A Note on Set-Valued Choquet Integrals

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Abstract

Recently, Zhang et al.(Fuzzy Sets and Systems 147(2004) 475-485) proved Fatou's lemma and Lebesgue dominated convergence theorem under some conditions of fuzzy measure. In this note, we show that these conditions of fuzzy measure is essential to prove Fatou's lemma and Lebesgue dominated convergence theorem by examples

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Recently, Zhang et al.(Fuzzy Sets and Systems 147(2004) 475-485) proved that fuzzy version Fatou's lemma and Lebesgue dominated convergence theorem for a sequence of measurable set-valued function under the assumption that μ is continuous fuzzy measure or μ is m-continuous. It is natural to ask whether one can abandon these assumptions.

The object of the note is to provide an examples to show that the assumptions that μ is continuous fuzzy measure or μ is m-continuous are essential for the conclusions of Theorem 3.12, 3.13 and 3.14 in [1].

We use the same notation as in [1]. Throughout the paper R^+ will denote the interval [0, ∞), $P_0(R^+)(P_f(R^+))$ denotes the class of all nonempty sets(closed, resp.) of R^+, X denotes an abstract nonempty set, Σ is a $\sigma-$algebra formed by subsets of X, (X, Σ) is the measurable space.

Fuzzy measure is a set function $\mu : \Sigma \rightarrow R^+$ with the property (i) $\mu(\phi)$=0, (ii) $A \subset B$ implies $\$\backslash \mu(A) \leq u(B)$. Fuzzy measure μ is said to be continuous, if

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it is both lower semi-continuous and upper semi-continuous.   A fuzzy measure μ is said to be m-continuous, if there exists a complete and finite measure m such that $\mu \ll m$ i.e., $m(A) = 0$ implies A is null set with respect to μ.

The Choquet integral of a measurable function $f : X \to R^+$ with respect to a fuzzy measure μ is defined as

$$(C)\int_A f \, d\mu = \int_0^\infty \mu((f \rangle \alpha) \cap A) d\alpha,$$

where the right-hand side is the Lebesgue integral.

A set-valued function is a mapping F from X to $P_0(R^+)$. By a measurable set-valued function F, it means that its graph is measurable, i.e.,

$$G_r(F) = \{(x ; r) \in X \times R^+ : r \in F(x)\} \in \Sigma \times B(R^+),$$

where $B(R^+)$ is the Borel field of R^+.

Let F be a set-valued function, $A \in \Sigma$. Then the Choquet integral of F on A is defined as

$$(C)\int_A f \, d\mu = \{(C)\int_A f \, d\mu : d \in S(F)\}$$

where $S(F) = \{f$ is measurable: $f(x) \in F(x)$, $x \in X$ m − a.e.$\}$, the family of μ – a.e. measurable selection of F. Instead of $\int_X F \, d\mu$, we will write $\int F \, d\mu$. Obviously, $\int F \, d\mu$, may be empty.

Let $\{A_n\} \subset P_f(R^+)$ be a sequence. We define

$$\lim \sup A_n = \{x : x = \lim_{k \to \infty} x_{n_k}, \ x_{n_k} \in A_{n_k}(k \geq 1)\},$$
$$\lim \inf A_n = \{x : x = \lim_{k \to \infty} x_n, \ x_n \in A_n(n \geq 1)\}$$

If $\lim \sup A_n = \lim A_n = A$, we say that $\{A_n\}$ is convergent to A, and it is simply written as $\lim A_n = A$ or $A_n \to A$.

Let $\{F_n\}$ be a sequences of closed-valued functions and F a closed a closed-valued function. We define

$$(\lim \ sup \ F_n)(x) = \lim \ sup \ F_n(x) \ \mu - a.e.,$$
$$(\lim \ \infty \ F_n)(x) = \lim \ \infty \ F_n(x) \ \mu - a.e.$$

and

$$F_n \rightarrow F \quad \text{if and only if} \quad F_n(x) \rightarrow F(x) \quad \mu - a.e.$$

Zhang et al.[1] proved the following theorem under assumption that the fuzzy measure μ is a continuous fuzzy measure.

Theorem 1[1]. (Fatou's lemma) Let $\{F_n\}$ be a sequence of closed-valued functions. If there exists a Choquet integrable function g, such that $F_n \leq g$ for $n \geq 1$, then

$$\limsup (C) \int F_n \, d\mu \subset (C) \int \limsup F_n \, d\mu.$$

In the following, it is assumed that μ is a m-continuous.

Theorem 2[1]. (Fatou's lemma) Let $\{F_n\}$ be a sequence of closed-valued functions. If there exists a Choquet integrable function g, such that $F_n \leq g$ for $n \geq 1$, then

$$(C) \int \liminf F_n \, d\mu \subset \liminf (C) \int F_n \, du.$$

Theorem 3[1]. (Lebesgue dominated convergence theorem) Let the same conditions as in Theorem 2 be given. If $F_n \rightarrow F$, then

$$(C) \int F_n \, d\mu \rightarrow (C) \int F \, d\mu.$$

We first consider the following example to show that the conclusion of Theorems 1 is not true when the continuity condition of μ is abandoned.

Example 1. Let $X = (0, \infty)$ and Σ be the Borel σ-algebra on X. Define μ as $\mu(A) = 1$ if $(0, 1) \subset A$ and 0, otherwise. Then μ is a fuzzy measure. Consider the set $A_n = (\frac{1}{n}, 1 - \frac{1}{n})$ for $n = 2, 3, \cdots$. Then $A_2 \subset A_3 \subset \cdots \uparrow (0, 1)$ but $1 = \mu((0,1)) \neq \lim_n \mu(A_n) = 0$ and hence μ is not continuous. We now define a sequence of closed-valued measurable function F_n from $(0, \infty)$ to $P_f(R^+)$ by

$$F_n(x) = \begin{cases} \{1\} & \text{if } x \in (\frac{1}{n}, 1 - \frac{1}{n}), \\ \{0\} & \text{otherwise}, \end{cases}$$

Then we have $(C)\int F_n \, d\mu = \int 0 \, d\alpha = 0$ for all n and hence $\limsup(C)\int F_n \, d\mu = 0$. On the other hand, since

$$\limsup F_n(x) = \begin{cases} \{1\} & \text{if } x \in (0,1), \\ \{0\} & \text{otherwise}, \end{cases}$$

$(C)\int \limsup F_n \, d\mu = \int_0^1 1 \, d\alpha = 1$. Therefore, the conclusion of Theorem 1 does not hold.

We now consider the following example to show that the conclusions of Theorems 2 and 3 are not true when the m-continuity condition of μ is abandoned.

Example 2. Let $X = (0, \infty)$ and Σ be the power set of X. Define μ and m as $m(A) = \mu(A) = 1$ if $A \neq \phi$ and 0, otherwise. Then m is complete and finite, and μ is m-continuous. We define a sequence of closed-valued measurable function F_n from $(0, \infty)$ to $P_f(R^+)$ by

$$F_n(x) = \begin{cases} \{1\} & \text{if } x \in (0, \frac{1}{n}), \\ \{0\} & \text{otherwise}, \end{cases}$$

Then, since $\liminf F_n(x) = 0$ we have $(C)\int \liminf F_n d\mu = 0$. On the other hand, since $(C)\int F_n d\mu = \int_0^1 1 d\alpha = 1$ for all n, $\liminf(C)\int F_n d\mu = 1$. Therefore, the conclusions of Theorem 2 and 3 do not hold.

References

1. Zhang, D., Guo, C. and Liu, D.(2004). Set-valued Choquet integrals revisited, *Fuzzy Sets and Systems* 147, 475-485.