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Quasi-Likelihood Estimation for ARCH Models[1)]

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Abstract

In this paper, the quasi-likelihood function was proposed and the estimators which are the solutions of the estimating equations for estimation of a class of nonlinear time series models. We compare the performances of the proposed estimators with those of the ML estimators under the heavy-railed distributions by simulation.

Keywords : ARCH, MLE, Quasi-Likelihood Function

1. Introduction

The Box-Jenkins linear time series models which were introduced in the 1970′s have been used for analysing various types of times series data. But they sometimes could not explain the important characteristics of data especially financial times ones. In general, the volatilities in the financial data means the risks and they have been considered very importantly. Engle(1982) proposed the ARCH(Auto regressive Conditional Heteroscadastic) models which could explained the volatilities in the data. Bollerslev(1986) generalized the ARCH models which are called GARCH(Generalized ARCH) models. On the other hand, the statistical estimation methods for the heteroscadastic time series models have been studied by numerous researcher. But one of the important assumptions for estimation of the parameters in the models was the normality of the distributions for the error terms in the models. This basic assumption has been violated by many examples in the financial data. One of the methods for solving this problem is to introduce the different estimation method such as the quasi-likelihood(QL) estimation proposed by Godambe(1988). Recently Chandra and Taniguchi(2001) pointed out that the generalized method of moments(GMM) estimation proposed by Hansen(1982) and the QL estimation methods are basically same and the two

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methods have been studied independently. In this paper, we first consider the QL estimation method and a class of nonlinear time series models such that as the autoregressive(AR) model with the ARCH errors. We compare the performance of the QL estimations with the maximum likelihood(ML) estimations under the normal and heavy-tailed distributions by simulation.

2. Quasi-likelihood estimation.

Godambe(1985) proposed the optimal estimating functions for a class of time series models and the estimating function is called the quasi-likelihood estimating function. Consider the class of quasi-likelihood estimating function with the mean and the conditional variance such as $E(X_k|F_{k-1})=\mu_k(\theta)$ and $V(X_k|F_{k-1})=V_k(\theta)$, where θ is the p×1 vector of parameters of interests and $\{X_t\}$ is the observed time series data and F_{k-1} is the σ-field generated by the past data $X_{k-1,\ldots}$. Define the quasi-likelihood estimating function as follows.

$$S_n(\theta)=\sum_{k=1}^{n} w_k(\theta)g_k(\theta) \tag{1}$$

where

$$w_k(\theta)=\left[\frac{d\mu_k(\theta)}{d\theta}\quad \frac{dV_k(\theta)}{d\theta}\right]\begin{bmatrix} V_k(\theta) & E(x_k-\mu_k(\theta)|F_{k-1})^3 \\ E(x_k-\mu_k(\theta)|F_{k-1})^3 & E(x_k-\mu_k(\theta)|F_{k-1})^4\end{bmatrix}$$

$$g_k(\theta)=\begin{bmatrix} x_k-\mu_k(\theta) \\ (x_k-\mu_k(\theta))^2-V_k(\theta)\end{bmatrix}$$

It is easily known that $E(g_k(\theta)|F_{k-1})=0$. The next theorem guarantees the consistency and asymptotic normality of the estimator which is the solution of the equation in (1).

<u>Theorem 2.1</u> Under the regularity conditions, we have

(1) $\widehat{\theta}_n \xrightarrow{p} \theta$,

(2) $\sqrt{n}(\widehat{\theta}_n-\theta) \xrightarrow{d} N(0, F_n(\theta))$

where $\widehat{\theta}_n$ is the solution of the estimating equation in (1) and $F_n(\theta)=\sum_{k=1}^{n} D_n(\theta)\sum_k^{-1}(\theta)D_k^T(\theta)$, $D_k(\theta)=\left[\frac{dg_k(\theta)}{d\theta}|F_{k-1}\right]$,

$\sum = [\, g_k(\theta) g_k(\theta)^T | F_{k-1}]$

<u>proof</u> Along the lines of Klinko and Nelson(1978), consistency and asymptotic normality can be proved. we omit it.

Example 2.1

Consider the following model,

$$y_t = \phi y_{t-1} + \varepsilon_t ,$$
$$\varepsilon_t = \sqrt{\ } of h_t e_t , \quad e_t \sim iid\ (0, \sigma^2) , \quad h_t = \alpha_0 + \alpha_1 \varepsilon^2_{t-1} \quad .$$

Then we have the quasi-likelihood function such as

$$S_n(\theta) = \sum_{t=1}^{n} DC^{-1} \begin{bmatrix} y_t - \phi y_{t-1} \\ (y_t - \phi y_{t-1})^2 - h_t \end{bmatrix} \tag{2}$$

where $D = \begin{pmatrix} \frac{d\mu_t(\theta)}{d\theta} & \frac{dh_t(\theta)}{d\theta} \end{pmatrix}^T = \begin{pmatrix} y_{t-1} & 0 \\ 0 & 1 \\ 0 & \varepsilon^2_{t-1} \end{pmatrix}$

and $C = \begin{pmatrix} h_t & h_t^{3/2} E(\varepsilon^3) \\ h_t^{3/2} E(\varepsilon^3) & E(\varepsilon^4) - h_t^2 \end{pmatrix}$

In this paper, we only consider the symmetric distribution of error terms and it makes $E(\varepsilon^3_t) = 0$. Then we have $C = \begin{pmatrix} h_t & 0 \\ 0 & h^2_t E(\varepsilon^4_t) - h^2_t \end{pmatrix}$.

Therefore, we have the quasi-likelihood estimator which is the solution of $S_n(\theta) = 0$ and the asymptotic variance of the estimator.

$$\sqrt{n}(\widehat{\theta}_n - \theta) \rightarrow N(0, F^{-1}), \tag{3}$$

where $S_n(\widehat{\theta}_n) = 0$ and

$$F = \begin{bmatrix} \frac{y^2_{t-1}}{h_t} & 0 & 0 \\ 0 & \frac{1}{h_t[E(\varepsilon^4_t) - 1]} & \frac{\varepsilon^2_{t-1}}{h_t[E(\varepsilon^4_t) - 1]} \\ 0 & \frac{\varepsilon^2_{t-1}}{h_t[E(\varepsilon^4_t) - 1]} & \frac{\varepsilon^4_{t-1}}{h_t[E(\varepsilon^4_t) - 1]} \end{bmatrix} \tag{4}$$

3. Simulation Results

In this section, we consider the AR(1)-ARCH(1) model and we compare the sample variances of the quasi-likehood(QL) estimators with the maximum likelihood(ML) estimators under the normal, student's t, and the double exponential distributions for the error terms in the model. For the initial values of ϕ, α_0 and α_1, we set ϕ=0.3, 0.5, α_0=0.3 and α_1=0.1, then we generate the simulated data for the AR(1)-ARCH(1) model. In the following tables, we see that the sample variances of the QL estimators and those of the ML estimators have almost same values. This means that the ML estimator work quite well under the normal distribution. In this case of the heavy-tailed distributions such as the student's-t, the double exponential distributions, the QL estimators work better than the ML estimators.

[Table 4.1] $e_t \sim N(0, 1)$

Sample size		ϕ=0.3		α_0=0.3		α_1=0.1		ϕ=0.5		α_0=0.3		α_1=0.1		α_1=0.1	
		mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.
1000	MLE	0.299	0.055	0.300	0.045	0.095	0.063	0.199	0.055	0.300	0.045	0.095	0.063	0.095	0.063
	QL	0.299	0.063	0.300	0.045	0.095	0.063	0.499	0.055	0.300	0.045	0.095	0.063	0.095	0.063
1500	MLE	0.302	0.045	0.301	0.032	0.096	0.055	0.503	0.045	0.301	0.032	0.096	0.055	0.039	0.055
	QL	0.302	0.055	0.301	0.055	0.096	0.063	0.503	0.045	0.301	0.045	0.096	0.063	0.096	0.063
2000	MLE	0.305	0.045	0.300	0.032	0.099	0.055	0.505	0.045	0.300	0.032	0.099	0.055	0.099	0.055
	QL	0.305	0.045	0.300	0.032	0.099	0.055	0.505	0.045	0.300	0.032	0.100	0.055	0.099	0.055

[Table 4.2] $e_t \sim student-t(3)$

Sample size		ϕ=0.3		α_0=0.3		α_1=0.1		ϕ=0.5		α_0=0.3		α_1=0.1		α_1=0.1	
		mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.
1000	MLE	0.301	0.055	0.313	0.045	0.087	0.071	0.497	0.055	0.316	0.045	0.087	0.071	0.088	0.071
	QL	0.297	0.055	0.316	0.045	0.087	0.055	0.497	0.055	0.316	0.045	0.087	0.071	0.088	0.055
1500	MLE	0.296	0.055	0.315	0.045	0.090	0.063	0.497	0.055	0.315	0.045	0.090	0.063	0.091	0.063
	QL	0.295	0.055	0.315	0.045	0.090	0.055	0.497	0.045	0.314	0.045	0.091	0.055	0.090	0.055
2000	MLE	0.296	0.055	0.315	0.045	0.090	0.063	0.499	0.045	0.313	0.045	0.093	0.063	0.093	0.063
	QL	0.295	0.055	0.315	0.045	0.090	0.055	0.499	0.045	0.313	0.045	0.093	0.055	0.093	0.055

[Table 4.3] $e_t \sim student-t(5)$

Sample size		ϕ=0.3		α_0=0.3		α_1=0.1		ϕ=0.5		α_0=0.3		α_1=0.1		α_1=0.1	
		mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.
1000	MLE	0.299	0.063	0.332	0.055	0.091	0.071	0.498	0.063	0.332	0.055	0.091	0.071	0.091	0.071
	QL	0.299	0.063	0.332	0.045	0.091	0.071	0.498	0.055	0.332	0.045	0.091	0.071	0.092	0.071
1500	MLE	0.298	0.055	0.332	0.045	0.093	0.071	0.498	0.055	0.332	0.045	0.093	0.071	0.092	0.071
	QL	0.298	0.055	0.332	0.045	0.094	0.063	0.498	0.055	0.332	0.045	0.093	0.063	0.093	0.063
2000	MLE	0.300	0.055	0.330	0.045	0.096	0.063	0.499	0.055	0.330	0.045	0.096	0.063	0.096	0.063
	QL	0.300	0.045	0.330	0.045	0.096	0.063	0.499	0.045	0.330	0.045	0.096	0.063	0.096	0.063

[Table 4.4] $e_t \sim$ double exponential

Sample size		ϕ=0.3		α_0=0.3		α_1=0.1		ϕ=0.5		α_0=0.3		α_1=0.1		α_1=0.1	
		mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.
1000	MLE	0.291	0.084	0.294	0.071	0.095	0.095	0.493	0.084	0.294	0.071	0.095	0.095	0.096	0.095
	QL	0.292	0.055	0.294	0.045	0.094	0.063	0.493	0.055	0.293	0.045	0.095	0.063	0.096	0.063
1500	MLE	0.298	0.077	0.302	0.084	0.096	0.095	0.498	0.077	0.302	0.084	0.097	0.095	0.098	0.095
	QL	0.299	0.055	0.302	0.055	0.097	0.063	0.499	0.045	0.301	0.055	0.099	0.063	0.097	0.063
2000	MLE	0.303	0.071	0.306	0.063	0.086	0.084	0.503	0.071	0.306	0.063	0.098	0.084	0.085	0.084
	QL	0.303	0.045	0.306	0.045	0.086	0.055	0.504	0.045	0.305	0.045	0.099	0.055	0.086	0.055

4. Conclusion

Godambe(1985) proposed the optimal estimating function which is called the quasi-likelihood estimating function for the linear time series models and the estimating function approach can be applied to a class of nonlinear time series models. We have shown that the QL estimators which are the solutions of the equations may work very well under the heavy-tailed distributions. The estimating function would also be extended for the other class of times series models and would be applied to analyse the real data such as stock market data in Korea.

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