weibull distribution based on Type-II censored samples by using the relationship of Weibull and extreme value distributions. We compare the proposed estimators in the sense of MSE.

2. Estimation of the parameters

Consider the Weibull distribution with the pdf (1.1) and the cumulative distribution function (cdf)

$$F(x) = 1 - \exp\left\{-\left(\frac{x}{\theta}\right)^{\delta}\right\}, \quad x > 0. \tag{2.1}$$

Let

$$X_{r+1:n} \le X_{r+2:n} \le \cdots \le X_{n-s:n} \tag{2.2}$$

be a doubly Type-II censored sample from the Weibull distribution with pdf (1.1), where the first r and the last s observations are censored.

The maximum likelihood method for the parameters of the Weibull distribution based on Type-II censored samples does not admit explicit solutions and it is difficult. So we will use the transformation which changes the Weibull distribution into the extreme value distribution.

Let X be a random variable with pdf (1.1), then the pdf of $Y = \ln X$ is

$$g(y) = \frac{1}{\sigma} e^{\frac{y-\mu}{\sigma}} \exp\left(-e^{\frac{y-\mu}{\sigma}}\right), \quad -\infty < y < \infty, \tag{2.3}$$

where $\sigma = \frac{1}{\delta}$, $\mu = \ln\theta$ and the cdf of Y is

$$G(y) = 1 - \exp\left(-e^{\frac{y-\mu}{\sigma}}\right), \quad -\infty < y < \infty. \tag{2.4}$$

That is, Y has the extreme value distribution with the location parameter μ and the scale parameter σ.

The likelihood function based on the doubly Type-II censored sample $Y_{r+1:n}$, $Y_{r+2:n}, \ldots, Y_{n-s:n}$ can be written as

$$L = \frac{n!}{r!s!}\{G(Y_{r+1:n})\}^r\{1-G(Y_{n-s:n})\}^s \prod_{i=r+1}^{n-s} g(Y_{i:n}). \tag{2.5}$$

By putting $Z_{j:n} = \dfrac{Y_{j:n}-\mu}{\sigma}$, the likelihood function (2.5) can be rewritten as

$$L = \frac{n!}{r!s!}\sigma^{-(n-r-s)}\{G(Z_{r+1:n})\}^r\{1-G(Z_{n-s:n})\}^s \prod_{i=r+1}^{n-s} g(Z_{i:n}), \tag{2.6}$$

where $g(z) = e^z \exp(-e^z)$ and $G(z) = 1-\exp(-e^z)$ are the pdf and cdf of the standard extreme value distribution, respectively.

Therefore, we have the log-likelihood function as follows;

$$\begin{aligned} \ln L = & \ln\frac{n!}{r!s!} + r\ln\{G(Z_{r+1:n})\} + s\ln\{1-G(Z_{n-s:n})\} \\ & + \sum_{i=r+1}^{n-s} \ln g(Z_{i:n}) - (n-s-r)\ln\sigma. \end{aligned} \tag{2.7}$$

First, we want to obtain the estimators of the shape parameter in the Weibull distribution when the scale parameter is known. We also simulate and compare the MSEs of the proposed estimators through the Monte Carlo simulations.

2.1 Estimation of the shape parameter when the scale parameter is known

We shall find the estimator of the scale parameter in the extreme value distribution with pdf (2.3) when the location parameter μ is known.

On differentiating the log-likelihood function (2.7) with respect to σ in turn and equating to zero, we obtain the likelihood equation as

$$\begin{aligned} \frac{\partial \ln L}{\partial\sigma} = & -\frac{1}{\sigma}\Bigg[r\frac{g(Z_{r+1:n})}{G(Z_{r+1:n})} Z_{r+1:n} - s\frac{g(Z_{n-s:n})}{1-G(Z_{n-s:n})} Z_{n-s:n} \\ & + \sum_{i=r+1}^{n-s} \frac{g'(Z_{i:n})}{g(Z_{i:n})} Z_{i:n} + (n-s-r)\Bigg] = 0. \end{aligned} \tag{2.8}$$

Since the likelihood equation (2.8) is very complicated, the equation does not admit an explicit solution for σ. But if we let

$$p_i = \frac{i}{n+1}, \quad G^{-1}(p_i) = \ln(-\ln(1-p_i)) \equiv \xi_i,$$

then we can expand the following functions in Taylor series around the points

ξ_{r+1}, ξ_{n-s}, and ξ_i respectively;

$$\frac{g(Z_{r+1:n})}{G(Z_{r+1:n})} Z_{r+1:n} \simeq \alpha_{1r} + \beta_{1r} Z_{r+1:n} \tag{2.9}$$

$$\frac{g(Z_{n-s:n})}{1-G(Z_{n-r:n})} Z_{n-r:n} \simeq -e^{\xi_{n-s}} \xi_{n-s}^2 + e^{\xi_{n-s}}(\xi_{n-s}+1) Z_{n-s:n} \tag{2.10}$$

and

$$\frac{g'(Z_{i:n})}{g(Z_{i:n})} Z_{i:n} \simeq e^{\xi_i}\xi_i^2 + [1-e^{\xi_i} - e^{\xi_i}\xi_i] Z_{i:n}. \tag{2.11}$$

where

$$\alpha_{1r} = -\frac{\exp(\xi_{r+1})\exp(-\exp^{\xi_{r+1}})[1-\exp(\xi_{r+1}) - \exp\{-\exp(\xi_{r+1})\}]}{[\exp(-\exp^{\xi_{r+1}})]^2} \xi_{r+1}^2$$

$$\beta_{1r} = \frac{\exp(\xi_{r+1})\exp(-\exp^{\xi_{r+1}})[1-\exp(\xi_{r+1}) - \exp\{-\exp(\xi_{r+1})\}]}{[1-\exp(-\exp^{\xi_{r+1}})]^2} \xi_{r+1} + \frac{\exp(\xi_{r+1})\exp(-\exp^{\xi_{r+1}})}{1-\exp(-\exp^{\xi_{r+1}})}.$$

By substituting the equations (2.9), (2.10), and (2.11) into the equation (2.8), we obtain the approximate likelihood equation for σ as follows;

$$\frac{d\ln L}{d\sigma} \simeq \frac{1}{\sigma}\Big[r(\alpha_{1r} + \beta_{1r} Z_{r+1:n}) - s\{-e^{\xi_{n-s}}\xi_{n-s}^2 + e^{\xi_{n-s}}(\xi_{n-s}+1) Z_{n-s:n}\} + \sum_{i=r+1}^{n-s}\{e^{\xi_i}\xi_i^2 + (1-e^{\xi_i} - e^{\xi_i}\xi_i) Z_{i:n}\} + (n-s-r)\Big] = 0. \tag{2.12}$$

Upon solving the equation (2.12) for σ, we can derive an estimator of σ as follows;

$$\hat{\sigma}_1 = \frac{1}{r\alpha_{1r} + se^{\xi_{n-s}} + \sum_{i=r+1}^{n-s} e^{\xi_i}\xi_i^2 + (n-s-r)} \Big[r\beta_{1r}(Y_{r+1:n} - \mu) - se^{\xi_{n-s}}(\xi_{n-s}+1)(Y_{n-s:n} - \mu) + \sum_{i=r+1}^{n-s}(1-e^{\xi_i} - e^{\xi_i}\xi_i)(Y_{i:n} - \mu)\Big]. \tag{2.13}$$

This estimator is a linear function of the order statistics, so we can calculate the moments of this estimator.

Now, we derive the single moments and the product moments of the order statistics by using the formulas of Gradshteyn and Ryzhik (1965) and Balakrishinan and Chan (1992). The results are given by

$$E(Z_{i:n}) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j} \times\left[-\frac{1}{n-j}(\gamma+\ln(n-j))\right], \tag{2.14}$$

$$E(Z^2_{i:n}) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{j=0}^{i-1}(-1)^{i-j-1} \times\binom{i-1}{j}\frac{1}{n-j}\left[\frac{\pi^2}{6}+(\gamma+\ln(n-j))^2\right], \tag{2.15}$$

$$E(Z_{i:n}Z_{j:n}) = \frac{n!}{(i-1)!}\sum_{l=0}^{j-i-1}(-1)^{i-j-1} \times\sum_{m=0}^{n-j}\frac{(-1)^{l+m}}{l!(j-i-1-l)!\,m!(n-j-m)!} \times\phi(i+l, j-i-l+\mu), \tag{2.16}$$

where the function ϕ is the double integral

$$\phi(t,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} xye^{x-te^x}e^{y-ue^y}dxdy, \quad t>0, \ u>0.$$

From the equations (2.14), (2.15), and (2.16), we can compute the expectation and the variance of the estimator $\hat{\sigma}_1$ as

$$\begin{aligned} E(\hat{\sigma}_1) &= \frac{1}{A_1}[-r\beta_{1r}E(Y_{r+1:n}-\mu)+se^{\xi_{n-s}}(\xi_{n-s}+1)E(Y_{n-s:n}-\mu) \\ &\quad -\sum_{i=r+1}^{n-s}(1-e^{\xi_i}-e^{\xi_i}\xi_i)E(Y_{i:n}-\mu)] \\ &= \frac{\sigma}{A_1}[-r\beta_{1r}E(Z_{r+1:n})+se^{\xi_{n-s}}(\xi_{n-s}+1)E(Z_{n-s:n}) \\ &\quad -\sum_{i=r+1}^{n-s}(1-e^{\xi_i}-e^{\xi_i}\xi_i)E(Z_{i:n})], \end{aligned} \tag{2.17}$$

$$\begin{aligned} \mathrm{Var}(\hat{\sigma}_1) &= \frac{\sigma^2}{A_1^2}\Big[r^2\beta_{1r}^2\{E(Z^2_{r+1:n})-(E(Z_{r+1:n}))^2\} \\ &\quad +(se^{\xi_{n-s}})^2(\xi_{n-s}+1)^2\{E(Z^2_{n-s:n})-(E(Z_{n-s:n}))^2\} \\ &\quad -\sum_{i=r+1}^{n-s}(1-e^{\xi_i}-e^{\xi_i}\xi_i)^2\{E(Z^2_{i:n})-(E(Z_{i:n}))^2\} \\ &\quad -2r\beta_{1r}(se^{\xi_{n-s}})(\xi_{n-s}+1)\{E(Z_{r+1:n}Z_{i:n})-E(Z_{r+1:n})E(Z_{i:n})\} \\ &\quad -2r\beta_{1r}\sum_{i=r+1}^{n-s}(1-e^{\xi_i}-e^{\xi_i}\xi_i)\{E(Z_{r+1:n}Z_{n-s:n}) \\ &\quad -E(Z_{r+1:n})E(Z_{n-s:n})\}-2(se^{\xi_{n-s}})(\xi_{n-s}+1) \\ &\quad \times\sum_{i=r+1}^{n-s}(1-e^{\xi_i}-e^{\xi_i}\xi_i)\{E(Z_{i:n}Z_{n-s:n})-E(Z_{i:n})E(Z_{n-s:n})\}\Big], \end{aligned} \tag{2.18}$$

where

$$A_{1}= - r\alpha_{1r} + se^{\xi_{n-s}} - \sum_{i=r+1}^{n-s} e^{\xi_i}\xi_i^2 + (n-s-r).$$

Since $\sigma=1/\delta$, we can obtain an estimator of the shape parameter δ in the Weibull distribution as follows;

$$\hat{\delta}_1 = 1/\hat{\sigma}_1. \tag{2.19}$$

It's difficult to find the expectation or variance of the estimator $\hat{\delta}_1$, so we will simulate the MSEs for the proposed estimator $\hat{\delta}_1$ of δ.

We can propose another estimator of the parameter δ by the similar method.

In the equation (2.8), we can expand the following functions in Taylor series around the points ξ_{r+1}, ξ_{n-s}, and ξ_i, respectively.

$$\frac{g(Z_{r+1:n})}{G(Z_{r+1:n})} \simeq \alpha_{2r} + \beta_{2r} Z_{r+1:n}, \tag{2.20}$$

$$\frac{g(Z_{n-s:n})}{1-G(Z_{n-s:n})} \simeq e^{\xi_{n-s}}(1-\xi_{n-s}) + e^{\xi_{n-s}} Z_{n-s:n}, \tag{2.21}$$

and

$$\frac{g'(Z_{i:n})}{g(Z_{i:n})} \simeq 1 - e^{\xi_i} + e^{\xi_i}\xi_i - e^{\xi_i} Z_{i:n}, \tag{2.22}$$

where

$$\alpha_{2r} = \frac{e^{\xi_{r+1}}\exp(-e^{\xi_{r+1}})}{1-\exp(-e^{\xi_{r+1}})} - \frac{e^{\xi_{r+1}}\exp(-e^{\xi_{r+1}})[1-e^{\xi_{r+1}}\exp(-e^{\xi_{r+1}})]}{[1-\exp(-e^{\xi_{r+1}})]^2}\xi_{r+1}$$

$$\beta_{2r} = \frac{e^{\xi_{r+1}}\exp(-e^{\xi_{r+1}})[1-e^{\xi_{r+1}}\exp(-e^{\xi_{r+1}})]}{[1-\exp(-e^{\xi_{r+1}})]^2}\xi_{r+1}.$$

By substituting the equations (2.20), (2.21), and (2.22) into the equation (2.8), we obtain the approximate likelihood equation for σ as follows;

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} \simeq -\frac{1}{\sigma}\Big[& r(\alpha_{2r}+\beta_{2r}Z_{r+1:n})Z_{r+1:n} - s\{e^{\xi_{n-s}}(1-\xi_{n-s}) \\ & + e^{\xi_{n-s}}Z_{n-s:n}\}Z_{n-s:n} + \sum_{i=r+1}^{n-s}\{(1-e^{\xi_i}+e^{\xi_i}\xi_i \\ & - e^{\xi_i}Z_{i:n})\}Z_{i:n} + (n-s-r)\Big] = 0.\end{aligned} \tag{2.23}$$

From the equation (2.23), we obtain the quadratic equation for σ as follows;

$$A_2\sigma^2 + B_2\sigma + C_2 = 0 \tag{2.24}$$

where

$$\begin{aligned}
A_2 &= n-r-s, \\
B_2 &= r\alpha_{2r}(Y_{r+1:n}-\mu) - se^{\xi_{n-s}}(1-\xi_{n-s})(Y_{n-s:n}-\mu) \\
&\quad + \sum_{i=r+1}^{n-s}(1-e^{\xi_i}-e^{\xi_i}\xi_i)(Y_{i:n}-\mu), \\
C_2 &= r\beta_{2r}(Y_{r+1:n}-\mu) - se^{\xi_{n-s}}(Y_{n-s:n}-\mu) - \sum_{i=r+1}^{n-s}e^{\xi_i}(Y_{i:n}-\mu).
\end{aligned}$$

On solving the equation (2.24) for σ, we derive another estimator of σ as

$$\hat{\sigma}_2 = \frac{-B_2 + \sqrt{B_2^2 - 4A_2C_2}}{2A_2}. \tag{2.25}$$

Therefore, we also obtain the other estimator of δ as follows;

$$\hat{\delta}_2 = 1/\hat{\sigma}_2. \tag{2.26}$$

2.2 Estimation of the shape and the scale parameters

Now, we shall obtain the estimators of the shape and scale parameters in the Weibull distribution with pdf (1.1).

On differentiating the log-likelihood function (2.7) with respect to μ, we obtain the likelihood equation as

$$\frac{\partial \ln L}{\partial \mu} = -\frac{1}{\sigma}\left[r\frac{g(Z_{r+1:n})}{G(Z_{r+1:n})} - s\frac{g(Z_{n-s:n})}{1-G(Z_{n-s:n})} + \sum_{i=r+1}^{n-s}\frac{g'(Z_{i:n})}{g(Z_{i:n})}\right] = 0. \tag{2.27}$$

The likelihood equations (2.8) and (2.27) do not admit explicit solutions for the parameters. To obtain the explicit solutions in the likelihood equations, we use the Taylor series. By substituting the equations (2.20), (2.21), and (2.22) into the equation (2.27), we obtain the approximate likelihood equation for μ as

$$\begin{aligned}
\frac{\partial \ln L}{\partial \mu} \simeq\ & r(\alpha_{2r}+\beta_{2r}Z_{r+1:n}) - s\{e^{\xi_{n-s}}(1-\xi_{n-s}) + e^{\xi_{n-s}}Z_{n-s:n}\} \\
& + \sum_{i=r+1}^{n-s}\{(1-e^{\xi_i}+e^{\xi_i}\xi_i - e^{\xi_i}Z_{i:n})\} = 0.
\end{aligned} \tag{2.28}$$

Upon solving the equations (2.12) and (2.28), we derive the approximate MLEs of μ and σ as

$$\hat{\mu} = \frac{D'F - DF'}{D'E - DE'} \tag{2.29}$$

and

$$\hat{\sigma} = \frac{1}{r\alpha_{1r} + se^{\xi_{n-s}} + \sum_{i=r+1}^{n-s} e^{\xi_i}\xi_i^2 + (n-s-r)} \Big[r\beta_{1r}(Y_{r+1:n} - \hat{\mu}) - se^{\xi_{n-s}}(\xi_{n-s}+1)(Y_{n-s:n} - \hat{\mu}) + \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} - e^{\xi_i}\xi_i)(Y_{i:n} - \hat{\mu}) \Big] \tag{2.30}$$

where

$$D = r\alpha_{2r} - s(e^{\xi_{n-s}} - e^{\xi_{n-s}}\xi_{n-s}) + \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} + e^{\xi_i}\xi_i),$$

$$D' = r\alpha_{1r} + se^{\xi_{n-s}}\xi_{n-s}^2 + \sum_{i=r+1}^{n-s} e^{\xi_i}\xi_i^2 + (n-r-s),$$

$$E = r\beta_{2r} - se^{\xi_{n-s}} - \sum_{i=r+1}^{n-s} e^{\xi_i},$$

$$E' = r\beta_{1r} - s(e^{\xi_{n-s}} + e^{\xi_{n-s}}\xi_{n-s}) + \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} - e^{\xi_i}\xi_i),$$

$$F = r\beta_{2r}Y_{r+1:n} + se^{\xi_{n-s}}Y_{n-s:n} - \sum_{i=r+1}^{n-s} e^{\xi_i}Y_{i:n},$$

and

$$F' = r\beta_{1r}Y_{r+1:n} - s(e^{\xi_{n-s}} + e^{\xi_{n-s}}\xi_{n-s})Y_{n-s:n} + \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} - e^{\xi_i}\xi_i)Y_{i:n}.$$

Since $\mu = \ln\theta$ and $\delta = 1/\sigma$, we can obtain the AMLEs of the scale parameter θ and the shape parameter δ in the Weibull distribution with pdf (1.1) as follows;

$$\hat{\theta} = e^{\hat{\mu}} \text{ and } \hat{\delta} = 1/\hat{\sigma}.$$

3. The simulated results

From the proposed estimators, we can see that $\hat{\sigma}_1$ is simpler than $\hat{\sigma}_2$ and $\hat{\sigma}_1$ is a linear function of the order statistics. It's difficult to find the moments of the estimators $\hat{\delta}_1$ and $\hat{\delta}_2$, so we simulate the MSEs of the proposed two estimators of δ by Monte Carlo simulation method. The simulation procedure is repeated 10,000 times for the sample size $n = 15, 20, 30, 50$ and various choice of censoring.

The MSEs of $\hat{\delta}_1$ and $\hat{\delta}_2$ are the same, regardless of value of the scale parameter, because the scale parameter for the Weibull distribution is the function of the location parameter for the extreme value distribution. The MSEs of $\hat{\delta}_2$ are smaller than those of $\hat{\delta}_1$, so $\hat{\delta}_2$ is more efficient than $\hat{\delta}_1$.

We simulate MSEs for the proposed estimators of δ and θ by Monte Carlo simulation method when the scale and shape parameters are both unknown. The MSEs of the proposed estimators decrease as the sample of size increases and the number of censoring decreases.

References

1. Balakrishnan, N. (1989). Approximate MLE of the scale parameter of the Rayleigh distribution with censoring, *IEEE Transactions on Reliability*, 38, 355-357.
2. Balakrishinan, N. and Chan P. S. (1992). Order statistics from extreme value distribution, I : Tables of means, variances and covariances. *Communications in Statistics,* 21(4), 1199-1217.
3. Balakrishnan, N. and Cohen, A. C. (1991). *Order Statistics and Inference* : Estimation Method, Academic press, Inc.
4. Balakrishnan, N., Gupta, S. S., and Panchapakesan, S. (1995). Estimation of the location and scale parameters of the extreme value distribution based on multiply Type-II censored samples. *Communications in Statistics, Theory and Methods,* 24, 2105-2125.
5. Fei, H. L. and Kong, F. H. (1995). Estimation for 2-parameter Weibull distribution and extreme-value distribution under multiply Type II censoring, *Communications in Statistics, Theory and Method,* 24, 2087-2104.
6. Gradshteyn, I. S. and Ryzhik, I. M. (1965). *Tables of Integrals*, *Series, and Products*, Academic Press, New York.
7. Kang S. B. (1996). Approximate MLE for the scale parameter of the double exponential distribution based on Type-II censoring, *Journal of Korean Mathematical Society,* 33(1), 69-79.
8. Kang S. B. (2003). Approximate MLEs for exponential distribution under multiple Type-II censoring. *Journal of Korean Data & Information Science Society,* 14, 983-988.
9. Kang S. B., Lee, H. J., and Han, J. T. (2004). Estimation of Weibull scale parameter based on multiply Tpye-II censored samples, *Journal of Korean Data & Information Science Society,* 15(3), 593-603.

Table 1. The relative MSEs of $\hat{\delta}_1$ and $\hat{\delta}_2$ when the scale parameter θ is known.

n		15		20	
r	s	$\hat{\delta}_1$	$\hat{\delta}_2$	$\hat{\delta}_1$	$\hat{\delta}_2$
0	0	0.088188	0.053826	0.054065	0.036727
0	1	0.099702	0.062486	0.059802	0.041733
0	2	0.119040	0.073537	0.066740	0.046613
0	3	0.142528	0.087976	0.074769	0.052062
1	0	0.098218	0.058933	0.057949	0.038845
2	0	0.109371	0.064040	0.062061	0.041324
3	0	0.127490	0.072071	0.067308	0.044196
1	1	0.112697	0.069067	0.064418	0.044321
1	2	0.137398	0.081949	0.072905	0.050174
1	3	0.167825	0.099416	0.082647	0.056539
2	1	0.128615	0.076704	0.069651	0.047498
2	2	0.160454	0.093005	0.079701	0.054277
2	3	0.199252	0.114191	0.091098	0.061397
3	1	0.154315	0.088705	0.076247	0.051188
3	2	0.197553	0.110093	0.087879	0.058652
3	3	0.255898	0.138938	0.102275	0.067290
n		30		50	
r	s	$\hat{\delta}_1$	$\hat{\delta}_2$	$\hat{\delta}_1$	$\hat{\delta}_2$
0	0	0.030512	0.022992	0.015205	0.012655
0	1	0.032348	0.024947	0.015922	0.013530
0	2	0.035084	0.027169	0.016675	0.014247
0	3	0.038199	0.029635	0.017595	0.015067
1	0	0.031796	0.023834	0.015523	0.012906
2	0	0.033012	0.024646	0.015891	0.013194
3	0	0.034498	0.025619	0.016185	0.013414
1	1	0.033801	0.025924	0.016250	0.013795
1	2	0.036712	0.028249	0.017028	0.014531
1	3	0.040156	0.030968	0.017979	0.015374
2	1	0.035181	0.026874	0.016652	0.014120
2	2	0.038388	0.029394	0.017473	0.014892
2	3	0.042156	0.032357	0.018452	0.015761
3	1	0.036844	0.027984	0.016960	0.014353
3	2	0.040315	0.030690	0.017818	0.015159
3	3	0.044541	0.034023	0.018842	0.016064

Table 2. The relative MSEs of $\hat{\delta}$ and $\hat{\theta}$

n		15		20	
r	s	$\hat{\theta}$	$\hat{\delta}$	$\hat{\theta}$	$\hat{\delta}$
0	0	0.077962	0.090433	0.057115	0.055954
0	1	0.080494	0.100332	0.057133	0.055847
0	2	0.084947	0.119138	0.057218	0.055667
0	3	0.091403	0.145386	0.057458	0.055452
1	0	0.077987	0.101326	0.057169	0.060135
2	0	0.078351	0.113600	0.057367	0.064762
3	0	0.078515	0.132631	0.057508	0.070666
1	1	0.080421	0.113672	0.057189	0.060014
1	2	0.084762	0.137470	0.057280	0.059813
1	3	0.091147	0.170781	0.057522	0.059574
2	1	0.080582	0.130092	0.057391	0.064625
2	2	0.084731	0.160484	0.057481	0.064403
2	3	0.090973	0.202747	0.057730	0.064138
3	1	0.080566	0.156005	0.057532	0.070512
3	2	0.084472	0.197417	0.057634	0.070259
3	3	0.090601	0.261552	0.057886	0.069955
n		30		50	
r	s	$\hat{\theta}$	$\hat{\delta}$	$\hat{\theta}$	$\hat{\delta}$
0	0	0.037565	0.031677	0.022142	0.016154
0	1	0.037570	0.031672	0.022144	0.016153
0	2	0.037582	0.031659	0.022146	0.016155
0	3	0.037582	0.031645	0.022137	0.016140
1	0	0.037591	0.033143	0.022177	0.016535
2	0	0.037588	0.034490	0.022202	0.016958
3	0	0.037697	0.036219	0.022214	0.017293
1	1	0.037596	0.033139	0.022179	0.016535
1	2	0.037610	0.033129	0.022182	0.016537
1	3	0.037608	0.033109	0.022173	0.016522
2	1	0.037593	0.034485	0.022204	0.016958
2	2	0.037607	0.034477	0.022207	0.016959
2	3	0.037605	0.034456	0.022197	0.016944
3	1	0.037702	0.036217	0.022216	0.017293
3	2	0.037717	0.036211	0.022219	0.017294
3	3	0.037716	0.036190	0.022210	0.017281

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