

Bayesian Test for the Difference of Exponential Guarantee Time Parameters

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Abstract

When X and Y have independent two parameter exponential distributions, we develop a Bayesian testing procedures for the equality of two location parameters. The reference prior in non-regular exponential model is derived. Under this reference prior, we propose a Bayesian test procedures for the equality of two location parameters using fractional Bayes factor and intrinsic Bayes factor. Simulation study and some real data examples are provided.

Keywords : Exponential Location Parameter, Fractional Bayes Factor, Intrinsic Bayes Factor, Reference Prior

1. INTRODUCTION

The two parameter exponential distribution plays an important role in the field of life testing and reliability theory since it is the only continuous distribution with a constant hazard function. The reciprocal of the scale parameter is the hazard rate. The location (threshold) parameter can translate the distribution along the time axis, so it is also known as the minimum life or guarantee time parameter. The guarantee time parameter can be used to model warranty periods for some products.

The present paper focuses on Bayesian testing procedure for the equality of two location parameters. In Bayesian testing problem, the Bayes factor under proper

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priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' priors or reference priors (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the fractional Bayes factor. To remove the arbitrariness in Bayes factor, he used to a portion of the likelihood with a so-called the fraction b (See Sang Gil Kang and Hee Chun Lee (2004)). These two approaches mentioned above have shown to be quite useful in many statistical areas.

For the statistical inference of the exponential distribution, Epstein and Tsao (1953) studied some statistical tests for two exponential distributions. Epstein and Sobel (1954) obtained the minimum variance unbiased estimator for the scale and location parameters. The shrinkage estimators for the scale parameter have been proposed by Bhattacharya and Srivastava (1974). Chiou and Han (1989) proposed a pre-test estimator and a pre-test shrinkage estimator for the location parameter. Chiou and Miao (2004) studied the shrinkage estimator for the difference between location parameters.

Almost all the work mentioned above is the analysis based on the classical point of view, there is a little work on this problem from the viewpoint of the objective Bayesian framework. Because the two parameter exponential distribution is the non-regular case, so the noninformative priors such as reference prior or probability matching prior were hard to derive. Since almost all the theory related to these priors were developed based on the assumption of regular distribution. Recently, Ghosal (1997, 1999) developed the procedures to derive the reference and matching priors for non-regular cases. Using his results, we feel a strong necessity to develop objective Bayesian testing procedure for the difference between location parameters. For dealing this problem, we use the fractional Bayes factor (O'Hagan, 1995) and the intrinsic Bayes factor (Berger and Pericchi, 1996).

The outline of the remaining sections is as follows. In Section 2, using the reference priors, we provide the Bayesian testing procedure based on the fractional Bayes factor and intrinsic Bayes factor for the testing equality of two location parameters. In Section 3, simulation study and some real examples are given. Section 4 devotes some conclusions of our Bayesian test procedure.

2. BAYESIAN TEST PROCEDURES

2.1 Preliminaries

Models (or Hypotheses) H_1, H_2, \dots, H_q are under consideration, with the data $\mathbf{x} = (x_1, x_2, \dots, x_n)$ having probability density function $f_i(\mathbf{x} | \boldsymbol{\theta}_i)$ under model $H_i, i=1, 2, \dots, q$. The parameter vectors $\boldsymbol{\theta}_i$ are unknown. Let $\pi_i(\boldsymbol{\theta}_i)$ be the prior distribution of model H_i , and let p_i be the prior probabilities of model $H_i, i=1, 2, \dots, q$. Then the posterior probability that the model H_i is true is

$$P(H_i | \mathbf{x}) = \left(\sum_{j=1}^q \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \tag{1}$$

where B_{ji} is the Bayes factor of model H_j to model H_i defined by

$$B_{ji} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})} = \frac{\int f_j(\mathbf{x} | \boldsymbol{\theta}_j) \pi_j(\boldsymbol{\theta}_j) d\boldsymbol{\theta}_j}{\int f_i(\mathbf{x} | \boldsymbol{\theta}_i) \pi_i(\boldsymbol{\theta}_i) d\boldsymbol{\theta}_i}. \tag{2}$$

The B_{ji} interpreted as the comparative support of the data for the model j to i . The computation of B_{ji} needs specification of the prior distribution $\pi_i(\boldsymbol{\theta}_i)$ and $\pi_j(\boldsymbol{\theta}_j)$. Usually, one can use the noninformative prior, often improper, such as uniform prior, Jeffreys prior, reference prior or probability matching prior. Denote it as π_i^N . The use of improper priors $\pi_i^N(\cdot)$ in (2) causes the B_{ji} to contain unspecified constants. To solve this problem, O'Hagan (1995) proposed the fractional Bayes factor for Bayesian testing and model selection problem as follow.

When the $\pi_i^N(\boldsymbol{\theta}_i)$ is noninformative prior under H_i , equation (2) becomes

$$B_{ji}^N = \frac{\int f_j(\mathbf{x} | \boldsymbol{\theta}_j) \pi_j^N(\boldsymbol{\theta}_j) d\boldsymbol{\theta}_j}{\int f_i(\mathbf{x} | \boldsymbol{\theta}_i) \pi_i^N(\boldsymbol{\theta}_i) d\boldsymbol{\theta}_i}.$$

Then the fractional Bayes factor of model H_j versus model H_i is

$$B_{ji}^F = B_{ji}^N \cdot \frac{\int f_i^b(\mathbf{x} | \boldsymbol{\theta}_i) \pi_i^N(\boldsymbol{\theta}_i) d\boldsymbol{\theta}_i}{\int f_j^b(\mathbf{x} | \boldsymbol{\theta}_j) \pi_j^N(\boldsymbol{\theta}_j) d\boldsymbol{\theta}_j},$$

and $f_i(\mathbf{x} | \boldsymbol{\theta}_i)$ is the likelihood function and b specifies a fraction of the likelihood which is to be used as a prior density. He proposed three ways for the choice of the fraction b . One frequently suggested choice is $b = m/n$, where m is the size of the minimal training sample, assuming this is well defined. (see O'Hagan, 1995, 1997 and the discussion by Berger and Mortera of O'Hagan, 1995).

Berger and Pericchi (1996) proposed the intrinsic Bayes factor (IBF) for Bayesian testing and model selection. The arithmetic intrinsic Bayes factor is given by

$$B_{ji}^I = B_{ji}^N \cdot \frac{1}{L} \sum_{l=1}^L B_{ij}^N(\mathbf{x}(l)),$$

where

$$B_{ij}^N(\mathbf{x}(l)) = \frac{m_i(\mathbf{x}(l))}{m_j(\mathbf{x}(l))} = \frac{\int f_i(\mathbf{x}(l) | \boldsymbol{\theta}_i) \pi_i^N(\boldsymbol{\theta}_i) d\boldsymbol{\theta}_i}{\int f_j(\mathbf{x}(l) | \boldsymbol{\theta}_j) \pi_j^N(\boldsymbol{\theta}_j) d\boldsymbol{\theta}_j}.$$

Here $\mathbf{x}(l)$ is minimal training sample and L is the number of all possible minimal training samples.

2.2 Bayesian Test

Let X be a two parameter exponential distribution with density function

$$f(x | \eta, \theta) = \frac{1}{\theta} \exp\left\{-\frac{x-\eta}{\theta}\right\}, \quad x > \eta, \theta > 0, \quad (3)$$

where η is the location parameter (guarantee parameter) and θ is the scale parameter. Suppose that $\mathbf{X} = (X_1, \dots, X_{n_1})$ is a random sample of size n_1 from a two parameter exponential distribution with location parameter η_1 and scale parameter θ_1 and $\mathbf{Y} = (Y_1, \dots, Y_{n_2})$ is a random sample of size n_2 from a two parameter exponential distribution with location parameter η_2 and scale parameter θ_2 . Then the joint probability density function is

$$f(\mathbf{x}, \mathbf{y} | \eta_1, \eta_2, \theta_1, \theta_2) = \theta_1^{-n_1} \theta_2^{-n_2} \exp\left\{-\frac{n_1(\bar{x} - \eta_1)}{\theta_1} - \frac{n_2(\bar{y} - \eta_2)}{\theta_2}\right\},$$

where $\theta_1 > 0$, $\theta_2 > 0$, $x_i > \eta_1$, $i = 1, \dots, n_1$ and $y_j > \eta_2$, $j = 1, \dots, n_2$, and \bar{x} and \bar{y} are the sample mean for each population.

We want to test the hypotheses $H_1: \eta_1 = \eta_2$ vs. $H_2: \eta_1 \neq \eta_2$. Our interest is to develop a Bayesian test based on the fractional and intrinsic Bayes factors for H_1 vs. H_2 under the noninformative priors. The two parameter exponential distribution is belong to non-regular cases. But for non-regular cases, the reference is developed by Ghosal (1997). In our model (3), the reference prior is given by $\pi(\eta, \theta) \propto 1/\theta$. For details, see Appendix 1.

2.2.1 Bayesian Test using the Fractional Bayes Factor

Under the hypothesis H_1 , the reference prior for $\eta (\equiv \eta_1 = \eta_2)$, θ_1 and θ_2 is

$$\pi_1(\eta, \theta_1, \theta_2) = \theta_1^{-1} \theta_2^{-1}.$$

The derivation of the reference prior for η , θ_1 and θ_2 , and the propriety of the posterior distribution under this reference prior are given in Appendix 1. The likelihood function under H_1 is

$$L(\eta, \theta_1, \theta_2 | \mathbf{x}, \mathbf{y}) = \theta_1^{-n_1} \theta_2^{-n_2} \exp\left\{-\frac{n_1(\bar{x} - \eta)}{\theta_1} - \frac{n_2(\bar{y} - \eta)}{\theta_2}\right\}$$

Then the element of fractional Bayes factor under H_1 is given by

$$\begin{aligned} m_1^b(\mathbf{x}, \mathbf{y}) &= \int_0^{m_{x,y}} \int_0^\infty \int_0^\infty L^b(\eta, \theta_1, \theta_2 | \mathbf{x}, \mathbf{y}) \pi_1(\eta, \theta_1, \theta_2) d\theta_1 d\theta_2 d\eta \\ &= (n_1 b)^{-n_1 b} (n_2 b)^{-n_2 b} \Gamma(n_1 b) \Gamma(n_2 b) \int_0^{m_{x,y}} (\bar{x} - \eta)^{-n_1 b} (\bar{y} - \eta)^{-n_2 b} d\eta, \end{aligned}$$

where $m_{x,y} = \min_{1 \leq i \leq n_1, 1 \leq j \leq n_2} \{x_i, y_j\}$.

For the H_2 , the reference prior for η_1, η_2, θ_1 and θ_2 is

$$\pi_2(\eta_1, \eta_2, \theta_1, \theta_2) = \pi(\eta_1, \theta_1) \pi(\eta_2, \theta_2) = \theta_1^{-1} \theta_2^{-1}.$$

The likelihood function under H_2 is

$$L(\eta_1, \eta_2, \theta_1, \theta_2 | \mathbf{x}, \mathbf{y}) = \theta_1^{-n_1} \theta_2^{-n_2} \exp\left\{-\frac{n_1(\bar{x} - \eta_1)}{\theta_1} - \frac{n_2(\bar{y} - \eta_2)}{\theta_2}\right\}.$$

Thus the element of fractional Bayes factor under H_2 gives as follows.

$$\begin{aligned}
& m_2^b(\mathbf{x}, \mathbf{y}) \\
&= \int_0^{m_x} \int_0^{m_y} \int_0^\infty \int_0^\infty L^b(\eta_1, \eta_2, \theta_1, \theta_2 \mid \mathbf{x}, \mathbf{y}) \pi_2(\eta_1, \eta_2, \theta_1, \theta_2) d\theta_2 d\theta_1 d\eta_2 d\eta_1 \\
&= \frac{1}{(n_1 b - 1)(n_2 b - 1)} (n_1 b)^{-n_1 b} (n_2 b)^{-n_2 b} \Gamma(n_1 b) \Gamma(n_2 b) \\
&\times [(\bar{x} - m_x)^{-n_1 b + 1} - (\bar{x})^{-n_1 b + 1}] [(\bar{y} - m_y)^{-n_2 b + 1} - (\bar{y})^{-n_2 b + 1}],
\end{aligned}$$

where $m_x = \min_{1 \leq i \leq n_1} \{x_i\}$ and $m_y = \min_{1 \leq j \leq n_2} \{y_j\}$. Therefore the B_{21}^N is given by

$$B_{21}^N = \frac{[(\bar{x} - m_x)^{-n_1 + 1} - (\bar{x})^{-n_1 + 1}] [(\bar{y} - m_y)^{-n_2 + 1} - (\bar{y})^{-n_2 + 1}]}{(n_1 - 1)(n_2 - 1)S(\mathbf{x}, \mathbf{y})}, \quad (4)$$

where

$$S(\mathbf{x}, \mathbf{y}) = \int_0^{m_{x,y}} (\bar{x} - \eta)^{-n_1} (\bar{y} - \eta)^{-n_2} d\eta.$$

And

$$\frac{m_1^b(\mathbf{x}, \mathbf{y})}{m_2^b(\mathbf{x}, \mathbf{y})} = \frac{(n_1 b - 1)(n_2 b - 1)S^b(\mathbf{x}, \mathbf{y})}{[(\bar{x} - m_x)^{-n_1 b + 1} - \bar{x}^{-n_1 b + 1}] [(\bar{y} - m_y)^{-n_2 b + 1} - \bar{y}^{-n_2 b + 1}]},$$

where

$$S^b(\mathbf{x}, \mathbf{y}) = \int_0^{m_{x,y}} (\bar{x} - \eta)^{-n_1 b} (\bar{y} - \eta)^{-n_2 b} d\eta.$$

Thus the fractional Bayes factor of H_2 versus H_1 is given by

$$\begin{aligned}
B_{21}^F &= \frac{(n_1 b - 1)(n_2 b - 1)S^b(\mathbf{x}, \mathbf{y})}{(n_1 - 1)(n_2 - 1)S(\mathbf{x}, \mathbf{y})} \\
&\times \frac{[(\bar{x} - m_x)^{-n_1 + 1} - (\bar{x})^{-n_1 + 1}] [(\bar{y} - m_y)^{-n_2 + 1} - (\bar{y})^{-n_2 + 1}]}{[(\bar{x} - m_x)^{-n_1 b + 1} - \bar{x}^{-n_1 b + 1}] [(\bar{y} - m_y)^{-n_2 b + 1} - \bar{y}^{-n_2 b + 1}]}.
\end{aligned}$$

Note that the calculation of the fractional Bayes factor of H_2 versus H_1 is requires only an one dimensional integration.

Remark. In the calculation of $m_1^b(\mathbf{x}, \mathbf{y})$, if $n_1 b$ is 1, then $m_1^b(\mathbf{x}, \mathbf{y})$ is $\log(\bar{x}/(\bar{x} - m_x))$. In the same manner, if $n_2 b$ is 1, then $m_2^b(\mathbf{x}, \mathbf{y})$ is $\log(\bar{y}/(\bar{y} - m_y))$.

2.2.2 Bayesian Test using the Intrinsic Bayes Factor

The element B_{21}^N , (4), of the intrinsic Bayes factor is computed in the fractional Bayes factor. So using minimal training sample, we only calculate the marginal densities $m^N(x_i, x_j, y_k, y_l)$ under H_1 and H_2 , respectively.

The marginal densities $m_1^N(x_i, x_j, y_k, y_l)$ under H_1 is given by

$$\begin{aligned} m_1^N(x_i, x_j, y_k, y_l) &= \int_0^{m_x} \int_0^\infty \int_0^\infty f(x_i, x_j, y_k, y_l | \eta, \theta_1, \theta_2) \pi_1(\eta, \theta_1, \theta_2) d\theta_1 d\theta_2 d\eta \\ &= \int_0^{m_x} (x_i + x_j - 2\eta)^{-2} (y_k + y_l - 2\eta)^{-2} d\eta \\ &\equiv T(x_i, x_j, y_k, y_l), \end{aligned}$$

where $1 \leq i < j \leq n_1, 1 \leq k < l \leq n_2$. And the marginal density $m_2^N(x_i, x_j, y_k, y_l)$ under H_2 is given by

$$\begin{aligned} &m_2^N(x_i, x_j, y_k, y_l) \\ &= \int_0^{m_x} \int_0^{m_y} \int_0^\infty \int_0^\infty f(x_i, x_j, y_k, y_l | \eta_1, \eta_2, \theta_1, \theta_2) \pi_2(\eta_1, \eta_2, \theta_1, \theta_2) d\theta_2 d\theta_1 d\eta_2 d\eta_1 \\ &= \frac{m_x m_y}{(x_i + x_j)(y_k + y_l)(x_i + x_j - 2m_x)(y_k + y_l - 2m_y)}. \end{aligned}$$

Therefore the IBF of H_2 versus H_1 is given by

$$\begin{aligned} B_{21}^I &= \frac{1}{L} \sum_{i,j} \sum_{k,l} \frac{(x_i + x_j)(y_k + y_l)(x_i + x_j - 2m_x)(y_k + y_l - 2m_y) T(x_i, x_j, y_k, y_l)}{(n_1 - 1)(n_2 - 1) S(\mathbf{x}, \mathbf{y})} \\ &\times \frac{[(\bar{x} - m_x)^{-n_1+1} - (\bar{x})^{-n_1+1}][(\bar{y} - m_y)^{-n_2+1} - (\bar{y})^{-n_2+1}]}{m_x m_y}, \end{aligned}$$

where $L = n_1(n_1 - 1)n_2(n_2 - 1)/4$. Note that the calculation of the IBF of H_2 versus H_1 is requires a one dimensional integration.

3. NUMERICAL STUDIES

In this section, we will show the usefulness of our test procedures by simulation and real data sets.

Example 1. To illustrate the Bayesian test procedures, we examine the cases when $(\theta_1, \theta_2) = (1, 3), (3, 1)$, $(\eta_1, \eta_2) = (1, 1), (1, 2), (1, 3)$ and $(n_1, n_2) = (5, 5)$

, (5, 10), (10, 10), (10, 20). The posterior probabilities of H_1 being true are computed assuming equal prior probabilities. The Table 1 shows that the results of the averages and the standard deviations in parentheses of posterior probabilities for each case based on 1,000 replications.

From the Table 1, when $(\eta_1, \eta_2) = (1, 1)$, the fractional Bayes factor does not select H_1 for some small sample size cases. However the intrinsic Bayes factor gives fairly reasonable answers. Also for moderate sample sizes, the fractional and intrinsic Bayes factors give fairly reasonable answers.

Table 1: The averages and the standard deviations in parentheses of posterior probabilities

(θ_1, θ_2)	(η_1, η_2)	(n_1, n_2)	$P^F(H_1 \mathbf{x}, \mathbf{y})$	$P^I(H_1 \mathbf{x}, \mathbf{y})$
1,3	1,1	10,10	0.5491(0.1572)	0.6984(0.1749)
		10,20	0.6245(0.1696)	0.7703(0.1713)
		20,10	0.5767(0.1704)	0.7232(0.1769)
		20,20	0.6683(0.1683)	0.8071(0.1592)
	1,2	10,10	0.1786(0.1038)	0.2721(0.1541)
		10,20	0.0376(0.0448)	0.0657(0.0713)
		20,10	0.1634(0.0938)	0.2686(0.1465)
		20,20	0.0304(0.0296)	0.0591(0.0590)
	1,3	10,10	0.0482(0.0432)	0.0865(0.0768)
		10,20	0.0014(0.0026)	0.0026(0.0051)
		20,10	0.0404(0.0380)	0.0852(0.0769)
		20,20	0.0012(0.0023)	0.0027(0.0052)
3,1	1,1	10,10	0.5449(0.1616)	0.6900(0.1799)
		10,20	0.5755(0.1649)	0.7210(0.1755)
		20,10	0.6179(0.1812)	0.7624(0.1812)
		20,20	0.6686(0.1757)	0.8050(0.1691)
	1,2	10,10	0.0826(0.1671)	0.1021(0.1981)
		10,20	0.0237(0.1042)	0.0290(0.1227)
		20,10	0.0258(0.0452)	0.0368(0.0635)
		20,20	0.0014(0.0110)	0.0019(0.0153)
	1,3	10,10	0.0037(0.0277)	0.0042(0.0315)
		10,20	0.0005(0.0121)	0.0006(0.0139)
		20,10	0.0012(0.0024)	0.0016(0.0035)
		20,20	0.0000(0.0000)	0.0000(0.0000)

Example 2. The data in Table 2 is taken from Bain and Engelhardt (1991). Suppose a certain additive is proposed for increasing the length of time of tread wear of a tire. Suppose 40 of the present and 40 tires made under the new process are placed in service and the experiment is continued until the 20 smallest observations are obtained for each sample.

The value of fractional Bayes factor and arithmetic intrinsic Bayes factor of H_2 versus H_1 is $B_{21}^F = 0.1314$ and $B_{21}^I = 0.0563$, respectively. We assume that the prior probabilities are equal. Then the posterior probability for H_1 is 0.8839 and 0.9467, respectively. Thus there are strong evidence for H_1 in terms of the posterior probability.

Table 2: Length of time of tread wear

Present	10.03	10.47	10.58	11.48	11.60	12.41	13.03	13.51
	14.48	16.96	17.08	17.27	17.90	18.21	19.30	20.10
	20.51	21.78	21.79	25.34				
Additive	10.10	11.01	11.20	12.95	13.19	14.81	16.03	17.01
	18.96	24.10	24.15	24.52	26.05	26.44	28.59	30.24
	31.03	33.51	33.61	40.68				

Example 3. The following data are a part of data which shows results of an experiment designed to compare the performances of high-speed turbine engine bearings made out of 5 different compounds (McCool 1979). The experiment tested 10 bearings of each type; the times to fatigue failure are given in units of millions of cycles. We did goodness-of-fit test of 5 types of data to 2-parameter exponential distribution and concluded that type II and type IV were accepted to this distribution. We used K-S statistic for the test.

Type II	3.19	4.26	4.47	4.53	4.67	4.69	5.78	6.79	9.37	12.75
Type IV	5.88	6.74	6.90	6.98	7.21	8.14	8.59	9.80	12.28	25.46

The value of fractional Bayes factor and arithmetic intrinsic Bayes factor of H_2 versus H_1 is $B_{21}^F = 12.771$ and $B_{21}^I = 10.0877$, respectively. We assume that the prior probabilities are equal. Then the posterior probability for H_1 is 0.0726 and 0.09019, respectively.

4. CONCLUSIONS

We developed a Bayesian test procedures for testing the equality of the difference of Guarantee time parameters in two parameter exponential distributions. The refence prior is developed and posterior propriety is proved. Using this reference prior, the intrinsic Bayes factor of Berger and Pericchi (1996) and the fractional Bayes factor of O'Hagan (1996) are computed.

Through the simulation we can conclude that even for the small sample sizes, intrinsic Bayes factor performs better than the fractional Bayes factor. As the sample size grows large, the two Bayes factors perform reasonably. So, we recommend the use of intrinsic Bayes factors in this case.

APPENDIX 1. Derivation of the Reference prior and Propriety of the Posterior Distribution

Following Ghosal (1997) and Sareen (2003), we derive the reference prior of η , θ_1 and θ_2 . The joint density is given by

$$f(\mathbf{x}, \mathbf{y} \mid \eta, \theta_1, \theta_2) = \theta_1^{-n_1} \theta_2^{-n_2} \exp\left\{-\frac{n_1(\bar{x}-\eta)}{\theta_1} - \frac{n_2(\bar{y}-\eta)}{\theta_2}\right\}.$$

Now since $l(\eta) = \eta$,

$$\begin{aligned} c(\eta, \theta_1, \theta_2) &= l'(\eta) f(l(\eta) \mid \eta, \theta_1, \theta_2) \\ &= \theta_1^{-n_1} \theta_2^{-n_2} \\ &= c_1(\eta) c_2(\theta_1, \theta_2), \end{aligned}$$

where $c_1(\eta) = 1$ and $c_2(\theta_1, \theta_2) = \theta_1^{-n_1} \theta_2^{-n_2}$. Moreover, letting $\theta = (\theta_1, \theta_2)$,

$$\begin{aligned} J^{\theta, \theta} &= E\left[\frac{\partial}{\partial \theta} \log f(\mathbf{x}, \mathbf{y} \mid \eta, \theta)\right] \left[\frac{\partial}{\partial \theta} \log f(\mathbf{x}, \mathbf{y} \mid \eta, \theta)\right]' \\ &= \begin{bmatrix} n_1 \theta_1^{-2} & 0 \\ 0 & n_2 \theta_2^{-2} \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \sqrt{\det J^{\theta, \theta}(\eta, \theta)} &= \sqrt{n_1 n_2 \theta_1^{-1} \theta_2^{-1}} \\ &= \lambda_1(\eta) \lambda_2(\theta), \end{aligned}$$

where $\lambda_1(\eta) = \sqrt{n_1 n_2}$ and $\lambda_2(\theta) = \theta_1^{-1} \theta_2^{-1}$. Therefore the reference prior η, θ_1 and θ_2 is given by

$$\pi(\eta, \theta_1, \theta_2) \propto c_1(\eta) \lambda_2(\theta) = \theta_1^{-1} \theta_2^{-1}.$$

Under this reference prior $\pi(\eta, \theta_1, \theta_2) = \theta_1^{-1} \theta_2^{-1}$, the joint posterior for η, θ_1 and θ_2 given \mathbf{x}, \mathbf{y} is

$$\pi(\eta, \theta_1, \theta_2 \mid \mathbf{x}, \mathbf{y}) \propto \theta_1^{-n_1-1} \theta_2^{-n_2-1} \exp\left\{-\frac{n_1(\bar{x}-\eta)}{\theta_1} - \frac{n_2(\bar{y}-\eta)}{\theta_2}\right\}. \quad (5)$$

Integrating with respect to θ_1 and θ_2 in (5), then

$$\pi(\eta \mid \mathbf{x}, \mathbf{y}) = C_1 (\bar{x} - \eta)^{-n_1} (\bar{y} - \eta)^{-n_2}, \quad \eta \leq m_{x,y}.$$

where $m_{x,y} = \min_{1 \leq i \leq n_1, 1 \leq j \leq n_2} \{x_i, y_j\}$ and C_1 is a constant. Since

$$(\bar{x} - \eta) \leq (\bar{x} - m_{x,y}) \quad \text{and} \quad (\bar{y} - \eta) \leq (\bar{y} - m_{x,y}), \quad \text{thus}$$

$$\int_0^{m_{x,y}} \pi(\eta \mid \mathbf{x}, \mathbf{y}) d\eta \leq m_{x,y} (\bar{x} - m_{x,y})^{-n_1} (\bar{y} - m_{x,y})^{-n_2}.$$

Thus the posterior distribution is proper if $n_1 \geq 1$ and $n_2 \geq 1$. This completes the proof. \square

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