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Lindley Type Estimators When the Norm is Restricted to an Interval[1)]

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Abstract

Consider the problem of estimating a $p\times 1$ mean vector $\theta\,(p\geq 4)$ under the quadratic loss, based on a sample $X_1,\ X_2,\ \cdots,\ X_n$. We find a Lindley type decision rule which shrinks the usual one toward the mean of observations when the underlying distribution is that of a variance mixture of normals and when the norm $\|\,\theta-\bar{\theta}1\,\|$ is restricted to a known interval, where $\bar{\theta}=\frac{1}{p}\sum_{i=1}^{p}\theta_i$ and 1 is the column vector of ones. In this case, we characterize a minimal complete class within the class of Lindley type decision rules. We also characterize the subclass of Lindley type decision rules that dominate the sample mean.

Keywords : Lindley type decision rule, Mean vector, Quadratic loss, Underlying distribution

1. Introcuction

The problem considered is that of estimating with quadratic loss function the mean vector of a compound multinormal distribution when the norm $\|\,\theta-\bar{\theta}1\,\|$ is restricted known interval. The class of estimation rules considered will consist of Lindley type estimators only. Such a class was introduced by James-Stein(1961) and Lindley(1962) in order to prove that some of its members dominate the

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sample mean in the multinormal case. Strawderman(1974) also derived a similar result for the more general case considered in this paper of a compound multinormal distribution.

The problem of estimation of a mean under constraint has an old origin and recently focussed again in the context of curved model in the works of Amari(1982), Kariya(1989), Perron and Giri(1989), Merchand and Giri(1993), and Baek(2000) among others. A study of compound multinormal distributions and the estimation of their location vectors was carried out by Berger(1975).

In section 2, we present the general setting of our problem and develop necessary notations. In section 3, we examine the estimation problem based on a Lindley type decision rule when the norm $\| \theta - \bar{\theta} 1 \|$ is restricted to a known interval. In this case, we give to the subclass of Lindley type estimators which dominate the sample mean when the norm is restricted to a known interval.

2. Notation and Preliminaries

Let $x = (x_1, \cdots, x_p)'$, $p \ge 4$, be an observation from a compound multinormal distribution with unknown location parameter $\theta(p \times 1)$ and mixture parameter $H(\cdot)$, where $H(\cdot)$ represents a known c.d.f defined on the interval $(0, \infty)$. In other words, we assume that the random variable X generating our observation x admits the representation,

$$L(X|Z=z) = N_p(\theta, zI_p), \quad \forall z > 0, \tag{2.1}$$

Z being the positive random variable with c.d.f. $H(\cdot)$.

Our problem concerns the estimation of the location parameter θ with loss function

$$L(\theta, \delta(x)) = (\delta(x) - \theta)'(\delta(x) - \theta),$$

with $\theta \in \Theta_{\lambda_2}^{\lambda_1} = \{ \theta \in R^p | \ \| \theta - \bar{\theta} 1 \| \in [\lambda_1, \lambda_2], \ 0 \le \lambda_1 \le \lambda_2 \le \infty \}$,

where $\bar{\theta} = \frac{1}{p} \sum_{i=1}^{p} \theta_i$, $1 = (1, \cdots, 1)'$ and the decision rule δ, $\delta(\cdot): R^p \to R^p$, is of the form

$$\delta(x) = \bar{x} 1 + \left(1 - \frac{c}{(x - \bar{x} 1)'(x - \bar{x} 1)} \right)(x - \bar{x} 1), \quad c \in R.$$

Restated in terms of the family of probability density functions of X, the distributional assumption give by expression (2.1) and the restriction on the location parameter θ indicate that the p.d.f. of X is

$$P_{\theta}(x) = \int_{(0,\infty)} (2\pi z)^{-p/2} \exp\left(\frac{\| x-\theta \|^2}{2z}\right) dH(z), \tag{2.2}$$

$x \in R^p$ and $\theta \in \Theta_{\lambda_2}^{\lambda_1}$. It will be also assumed that $E(Z) \langle \infty$ which will guarantee the existence of the covariance matrix $\sum = Cov(X) = E(Z) I_p$ and the mean vector $E(X) = \theta$. The performance of the estimator δ will be measured by its risk function

$$R(\theta, \delta) = E_{\theta}[L(\theta, \delta(X))] = E_{\theta}[(\delta(X) - \theta)'(\delta(X) - \theta)], \quad \theta \in \Theta_{\lambda_1}^{\lambda_2}$$

Define

$$D_{Lind} = \left\{ \delta : R^p \to R^p \mid \delta^c(X) = \overline{X}1 + \left(1 - \frac{c}{(X-\overline{X}1)'(X-\overline{X}1)}(X - \overline{X}1)\right), \quad c \in R \right\},$$

where the parameter space is of the form

$$\Theta_{\lambda}^{\lambda} = \Theta_{\lambda} = \{ \theta \in R^p \mid \| \theta - \overline{\theta} 1 \| = \lambda \}, \quad \lambda \geq 0.$$

Then under the assumptions $\theta \in \Theta_{\lambda}$, $p \geq 4$ and $E[Z] \langle \infty$, we can show that

$$\begin{aligned} R(\theta, \delta^c) &= E_{\theta}[(\delta^c(X) - \theta)'(\delta^c(X) - \theta)] \\ &= pE(Z) + \left\{ \int_{(0,\infty)} \left[\frac{c}{z} - 2c(p-3)\right] f_p(\lambda, z) dH(z) \right\}, \end{aligned} \tag{2.3}$$

using the method by Baek(2000). By expression (2.3), the unique best estimator within the class D_{Lind} is given by $\delta^{c^*(\lambda)}$ where

$$c^*(\lambda) = (p-3) \frac{\int_{(0,\infty)} f_p(\lambda, z) dH(z)}{\int_{(0,\infty)} f_p(\lambda, z) \frac{dH(z)}{z}} \tag{2.4}$$

and its risk is

$$R(\theta,\ \delta^{c^*(\lambda)}) = pE(Z) - (p-3)^2 \frac{\left[\int_{(0,\ \infty)} f_p(\lambda,\ z) dH(z)\right]^2}{\int_{(0,\ \infty)} f_p(\lambda,\ z) \frac{dH(z)}{z}}, \qquad \theta \in \Theta_\lambda.$$

When $\| \theta - \overline{\theta} 1 \| = \lambda$, the use of other estimators of the Lindley class other estimators of the Lindley class other that will incur risk which is a strictly increasing function of distance $| c - c^*(\lambda) |$. To see this, we can define $t(\lambda)$ such that $c = t(\lambda) c^*(\lambda)$ and, using expression (2.3), express $R(\theta,\ \delta^c)$ as

$$pE(Z) + (p-3)^2 [t^2(\lambda) - 2t(\lambda)] \frac{\left[\int_{(0,\ \infty)} f_p(\lambda,\ z) dH(z)\right]^2}{\int_{(0,\ \infty)} f_p(\lambda,\ z) \frac{dH(z)}{z}}. \tag{2.5}$$

From this we can write

$$R(\theta,\ \delta^c) - R(\theta,\ \delta^{c^*(\lambda)}) = | c - c^*(\lambda) |^2 \int_{(0,\ \infty)} f_p(\lambda,\ z) \frac{dH(z)}{z}. \tag{2.6}$$

The natural estimator $\delta^0(X) = X$ is a member of the Lindley class and has a constant risk function equal to $pE(Z)$. Using the expression (2.5), we can verify that the Lindley type estimator δ^c dominates the natural estimator δ^0 if and only if $0 < c < 2 < c^*(\lambda)$ for $\theta \in \Theta_\lambda$.

3. Estimation when the Norm is Restricted to an Interval

In this section, we study the case where the mean θ is restricted to a known interval $[\lambda_1,\ \lambda_2]$ case, no optimal Lindley type decision rule will exist whenever $\lambda_1 \le \lambda_2$(but see the discussion following Corollary 3.7 for asymptotic considerations). We can also characterize the subclass of Lindley type decision rules that dominate the natural estimator $\delta^0 = X$ when $\theta \in \Theta_{\lambda_2}^{\lambda_1}$. In the following, we will denote

$$\underline{c}^*[\lambda_1,\ \lambda_2] = \inf_{\lambda \in [\lambda_1,\ \lambda_2]} c^*(\lambda) \text{ and } \overline{c}^*[\lambda_1,\ \lambda_2] = \sup_{\lambda \in [\lambda_1,\ \lambda_2]} c^*(\lambda).$$

Theorem 3.1 Let x be a single observation from a p-dimensional location parameter with p.d.f. of the form given by expression (2.1). Under the assumptions

$\theta \in \Theta_{\lambda_1}^{\lambda_2}$, $0 \le \lambda_1 \le \lambda_2 \le \infty$; $p \ge 4$ and $E(Z) < \infty$,

(a) the subclass $\{\delta^c \in D_{Lind} \mid \underline{c}^*[\lambda_1, \lambda_2] \le c \le \{\overline{c}^*[\lambda_1, \lambda_2]\}$ is a minimal complete class within the class D_{Lind}, and

(b) the decision rule δ^c will be dominate the natural estimator δ^0 if $0 < c < 2\underline{c}^*[\lambda_1, \lambda_2]$.

Proof. (a) Let c_0 be a real number such that $c_0 \notin [\underline{c}^*[\lambda_1, \lambda_2], \overline{c}[\lambda_1, \lambda_2]]$. Then, using expression (2.6), if $c_0 < \underline{c}^*[\lambda_1, \lambda_2]$, we may write the difference in risks

$$\begin{aligned} & R(\theta, \delta^{c_0}) - R(\theta, \delta^{\underline{c}^*[\lambda_1, \lambda_2]}) \\ & = [R(\theta, \delta^{c_0}) - R(\theta, \delta^{c^*(\|\theta - \overline{\theta}1\|)})] - [R(\theta, \delta^{\underline{c}^*[\lambda_1, \lambda_2]}) - R(\theta, \delta^{c^*(\|\theta - \overline{\theta}1\|)})] \\ & = \int_{(0,\infty)} f_p(\lambda, z) \frac{dH(z)}{z} \{ |c_0 - c^*(\|\theta - \overline{\theta}1\|)|^2 \\ & \qquad - |\underline{c}^*[\lambda_1, \lambda_2] - c^*(\|\theta - \overline{\theta}1\|)|^2 \}; \end{aligned}$$

this last expression being positive for all $\theta \in \Theta_{\lambda_1}^{\lambda_2}$ given that $c_0 < \underline{c}^*[\lambda_1, \lambda_2]$. In the same manner, the decision rule δ^c with $c = \overline{c}^*[\lambda_1, \lambda_2]$ will dominate the decision rule δ^{c_0} if $c_0 > \overline{c}^*[\lambda_1, \lambda_2]$. Also if $c_0 \in [\underline{c}^*[\lambda_1, \lambda_2], \overline{c}^*[\lambda_1, \lambda_2]]$, the intermediate value theorem ($c^*(\lambda)$ is easily shown to be continuous) assures us that

$$R(\theta, \delta^c) - R(\theta, \delta^{c_0}) > 0, \quad \forall c \ne c_0,$$

when $c^*(\|\theta - \overline{\theta}1\|) = c_o$. These last results guarantee that all the rules δ^c with $c \notin [\underline{c}^*[\lambda_1, \lambda_2], \overline{c}^*[\lambda_1, \lambda_2]]$ are inadmissible within the class D_{Lind} and the rules δ^c with c belonging to the interval $[\underline{c}^*[\lambda_1, \lambda_2], \overline{c}^*[\lambda_1, \lambda_2]]$ cannot be improved upon by another rule of the class D_{Lind} Thus, the result of part (a) follows.

(b) Similar to last part in Section 2, the decision rule δ^c will dominate the decision rule δ^0 if

$$\begin{aligned} & R(\theta, \delta^c) < R(\theta, \delta^0), \quad \forall \theta \in \Theta_{\lambda_1}^{\lambda_2} \\ & \Leftrightarrow 0 < c < 2c^*(\|\theta - \overline{\theta}1\|), \quad \forall \|\theta - \overline{\theta}1\| \in [\lambda_1, \lambda_2] \\ & \Leftrightarrow 0 < c < 2\underline{c}^*[\lambda_1, \lambda_2]. \end{aligned}$$

It may also be remarked that the rule δ^c with $c = 2\underline{c}^*[\lambda_1, \lambda_2]$ will also dominate δ^0 under the conditions of the theorem when $\lambda_1 < \lambda_2$ and that all the decisions rules δ^c with $c > 2c^*[\lambda_1, \lambda_2]$ do not dominate δ^0 under the conditions of the theorem. The results above would be more explicit if the function $\underline{c}^*[\lambda_1, \lambda_2] = c^*(\lambda_1)$ and $\overline{c}^*[\lambda_1, \lambda_2] = c^*(\lambda_2)$.

The case with no restrictions on the norm $\| \theta - \overline{\theta}1 \|$ ($i.e.$, $\lambda_1 = 0$ and $\lambda_2 = \infty$) can be expanded using by Strawderman's result(1974) and it can be showed that the decision rules δ^c with $0 \le c \le 2(p-3)E^{-1}(Z^{-1})$ are minimax rules by showing that their risk functions are uniformly less than or equal to the risk function $(= pE(Z))$ of the minimax decision rule δ^c. This result is derived below as a particular case of Theorem 3.1. To do so, we need to determine the quantity $\underline{c}^*[0, \infty]$. The following three Lemmas will prove useful in determining $\underline{c}^*[0, \infty]$ and, also, $\overline{c}^*[\lambda_1, \lambda_2]$.

Lemma 3.2. Let X be an arbitrary random variable and let f and g be two real nondecreasing functions on the support of X. Then, if the quantities $E[f(X)]$ and $E[g(X)]$ exist, $Cov(f(X), g(X)) \ge 0$ with the inequality being strict if f and g are strictly increasing and X is nondegenerate.

Proof. A neat proof of Lemma 3.2. is given by Chow and Wang(1990).

Lemma 3.3. Let L be a Poisson random variable with mean γ, $\gamma > 0$, and $f^*_p(\gamma) = E^L[(p + 2L - 3)^{-1}]$; $p \ge 4$; then

(i) $f^*_p(\gamma) = e^{-\gamma} \int_{[0,1]} t^{p-4} e^{\gamma t^2} dt$, and

(ii) $f^*_{p+2}(\gamma) = (2\gamma)^{-1}[1 - (p-3)f^*_p(\gamma)]$. (3.1)

Proof. We can prove this lemma using the method by Egerton and Laycock(1982).

Lemma 3.4. Let $f^*_p(\cdot)$, $p \ge 4$, be a function defined on $[0, \infty]$ and equal to

$$f^*_p(\gamma) = E^L[(p + 2L - 3)^{-1}], \quad \gamma \ge 0,$$

where L is a Poisson random variable with mean γ. Then,

(i) $f^*_p(\cdot)$ is a strictly decreasing function,

(ii) $\lim_{\gamma \to 0^+} f^*_p(\gamma) = (p-3)^{-1}$, $\lim_{\gamma \to 0^+} f^*_p(\gamma) = 0$,

(iii) if $p \geq 5$, $\gamma f^*_p(\gamma)$ is strictly increasing function for $\gamma \geq 0$.

Proof. (i) Using part (i) of Lemma 3.3, we have for $\gamma_2 > \gamma_1 > 0$,

$$f^*_p(\gamma_2) - f^*_p(\gamma_1) = \int_{[0,\,1]} t^{p-4}\left(e^{\gamma_2(t^2-1)} - e^{\gamma_1(t^2-1)}\right) dt < 0.$$

(ii) By the dominated convergence theorem,

$$\begin{aligned}\lim_{\gamma \to 0^+} f^*_p(\gamma) &= \lim_{\gamma \to 0^+} \int_{[0,\,1]} t^{p-4} e^{\gamma(t^2-1)} dt \\ &= \int_{[0,\,1]} t^{p-4} \lim_{\gamma \to 0^+} \left(e^{\gamma(t^2-1)}\right) dt \\ &= \int_{[0,\,1]} t^{p-4}\, dt = (p-3)^{-1},\end{aligned}$$

and

$$\begin{aligned}\lim_{\gamma \to \infty} f^*_p(\gamma) &= \lim_{\gamma \to \infty} \int_{[0,\,1]} t^{p-4} e^{\gamma(t^2-1)} dt \\ &= \int_{[0,\,1]} t^{p-4}\left(\lim_{\gamma \to \infty} e^{\gamma(t^2-1)}\right) dt = 0.\end{aligned}$$

(iii) Using Lemma 3.3, we have

$$\gamma f^*_5(\gamma) = \frac{1}{2}(1 - e^{-\gamma}),$$

which is easily seen to be strictly increasing. For $p \geq 6$ we obtain by the recurrence formula given by expression (3.1),

$$\gamma f^*_p(\gamma) = \frac{1}{2}\left(1 - (p-5) f^*_{p-3}(\gamma)\right), \quad \gamma > 0,$$

which must be strictly increasing given that function $f^*_{p-3}(\cdot)$ is strictly decreasing by part (i).

In the following, we will set $E^{-1}[Z^{-1}]$ equal to zero if the expectation $E[Z^{-1}] = \infty$.

Theorem 3.5. The function $c^*(\cdot)$ defined by expression (2.4) satisfies the following properties :

(a) $\inf_{\lambda \geq 0} c^*(\lambda) = (p-3) E[Z^{-1}]$,

(b) $c^*(\lambda) = k \Rightarrow Z$ is constant with probability one

and,

(c) for $p \geq 5$, $\sup_{\lambda \geq 0} c^*(\lambda) = (p-3)E(Z)$

Proof. (a) Expression (2.4) can be rewritten as

$$c^*(\lambda) = (p-3)\frac{E^Z[f_p(\lambda, Z)]}{E^Z[Z^{-1}f_p(\lambda, Z)]}, \quad \lambda \geq 0.$$

By applying Lemma 3.2 to the functions $f_p(\lambda, Z)$ and Z^{-1}, the function $f_p(\lambda, z)$ being an increasing function by part (i) of Lemma 3.4, we have for $\lambda \geq 0$,

$$\begin{aligned}
&Cov(f_p(\lambda, Z), -Z^{-1}) \geq 0 \\
&\Rightarrow E^Z[Z^{-1}f_p(\lambda, Z)] \geq E[Z^{-1}]E^Z[f_p(\lambda, Z)] \\
&\Rightarrow c^*(\lambda) \geq (p-3)E^{-1}[Z^{-1}] \\
&\Rightarrow \inf_{\lambda \geq 0} c^*(\lambda) \geq (p-3)E^{-1}[Z^{-1}]
\end{aligned}$$

The reverse inequality is obtained by observing that $c^*(0) = (p-3)E^{-1}[Z^{-1}]$.
(b) The constancy of $c^*(\lambda)$ implies $c^*(\lambda) = k = c^*(0) = (p-3)E^{-1}(Z^{-1})$ $\forall \lambda > 0$, and

$$\int_{(0,\infty)} (p-2-\frac{k}{z})f_p(\lambda, z)dH(z) = 0.$$

Since both $f_p(\lambda, z)$ and $-kz^{-1}$ are strictly increasing function of z, we have by Lemma 3.2, for nondegenerate Z,

$$\begin{aligned}
&Cov(f_p(\lambda, Z), \quad p-3-kZ^{-1}) > 0 \\
&\Rightarrow E[(p-3-kZ^{-1})f_p(\lambda, Z)] > E[(p-3-kZ^{-1})]E[f_p(\lambda, Z)] = 0,
\end{aligned}$$

which results in a contradiction implying Z is constant with probability one.
(c) By applying Lemma 3.2 to the functions $-z^{-1}f_p(\lambda, z)$ and z, the function $-z^{-1}f_p(\lambda, z)$ being an increasing function by virtue of part (iii) of Lemma 3.4, we have for $p \geq 5$ and $\lambda \geq 0$,

$$\begin{aligned}
&Cov(-Z^{-1}f_p(\lambda, Z), Z) \geq 0 \\
&\Rightarrow E^Z[f_p(\lambda, Z)(] \leq E[Z^{-1}f_p(\lambda, Z)]E[Z] \\
&\Rightarrow c^*(\lambda) \leq (p-3)E[Z] \\
&\Rightarrow \sup_{\lambda \geq 0} c^*(\lambda) \leq (p-3)E[Z].
\end{aligned}$$

The reverse inequality is obtained by verifying that $\lim_{\lambda\to\infty} c^*(\lambda)=(p-3)E[Z]$ whenever $p\geq 5$. To do so, it will be useful to express the function $c^*(\cdot)$ in the following way,

$$c^*(\lambda)=(p-3)\frac{\int_{(0,\infty)}\sum_{j=0}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^{j+1}}{j!(p+2y-3)}zdH(z)}{\int_{(0,\infty)}\sum_{j=0}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^{j+1}}{j!(p+2y-3)}dH(z)},\quad \lambda>0$$

$$=(p-3)\frac{\int_{(0,\infty)}\sum_{j=1}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^{j}}{j!}\frac{2j}{(p+2y-5)}zdH(z)}{\int_{(0,\infty)}\sum_{j=1}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^{j}}{j!}\frac{2j}{(p+2y-5)}dH(z)},\quad \lambda>0$$

Moreover, we can write

$$\lim_{\lambda\to\infty} c^*(\lambda)=(p-3)\frac{\lim_{\lambda\to\infty}\left\{\int_{(0,\infty)}\sum_{j=1}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^{j}}{j!}\frac{2j}{(p+2y-5)}zdH(z)\right\}}{\lim_{\lambda\to\infty}\left\{\int_{(0,\infty)}\sum_{j=1}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^{j}}{j!}\frac{2j}{(p+2y-5)}dH(z)\right\}}$$

if both limits exist and the denominator is not equal to zero. By the dominated converge theorem, we can then write $\lim_{\lambda\to\infty} c^*(\lambda)$ as

$$(p-3)\frac{\int_{(0,\infty)}\lim_{\lambda\to\infty}E^{L_z}\left[\frac{2L_z}{p+2L_z-5}1_{(1,2,\ldots)}(L_z)\right]zdH(z)}{\int_{(0,\infty)}\lim_{\lambda\to\infty}E^{L_z}\left[\frac{2L_z}{p+2L_z-5}1_{(1,2,\ldots)}(L_z)\right]dH(z)},$$

where, for $z>0$, L_z is a Poisson random variable with mean $\lambda^2/2z$. Finally by noting that,

$$\forall z>0,\quad \lim_{\lambda\to\infty}E^{L_z}\left[\frac{2L_z}{p+2L_z-5}1_{(1,2,\ldots.)}(L_z)\right]=1,$$

because the integrand tends $2L_z(p+2L_z-5)^{-1}$ tends to one when $L_z\to\infty$, we obtain

$$\lim_{\lambda\to\infty} c^*(\lambda)=(p-3)\frac{\int_{(0,\infty)} z dH(z)}{\int_{(0,\infty)} dH(z)}=(p-3)E(Z).$$

Having evaluated the quantities $\underline{c}^*[0,\ \infty]$ and $\overline{c}^*[0,\ \infty]$, and Theorem 3.1 yields the following result.

Corollary 3.6. Let x be a single observation from a p-dimensional location parameter family with p.d.f. of the form given by expression (2.1). with $p\geq 4$, and under the assumption $\theta\in R^p$ and $E[Z]<\infty$,

(a) the subclass $\{\delta^c\in D_{Lind}\mid (p-3)E^{-1}[Z^{-1}]\leq c\leq (p-3)E[Z]\}$ is a minimal complete class D_{Lind} for $p\geq 5$,

(b) the decision rule δ^c will dominate the decision rule δ^0 if $0<c<2(p-3)E^{-1}[Z^{-1}]$.

Proof. These results above are a direct application of Theorem 3.1 and 3.5.

We pursue with some remarks.

Remark 3.1. Under the conditions of Corollary 3.6, the decision rule δ^c is a minimax rule if and only if $0\leq c\leq 2(p-3)E^{-1}[Z^{-1}]$. This condition can also be obtained using part(a) of Theorem 3.5 and similar to last part in Section 2 which, under the same conditions, would specify that

$$R(\theta,\ \delta^c)\leq p \Leftrightarrow 0\leq c\leq 2c^*(\|\ \theta-\overline{\theta}1\ \|).$$

It is interesting to note that the natural estimator δ^0 represents the only minimax rule within the class D_{Lind} when the quantity $E[Z^{-1}]$ does not exist.

Remark 3.2. The results above of Theorem 3.1 and Corollary 3.6 can be extended to the case where the experimental information consist of a sample $X_1,\ \ldots\ X_n$ with p.d.f. of the form in(2.1) and the class of decision rules considered consists of the decision rules of the form

$$\delta^c(X_1,\ \ldots,\ X_n)=\overline{\overline{X}}1+\left(1-\frac{c}{(\overline{X}-\overline{\overline{X}}1)'(\overline{X}-\overline{\overline{X}}1)}\right)(\overline{X}-\overline{\overline{X}}1),\quad c\in R,$$

where $\overline{X}$ is the sample mean and $\overline{\overline{X}} = (np)^{-1} \sum_{i=1}^{p} \sum_{j=1}^{n} X_{ij} = P^{-1} \sum_{i=1}^{n} \overline{X}_i$. This can be seen by nothing that the probability law of sample mean $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$; $X_1, \ldots, X_n$ being n independently and identically distributed random vectors admitting the representations.

$$L(X_j \mid Z_j = z_j) = N_p(\theta,\ z_j I_p), \quad j = 1, \cdots, n,$$

for all values $z_1, \ldots, z_n$ of n independent copies $Z_1, \ldots, Z_n$ of a positive random variable Z ; admits the representation

$$L(Z \mid Z_1 = z_1, \cdots, Z_n = z_n) = N_p(\theta,\ n^{-2} \sum_{j=1}^{n} z_j I_p),$$

or

$$L(\overline{X} \mid W = w) = N_p(\theta,\ wI_p), \quad \forall w > 0,$$

where W is a random variable such that

$$L(W) = L(n^{-2} \sum_{j=1}^{n} Z_j). \tag{3.2}$$

Thus the optimal estimator of the Lindley type is; with the conditions $\theta \in \Theta_\lambda$, $E[Z] < \infty$, $p \geq 4$; given by expression (2.4), and is equal to

$$\delta^{c_n^*(\lambda)} = \overline{\overline{X}}1 + \left(1 - \frac{c_n^*(\lambda)}{(\overline{X} - \overline{\overline{X}}1)'(\overline{X} - \overline{\overline{X}}1)}\right)(\overline{X} - \overline{\overline{X}}1)$$

where

$$c_n^*(\lambda) = (p-3) \frac{\int_{(0,\ \infty)} f_p(\lambda,\ w) dH_n^*(w)}{\int_{(0,\ \infty)} f_p(\lambda,\ w) \frac{dH_n^*(w)}{w}},$$

$H_n^*(\cdot)$ representing the c.d.f. of the random variable W defined by expression (3.2). Furthermore, the result specifying a minimal complete class within the class

$$D_{Lind} = \left\{ \delta^c : R^p \to R^p \mid \delta^c(\overline{X}) = \overline{\overline{X}}1 + \left(1 - \frac{c}{(\overline{X} - \overline{\overline{X}}1)'(\overline{X} - \overline{\overline{X}}1)}\right)(\overline{X} - \overline{\overline{X}}1) \right\}$$

as well as the result giving a subclass of Lindley type rues that dominate the sample mean $\delta^0(\overline{X}) = \overline{X}$ and be applied to the case where the experimental information consists of a sample. In particular, by rewriting Corollary 3.6, we obtain the following result. Part(b) of this corollary has been proved by Bravo and MacGibbon(1988) under a more general setting.

Corollary 3.7. Let $X_1, \ldots X_n$ be a sample generated by a common random vector X which admits the representation given by expression (2.1). Under the conditions $\theta \in R^p$, $p \geq 4$ and $E[Z] < \infty$

(a) for $p \geq 5$, the subclass

$\left\{ \delta^c \in D_{Lind} \mid n^{-2}(p-3)E^{-1}\left[\left(\sum_{i=1}^{n} Z_i \right)^{-1} \leq c \leq n^{-1}(p-3)E[Z] \right] \right\}$ is a minimal complete class with the class D_{Lind}, and

(b) the decision rule δ^c will dominate the sample mean if

$$0 < c < 2n^{-2}(p-3)E^{-1}\left[\left(\sum_{i=1}^{n} Z_i \right)^{-1} \right]. \tag{3.3}$$

Proof. These results are a direct application of Corollary 5.6 and the discussion above expression (3.2).

However, the results concerning the minimax criteria given by Strawerman cannot be applied to the decision rules $\delta^c(\bar{x})$ since the statistic $\bar{X}$ does not represent in general a sufficient statistic(the multinormal case being a well known exception). Finally it is interesting to note that,

$$E^{-1}\left[\left(\sum_{i=1}^{n} Z_i \right)^{-1} \right] \leq E\left[\sum_{i=1}^{n} Z_i \right] = nE[Z],$$

(the above inequality can be seen us a consequence of Lemma 3.2), implying that the interval

$$\left(0,\ 2n^{-1}(p-3)E^{-1}\left[\left(\sum_{i=1}^{n} Z_i \right)^{-1} \right]\right) \rightarrow \varnothing \text{ as } n \rightarrow \infty,$$

which, by expression (3.3), indicates that the subclass of Lindley type decision rules dominating the sample mean can be made arbitrarily small by increasing the sample size n.

Reference

1. Amari, S(1982). differential geometry of curved exponential families, curvature and information loss, *Annals of Statistics*, 10, 357-385.
2. Baek, H.Y.(2000). Lindley type estimators with the known norm, *Journal of the Korean Data and Information Science Society*, 11(1), 37-45.
3. Berger, J.(1975). Minimax estimation of location vectors for a wide class of densities, *Annals of Statistics*, 3, 1318-1328.
4. Bravo, G. and MacGibbon, G.(1988). Improved shrinkage estimators for the mean of a scale mixture of normals with unknown variance, *The Canadian Journal of statistics*, 16,237-245.
5. Chow, S.C., and Wang, S.C.(1990). A note an adaptive generalized ridge regression estimator, *Statistics and Probability Letters*, 10, 17-21.
6. Egerton, M.F. and Laycock, P.J.(1982). An explicit formula for the risk of James-Stein estimators, *The canadian Journal of Statistics*, 10, 199-205.
7. James, W. and Stein D.(1961). Estimation with guadratic loss, *In Proceedings Fourth Berkeley Symp. Math. Statis. Probability*, 1, University of California Press, Berkeley, 361-380.
8. Kariya, T.(1989). Equivariant estimation in a model with ancillary statistics, *Annals of Statistics*, 17, 920-928.
9. Lindley, D.V.(1962). Discussion of paper by C. Stein, *Journal of The Royal Statistical society*, B, 2, 265-296.
10. Marchand, E. and Giri, N.C.(1993). James-Stein estimation with constraints on the norm, *communication in statistics-Theory and Methods*, 22(10), 2903-2924.
11. Perron, F. and Giri, N.(1989). On the best equivariant estimator of mean of a multivariate normal population, *Journal of Multivariate Analysis*, 32,1-16.
12. Strawderman, W. E(1974). Minimax estimation of location parameters for certain spherically symmetric distributions, *Journal of Multivariate analysis*, 4, 255-264.

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