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Revisiting the Kendall Tau Statistic and Kendall's Tau Table Giving P-values

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Abstract

This paper gives a computer code that gives p-value's for Kendall's Tau table. The code centers around finding the cumulative frequency of a given Sigma value. Current tables usually give critical values for given cut-off percentages. For small sample sizes the critical values may be inexact, and for large samples the values are not be published. SPSS has a built in function for finding the Kendall rank correlation, but the output does not include the Sigma value. Before the code is given, this paper will concentrate around how to find the Sigma value, and the related tau statistic. The computer code is written in MATLAB, but the ideas can be easily transferred to different languages.

1. Background and an example of the Kendall Tau Statistic

The Kendall tau statistic is a measure of rank correlation. That is, are two samples of rankings independent or do they show signs of being dependent. Daniel(1990) describes Kendall's Tau as a measure of the strength of the association between two variables. The Kendall Tau statistic can be seen in many real life situations. For example, assume two political candidates from the same political party are each asked to rate 10 issues on a scale from 1-10. Here 1 is the most important and 10 is the least important. Assume that the rankings are

I_S	A	B	C	D	E	F	G	H	I	J
C_1	3	5	7	2	1	4	8	9	10	6
C_2	4	5	6	3	2	8	10	1	7	9

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The first row is the issue being ranked, the second row is the rankings of candidate one, and the third row is the rankings of candidate two. After looking at the results, one may question whether or not the rankings are independent of each other. If the rankings are similar, then the rankings may be suspected of being dependent. Dependence among the results might mean that party affiliation influences how the two candidates rate issues. It is important to note that this problem can be converted into an equivalent where the rankings of candidate one and candidate two are each simultaneously reordered so that the rankings of one of the candidates is all consecutive integers. For example, assume the issues are reordered so that the rankings of candidate one are in the order $1,2,3,\cdots,10$.

I_S	E	D	A	F	B	J	C	G	H	I
C_1	1	2	3	4	5	6	7	8	9	10
C_2	2	3	4	8	5	9	6	10	1	7

This rearranging can be thought of as moving the locating of eachissue. The order of the issues is immaterial, what matters is that each person's ranking correspond to the same issue. The problem is now, do the second candidate's rankings show dependence with the rankings $1,2,\cdots,10$? This particular example is examined in section 8. Reordering the issues generally makes hypothesis testing easier to compute. Therefore, this paper assumes that the two rankings that are being compared have been properly transformed so that one of the rankings is in the order $1,2,\cdots,n$

Other real-life examples are observed in many different areas. Daniel(1990) suggests that a business manager might suspect that job satisfaction and worker productivity are dependent. Kendall's Tau could be used to determine how strong the relationship is between the two variables. A philosophical statistician might want to determine if how a person ranks their personal happiness is dependent on the ranking of their income level. In education, a teacher might use Kendall's Tau to determine if the ranking of scores on a first test and the ranking of scores on a second test are dependent. This paper will examine this hypothesis for one particular instance, and it will be shown that there is significant evidence to say that the rankings of two tests are dependent. Details of this case study are given in section 7.

Other Authors have found uses for the Kendall Tau Statistic. Sen(1965) gives non-parametric alternatives to the commonly used least squares method, and uses Kendall's Tau in order obtain the estimates of the parameters in the regression model.

2. Introduction to the Tau statistic

Consider the arbitrary ordering of the four elements 1-4: 4,2,1,3. Also notice the natural ordering, 1,2,3,4. There is a central question that arises when dealing with the arbitrary ranking. Is it possible that this arbitrary ranking arose by chance, or is it dependent on the natural ordering? For example, we might suspect that the other rankings 1,2,4,3, and 4,3,2,1 are dependent on the natural ranking. The first because only the 4 and 3 have been switched. The second because it is the exact reverse of the natural ranking. The Kendall Tau statistic answers the question of independence by finding the correlation between the two rankings, and determines if it is statistically significant. Throughout this paper we will generally use the arbitrary ranking 4,2,1,3 for the use of examples.

3. Finding the Tau statistic

Before finding the tau statistic, a definition is required. We say that two numbers i and j of an ordered pair, (i,j), are in natural order if $i<j$. If an ordered pair is in natural order it is assigned a value of +1, otherwise it is given a value of -1. Consider our initial arbitrary ranking of the elements 1-4, 4,2,1,3. Look at each of the 6 ordered pairs where i is left of j in the arbitrary ranking. Consider 4 as the first entry in the ordered pair. This results in 3 distinct ordered pairs: (4,2),(4,1),(4,3). All of these ordered pairs are not in natural order, so each is assigned -1. The total is therefore $-1-1-1=-3$. Next we look at the ordered pairs with 2 as the first number and the entries to its right. We have (2,1),(2,3). Here we see that (2,3) is in natural order, and (2,1) is not. Thus the total score is $-1+1=0$. Finally, the remaining ordered pair is (1,3), as this is in natural order it is assigned +1. If we sum up each of the three scores we get a total of $-3+0+1=-2$. We will temporarily refer to this sum as the computed score.

When testing to see if an arbitrary ranking shows dependence we need to compute the number of ordered pairs that are in natural order and the number of pairs that are not in natural order. Clearly, if there is a strong dependent relationship between the two rankings, there will either be many ordered pairs in natural order or many ordered pairs not in natural order. This means that the absolute value of the computed score will be large. On the other hand if there is no dependence, then there will generally be as many ordered pairs in natural order as there are not in natural order. This means that the absolute value of the computed score will be small. The Tau statistic, τ, is therefore defined to be

$$\tau = \frac{computed\ score}{\max imum\ score\ possible}. \qquad (1)$$

Notice now that the maximum score will occur when the arbitrary ranking is the natural order. This is true since every ordered pair would be in natural order, and thus assigned a score of +1. In our example we would have the natural ranking 1,2,3,4. The 6 ordered pairs, $(1,2),(1,3),(1,4)\&(2,3),(2,4)\&(3,4)$ each are assigned a value of +1. Therefore the computed score is $3+2+1=6$. Since the computed score will always be less than or equal to the maximum score, τ will never be greater than 1.

Finding the maximum score can be easily extended for any arbitrarily large set of n elements in two ways, and can be found by calculating the computed score of the natural ordering, $1,2,\cdots,n$. First, Notice that there will be n-1 ordered pairs with the first element and the elements to its right, $(1,2),(1,3),\cdots,(1,n)$. There will be n-2 ordered pairs with the second element and the elements to its right, $(2,3),(2,4),\cdots,(2,n)$. There will be n-3 ordered pairs with the third element and the elements to its right, $(3,4),(3,5),\cdots,(3,n)$. This pattern will continue with the remaining elements. Now, since each ordered pair is from a natural ordering, each will be assigned a value of +1. Therefore, the maximum scores simplifies to $(n-1)+(n-2)+\cdots+1=\frac{n(n-1)}{2}$.

We can find the maximum score in a different manner. Notice that we are concerned with all distinct pairs of n elements. This means we will have C_2^n pairs. Where C is the combination function. If each pair has a value of +1, then the score will be $\frac{n!}{(n-2)!2!}=\frac{n(n-1)}{2}$. Here again we have the same maximum score. Calling the computed score Σ, and substituting $\frac{n(n-1)}{2}$ in for the maximum score we find that

$$\tau = -\frac{2\Sigma}{n(n-1)}.$$

This is the definition given by Kendall(1938) for τ, and it will be used in this paper.

It should also be noted that τ will be a minimum when the arbitrary ranking is in the reverse natural order. In this case, each of the $\frac{n(n-1)}{2}$ ordered pairs will be in not be in natural order. This means that $\Sigma=-\frac{n(n-1)}{2}$. Substituting the calculated Σ yields, $\tau = \frac{2n(n-1)}{2n(n-1)} = -1$. Therefore τ will always be a number between -1 and 1, much like the Pearson's correlation coefficient commonly used in statistics. In our example Σ = computed score = -2, and the subsequent τ statistic = $\frac{-2}{6} = \frac{-1}{3} \approx -.3333$.

In this section only one method has been given to find the Σ value and subsequent τ statistic, and this method is not particularly efficient when preformed by hand. It should be noted that the repetitive and incremental aspects of make this method easy for a computer to perform. In order to compute Σ by hand, there are a number of shorter methods that speed up the process which are given in Kendall(1938). We will outline the idea of one of these procedures with our ranking. If we look at the numbers 4,2,1,3, we see that 1 has two numbers to its left and one to its right. Since every number is greater than 1, we know that two ordered pairs will be in reverse natural order and one ordered pair will be in natural order. Therefore the sum of the two values would be $-2+1=-1$. Cross out 1 and look at the remaining numbers, 4,2,3. 2 has one number to its left and one to its right. Since we have removed 1, every number is now bigger than 2. Thus the sum of the values of the two ordered pairs will be $-1+1=0$. Cross out 2, and we have 4,3. Finally, 3 has only one number to its left, so it would have one ordered pair with a value of -1. Thus, the sum of these sums is $-1+0-1=-2$, which is the same Σ score that was found earlier. Meaning we again see that $\tau \approx -.3333$. It should be noted that finding τ is mostly a matter of finding Σ. In fact, exact Σ values can be found, but often times τ values are only approximations. Therefore the rest of this paper will be centered around the calculation of Σ.

Recall the original question concerning arbitrary rankings of numbers. Is it possible that this ranking arose by chance? In other words, how different is one arbitrary ranking from the know natural order? This is a question that arises many times in statistics when trying to compare two samples, for example, the t-statistic. The original question can be simplified and answered if we consider the following, in how many ways can we arrange n elements which result in a given Σ Value? Knowing this will in turn determine the probability of such an arbitrary ranking arising by chance.

We now need to introduce some notation. We will call $f_{n,\Sigma}$ the frequency of a given Σ value obtained from n elements. In other words, $f_{n,\Sigma}$ is the number of arrangements of n numbers that result in a given Σ value.

Consider now any arbitrary ranking of the first n integers. This ranking will have a known Σ value. What happens when the integer n+1 is inserted into this ranking? Since n+1 is greater than all of the other elements, Σ will vary depending on where n+1 is placed. If n+1 is inserted at the beginning, then there will be n new ordered pairs with n+1 as the left element and a smaller number as the right element. Therefore each of the n new ordered pairs will be assigned a value of -1. Thus n will be subtracted from Σ. If n+1 is inserted in the second position, there will be one ordered pair in natural order and n-1 ordered pairs not in natural order. Therefore one will be added and n-1 will be subtract from Σ for a total of $\Sigma-(n-2)$. If n+1 inserted in the third position, then n-4 is subtracted from Σ. If it is inserted in the last position then, +n will be added to Σ. Given this information, a recursive formula can be given to find the frequency of a given Σ value. That is

$$f_{n+1,\Sigma}= f_{n,\Sigma-n}+f_{n,\Sigma-(n-2)}+f_{n,\Sigma-(n-4)}+\cdots+f_{n,\Sigma+(n-4)} + f_{n,\Sigma+(n-2)}+f_{n,\Sigma+n} \tag{3}$$

We can now use this formula to find different frequencies of Σ values. Starting with $n=2$ there are only two possible rankings: 1,2 and 2,1. Therefore either $\Sigma=+1$ or -1 in both cases we see that $.f_{2,-1}=f_{2,1}=1$. Meaning, there is only one way to have a Σ value equal to +1 and -1, when n=2. Using equation 3, it is found that

$$f_{3,\Sigma}=f_{2,\Sigma-2}+f_{2,\Sigma}+f_{2,\Sigma+2}$$

Using this formula we can find the different values of $f_{3,\Sigma}$. For example, if $\Sigma=3$ then we have that

$$\begin{aligned} f_{3,3} &= f_{2,3-2}+f_{2,3}+f_{2,3+2} \\ &= f_{2,1}+f_{2,3}+f_{2,5} \\ &= 1+0+0=1 \end{aligned}$$

This means that there is only one arrangement of the three numbers 1,2,3 that will yield a Σ value of 3, which is the natural order. Similarly,

$$\begin{aligned} f_{3,2} &= f_{2,2-2}+f_{2,2}+f_{2,2+2} \\ &= f_{2,0}+f_{2,2}+f_{2,4} \\ &= 0+0+0=0 \end{aligned}$$

This tells us that there is no arrangement of three numbers having a Σ value of $+2$. Continuing this procedure we see that $f_{3,1}=2$ and $f_{3,0}=0$. Now notice what happens when trying to find $f_{3,-3}$. The formula becomes

$$\begin{aligned} f_{3,-3} &= f_{2,-3-2}+f_{2,-3}+f_{2,-3+2} \\ &= f_{2,-5}+f_{2,-3}+f_{2,-1} \\ &= 0+0+1=1 \end{aligned}$$

It is no coincidence that $f_{3,3}=f_{3,-3}$. This notion of symmetry is true for all values of n and all Σ, $-\Sigma$ values. In fact, this can be extended to the following for all $n \geq 2$

$$f_{n,\Sigma}=f_{n,-\Sigma}$$

Using the recursive formula to find the next set of frequencies we find

$$f_{4,6}=1, f_{4,5}=0, f_{4,4}=3, f_{4,3}=0, f_{4,2}=5, f_{4,1}=0, f_{4,0}=6$$

By symmetry, we know what the frequency for each $-\Sigma$ value is. That is,

$$f_{4,-6}=1, f_{4,-5}=0, f_{4,-4}=3, f_{4,-3}=0, f_{4,-2}=5, f_{4,-1}=0$$

In order to make future calculations easier, we will only be concerned with the frequency values that are not 0.

This recursive formula leads to two methods for finding the frequencies of Σ values. Kendall (1938) gives each method, and I will elaborate on them. It should be noted that the second method, which is an improvement of the first, is the one that is used in this paper to create the MATLAB program.

The first Method for finding the Σ-frequencies of n elements

In order to find the n+1 array of non-zero frequencies write the n^{th} array n+1 times. Each time start 1 row down and 1 column to the right. For example, in order to find the array of non-zero frequencies corresponding to n+1=3, create the following array. Each row represents the Σ-frequencies for n=2 elements. Recall that the possible frequencies when n=2 are $f_{2,-1}=f_{2,1}=1$.

$$\begin{pmatrix} f_{2,-1} & f_{2,1} & & \\ & f_{2,-1} & f_{2,1} & \\ & & f_{2,-1} & f_{2,1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{pmatrix}$$

Now, find each of the column sums, and this will be the Σ-frequency for the n+1 array.

$$\begin{array}{c} \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{pmatrix} \\ \hline \begin{matrix} 1 & 2 & 2 & 1 \end{matrix} \end{array}$$

These are the frequencies corresponding to n=3, and they are the same as the frequencies that were found earlier if zero-frequencies are omitted.

Similarly the frequencies for the array when n+1=4 are

$$\begin{array}{c} \begin{pmatrix} 1 & 2 & 2 & 1 & & & \\ & 1 & 2 & 2 & 1 & & \\ & & 1 & 2 & 2 & 1 & \\ & & & 1 & 2 & 2 & 1 \end{pmatrix} \\ \hline \begin{matrix} 1 & 3 & 5 & 6 & 5 & 3 & 1 \end{matrix} \end{array}$$

This process can be simplified into the second method.

The Second Method for finding the Σ-frequencies of n elements

Notice that each column sum is the sum of n+1 numbers where empty spaces are assumed to be zero. Also, the n+1 numbers follow a predictable pattern. Looking at table 5 we can see how the first method can be simplified.

$$\begin{matrix} 1) \\ 2) \\ 3) \\ 4) \end{matrix} \begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 2 & 1 & 0 & & \\ 1 & 3 & 5 & 6 & 5 & 3 & 1 \end{pmatrix}$$

These are the non-zero frequencies that we have already found. Notice now that the element in the n^{th} row and k^{th} column is the sum of $(n-1, k), (n-1, k-1), \cdots, (n-1, k-n+1)$. If no element is listed then the value is 0. For example, the "5"' in the position (4,5) is the sum of 0,1,2,2. Which is exactly how the "5" was computed in the first method.

Examination of equation 3 reveals that we are always adding an even amount to Σ in order to find the next Σ frequency. This means that, for a particular n, Σ will be all odd numbers or all even numbers including zero. Also, when examining matrix 5 we see that sometimes there is a unique middle number, n=4, and sometimes there is not a unique middle number, n=3. If there is a unique middle number, then that entry corresponds to $f_{n,0}$. $f_{n,0}$ is the frequency of $\Sigma=0$ for some n. The Σ values to the right increase as even numbers up to $\frac{n(n-1)}{2}$. The Σ values to the left decrease as negative even numbers down to $-\frac{n(n-1)}{2}$. If there is not a unique middle number, then the two middle entries correspond to $f_{n,\pm1}$. $f_{n,\pm1}$ is the frequency of $\Sigma=\pm1$ for a particular n. The Σ values to the right increase as odd numbers up to $\frac{n(n-1)}{2}$. The Σ values to the left decrease as negative odd numbers down to $-\frac{n(n-1)}{2}$.

4. Finding the P-value

Now we want to show how to find the p-value associated with the Σ values. Recall that there are $n!$ distinct ways to arrange the numbers $1,2,3,\cdots,n$. Therefore, the probability of a given Σ score is $\frac{f_{n,\Sigma}}{n!}$. For example, the probability of $\Sigma=0$, $prob(\Sigma=0)$, with $n=4$ is $\frac{6}{4!}=.25$. This probability is not generally used, instead a more interesting probability is $prob(\Sigma\le S)$, for given S,n. This represents the probability of Σ being as extreme or more extreme than a given S. For example, if n=4 then $prob(\Sigma\le 0)$ is $prob(\Sigma=0,-2,-4,-6)$ $=\frac{f_{4,0}+f_{4,-2}+f_{4,-4}+f_{4,-6}}{4!}=\frac{6+5+3+1}{4!}=\frac{15}{24}=.625$. Which means that 62.5% of all arrangements of the numbers 1,2,3,4 will have a Σ value of zero or less. We call this probability the p-value, and this is one of the statistics that the MATLAB code gives. Since each row is symmetric, and total probability must be 1, it is relatively quick to find the probabilities $prob(\Sigma \geq S)$, $prob(\Sigma > S)$, $prob(\Sigma < S)$.

5. Large Sample Approximation

Appendix A only gives the frequencies for $\Sigma \leq 25$. For n larger than 25 a different approach is necessary. Kendall(1938) shows that τ tends to normal for large n. In fact, even for small n $(n > 5)$, τ is approximated by the normal distribution surprisingly well. Recall that $\tau = \frac{2\Sigma}{n(n-1)}$ is a function of Σ, therefore Σ will also tend to normal. This fact is useful since we are more concerned with the Σ value in this paper.

One way to show the similarities between the normal distribution and the Σ distribution is to compare their cumulative distribution functions, CDF's. Since the Σ distribution is a population, we can find the population variance and the subsequent population standard deviation. Our calculations are simplified by the fact that the mean, $E(\Sigma)$, will always be zero. This comes from the symmetry of the distribution. Here, E, is the expected value function. Therefore the population variance is simply $E(\Sigma^2)$, a calculation that a computer can do rather quickly. The standard deviation is then $\sqrt{E(\Sigma^2)}$, again this is found quickly with the aid of the computer. After the standard deviation is found, Σ can be standardized by dividing Σ by its standard deviation. The result is a value that tends to standard normal for large n. As would be expected, MATLAB has a built in function for finding the CDF values for the standard normal distribution. Appendix C gives the max error for the comparisons of the standard normal CDF and the CDF of the standardized Σ distribution, for $n \leq 26$. As you can see when $n > 25$ the max error is less than .01. Therefore we have included the cumulative Σ frequencies for $n \leq 25$ in Appendix B. For $26 < n < 40$, the MATLAB code works rather quickly. For $n > 40 \Sigma$ values can be found by using the MATLAB code or they can be converted to an approximate normal score using the following formula.

$$Z = \Sigma \sqrt{\frac{18}{n(n-1)(2n+5)}}$$

where $\sqrt{\frac{n(n-1)(2n+5)}{18}}$ is the standard deviation of the Σ population. This formula can be used when $n > 25$ if the MATLAB program is not accessible.

Kendall(1938) gives this formula in the derivation of an approximation for the standard deviation of τ, $\sigma_{\tau} \sim \frac{2}{3}\frac{1}{\sqrt{n}}$. This formula makes calculating the standard deviation of τ easy. However, because of current technology, equation 6 can be used to give more accurate normal approximations for Σ. Since this paper is concerned with the Σ aspect of the τ statistic, equation 6 is preferred.

6. Ties

Previously, when objects were ranked in this paper, ties did not occur. However, often times when objects are ranked there will be ties. For example, In the case study performed in section 7, students test scores are examined. Since 42 scores are used, and by way of grading most scores are integers between 70% and 100%, it is entirely possible that more than one student will receive the same score. In such a case a method is necessary to deal with ties.

First, if their are only a few number of ties, then the mean rank of the ties can be assigned to each of the objects. for example the observations 19,19,13,12,17,14,17,17 would be ranked 7.5,7.5,2,1,5,3,5,5. Therefore, when computing Σ, ordered pairs comparing identical rankings are assigned a value of 0. If there are many ties, then Daniel(1990) offers a correction factor in addition to assigning the mean rank.

$$\hat{\tau} = \frac{\Sigma}{\sqrt{\frac{1}{2}n(n-1) - T_x}\sqrt{\frac{1}{2}n(n-1) - T_y}}$$

where $T_x = \frac{1}{2}\sum t_x(t_x - 1)$, $T_y = \frac{1}{2}\sum t_y(t_y - 1)$. In T_x and T_y, $\sum$, is the sum function. t_x is the number of objects that are tied in one particular set of rankings. t_y is the number of objects that are tied in the other set of rankings. The problem is that this correction factor is designed for τ. Therefore if we substitute $\frac{2\Sigma}{n(n-1)}$ for $\hat{\tau}$ we get the new correction formula,

$$\tilde{\Sigma} = \frac{n(n-1)\Sigma}{2\sqrt{\frac{1}{2}n(n-1) - T_x}\sqrt{\frac{1}{2}n(n-1) - T_y}}$$

Here T_x and T_y are defined the same way. Generally the correction factor will not effect the test statistic significantly if there are only a few ties. Therefore, it is generally acceptable to assign the mean of ranks and compute Σ directly.

7. Case Study

In section 1 a real life example of the Kendall-tau statistic was presented. At that time it was proposed that a teacher might want to determine if the scores that students received on two tests are dependent.

For a class of 42 students taking College Algebra the score on the first test and second test are recorded. The tests cover different material, chapter 1 and chapter 2 of the courses text respectively. The tests are constructed and graded by the same teacher. After all of the tests are corrected the scores are ranked from 1-42, where 1 is the highest and 42 is the lowest. Because of the large number of students in the study, the large sample approximation discussed in section 5 is used. In order to break ties, the mean rank is assigned to each of the tied ranks, as is suggested in section 6. Therefore the null hypothesis forthe experiment is,

$$H_0: test\ 1\ score \text{ and } test\ 2\ score\ are\ indepedent$$
$$H_1: test\ 1\ score\ and\ test\ 2\ score\ are\ not\ independent$$

Or it could be stated

$$H_0: \Sigma = 0$$

$$H_1: \Sigma \neq 0$$

Using Kendall's tau the Σ score is calculated to be 406. The corresponding z-value is, $z = 406\sqrt{\frac{18}{42(41)(89)}}$. $z = 4.40$. The p-value is $< .0001$. There is strong statistical evidence to reject to null at any reasonable level of significance. We conclude that the score a person receives on the first test and the score they receive on the second test are dependent. Another approach is to use the MATLAB program given in Appendix A. Using this program it can be seen that the p-value is $< .0000002$. In either case the evidence is overwhelming that the two scores are not independent.

These results are not surprising. It should be expected that a student that does well on their first test will tend to do well on the second test. In fact, a lack evidence to show dependence might in fact raise questions about the consistency of the instructor and their grading policies

8. Political Candidates Issue Ranking

In section 1 the idea of political candidates ranking issues was mentioned. At that time is was proposed that hypothetical rankings might be as follows,

I_S	E	D	A	F	B	J	C	G	H	I
C_1	1	2	3	4	5	6	7	8	9	10
C_2	2	3	4	8	5	9	6	10	1	7

The hypotheses might be

$$H_0: candidate\ 1\ \ rankings\ and\ candidate\ 2\ \ rankings\ are\ independent$$
$$H_1: candidate\ 1\ \ rankings\ and\ candidate\ 2\ \ rankings\ are\ not\ independent$$

A .05 level of significance is used to make the decision. If we find the Σ value for this particular set of rankings we get, $\Sigma = 17$. By Symmetry, $p(\Sigma \geq 17) = p(\Sigma \leq -17)$. If we refer to Appendix B we see that $p(\Sigma \leq -17) = .0779$. Therefore the two-sided p-value is $2 \times .0779 = .1558$. Therefore, there is not enough evidence at the .05 level of significance to conclude that the rankings of the two political candidates are not independent.

9. Justification of the Kendall Tau Statistic

In previous sections the Kendall Tau statistic is used, but no explanation is given as to the logic behind its formulation or the reason for its use. When finding the correlation between two variables linear regression could be used to find the strength of the association. The benefit of using linear regression is that it is commonly known, it can be expanded to multiple variables relatively easily, and it is included in most statistical software packages. The drawback is that it is based on assumptions that might not always be reasonable, for example, normality of the errors. Therefore, when it is unreasonable to make these assumptions, another approach is necessary. In general, methods that are not based on restricting assumption are called, non-parametric methods. Kendall's Tau is an example of one such method, and it is based on the logical idea of comparisons.

Kendall(1938) and Daniel(1990) each describe the rationalization for Σ in a similar manner using different examples. A contemporary example is used here. Assume that two film critics are each asked to rank ten films on a scale from

1-10. Where 1 is the best and 10 is the worst. Each critic then reviews and ranks each of the ten films. After the rankings are determined, all of the possible comparisons of two films are examined. If $\Sigma = 0$ initially, then very time that the two critics agree, 1 is added to Σ. Every time that they disagree, 1 is subtracted from Σ. Here a large Σ value indicates that there is significant agreement between the two critics, and therefore dependence. A large $-\Sigma$ value indicates that there is significant disagreement, and thus again dependence. A Σ value near 0 means that there is as much agreement as there is disagreement, and therefore there is no dependence. It should be noted that this particular example was inspired by the real-life critiquing of "the ten worst films of the year" by two critics.

10. Conclusion

The Kendall Tau Statistic is a nonparametric method used to determine the rank correlation of two samples of ranking. Current methods center around finding the τ statistic. Due to round off error, the tau statistic loses accuracy. This paper avoids the inaccuracies of the τ statistic by instead finding the related Σ value. The Σ values can be found exactly, and used to give more accurate p-values.

The MATLAB code uses a method discussed by Kendall(1938) in order to find the distribution of the Σ frequencies. The exact cumulative Σ frequencies, and the coresponding p-values, are given in Appendix B. The MATLAB code is given in Appendix A.

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Appendix A: MATLAB Code

```
clear; clc; format long;

n = input('What is the size of n? ');

number_of_elements = n*(n-1)/2+1;

if rem(number_of_elements,2) == 0
    number_of_neg_elements = (number_of_elements)/2;
else
    number_of_neg_elements = (number_of_elements+1)/2;
end

sigma_frequency(1,1) = 1;

for i = 2:1:n
    if rem(i,2) ==0
        for j = 1:i*(i-1)/2+1
            sigma_frequency(2,j)=0;
            for k = max(1,j-i+1):1:j
                sigma_frequency(2,j) = sigma_frequency(2,j)+...
                sigma_frequency(1,k);
            end
        end
   else
        for j = 1:i*(i-1)/2+1
            sigma_frequency(1,j)=0;
            for k = max(1,j-i+1):1:j
                sigma_frequency(1,j) = sigma_frequency(1,j)+...
                sigma_frequency(2,k);
            end
        end
   end
end
```

```
if rem(n,2)==0
    row = 2;
else
    row = 1;
end

neg_sigma_frequency(1,1) = 0;

for i = 1:number_of_neg_elements
    neg_sigma_frequency(1,i) = sigma_frequency(row,i);
end

sigma = -1*(number_of_elements-1):2:0;

sum_sigma_frequency=cumsum(sigma_frequency(row,:));

p_value = (neg_sigma_frequency)/factorial(n);

for g=1:number_of_neg_elements
    for f= 1:3
        spacer(g,f) = blanks(1);
    end
end

p_value = str2num(int2str(round(10000*p_value)))/10000;
disp('Sigma      Tau        Frequencies       p-value   ') disp('Values
[P(s <,= Sigma)]')
disp('_____________________________________________________')
table2 = [num2str(sigma',10)   spacer
num2str(neg_sigma_frequency',19)... spacer num2str((p_value),4)];
disp(table1);
disp('_____________________________________________________')
disp('Sigma      Tau        Frequencies       p-value') disp('Values
[P(s <,= Sigma)]')
```

Table 1 : Kendall Table, n=4-12

Σ	$f_{\Sigma, n}$	p
n=4		
-6	1	.0417
-4	4	.1667
-2	9	.375
0	15	.625
n=5		
-10	1	.0083
-8	5	.0417
-6	14	.1167
-4	29	.2417
-2	49	.4083
0	71	.5917
n=6		
-15	1	.0014
-13	6	.0083
-11	20	.0278
-9	49	.0681
-7	98	.1361
-5	169	.2347
-3	259	.3597
-1	360	.5
n=7		
-21	1	.0002
-19	7	.0014
-17	27	.0054
-15	76	.0151
-13	174	.0345
-11	343	.0681
-9	602	.1194
-7	961	.1907
-5	1416	.281
-3	1947	.3863
-1	2520	.5
n=8		
-28	1	.0000
-26	8	.0002
-24	35	.0009
-22	111	.0028
-20	285	.0071
-18	628	.0156
-16	1230	.0305
-14	2191	.0543
-12	3606	.0894
-10	5546	.1375
-8	8039	.1994
-6	11056	.2742
-4	14506	.3598
-2	18242	.4524
0	22078	.5476
n=9		
-36	1	.0000
-34	9	.0000
-32	44	.0001
-30	155	.0004
-28	440	.0012
-26	1068	.0029
-24	2298	.0063
-22	4489	.0124
-20	8095	.0223
-18	13640	.0376
-16	21671	.0597
-14	32692	.0901
-12	47087	.1298
-10	65044	.1792
-8	86494	.2384
-6	111078	.3061
-4	138151	.3807
-2	166826	.4597
0	196054	.5403
n=10		
-45	1	.0000
-43	10	.0000
-41	54	.0000
-39	209	.0001
-37	649	.0002
-35	1717	.0005
-33	4015	.0011
-31	8504	.0023
-29	16599	.0046
-27	30239	.0083
-25	51909	.0143
-23	84592	.0233
-21	131635	.0363
-19	196524	.0542
-17	282578	.0779
-15	392588	.1082
-13	528441	.1456
-11	690778	.1904
-9	878737	.2422
-7	1089826	.3003
-5	1319957	.3637
-3	1563651	.4309
-1	1814400	.5
n=11		
-55	1	.0000
-53	11	.0000
-51	65	.0000
-49	274	.0000
-47	923	.0000
-45	2640	.0001
-43	6655	.0002
-41	15159	.0004
-39	31758	.0008
-37	61997	.0016
-35	113906	.0029
-33	198497	.005
-31	330122	.0083
-29	526592	.0132
-27	808961	.0203
-25	1200900	.0301
-23	1727624	.0433
-21	2414387	.0605
-19	3284620	.0823
-17	4357847	.1092
-15	5647565	.1415
-13	7159307	.1794
-11	8889115	.2227
-9	10822629	.2711
-7	12934948	.324
-5	15191344	.3806
-3	17548819	.4396
-1	19958400	.5
n=12		
-66	1	.0000
-64	12	.0000
-62	77	.0000
-60	351	.0000
-58	1274	.0000
-56	3914	.0000
-54	10569	.0000
-52	25728	.0001
-50	57486	.0001
-48	119483	.0002
-46	233389	.0005
-44	431886	.0009
-42	762007	.0016
-40	1288588	.0027
-38	2097484	.0044
-36	3298110	.0069
-34	5024811	.0105
-32	7436558	.0155
-30	10714523	.0224
-28	15057211	.0314
-26	20673018	.0432
-24	27770328	.058
-22	36545537	.0763
-20	47169669	.0985

Table 2: Kendall Table n=12-15

Σ	$f_{\Sigma, n}$	p
-18	59774495	.1248
-16	74439247	.1554
-14	91179105	.1904
-12	109936605	.2295
-10	130576962	.2726
-8	152888031	.3192
-6	176585263	.3687
-4	201321587	.4203
-2	226701707	.4733
0	252299893	.5267
n=13		
-78	1	.0000
-76	13	.0000
-74	90	.0000
-72	441	.0000
-70	1715	.0000
-68	5629	.0000
-66	16198	.0000
-64	41926	.0000
-62	99412	.0000
-60	218895	.0000
-58	452284	.0001
-56	884170	.0001
-54	1646177	.0003
-52	2934764	.0005
-50	5032236	.0008
-48	8330269	.0013
-46	13354729	.0021
-44	20790013	.0033
-42	31500622	.0051
-40	46547264	.0075
-38	67194554	.0108
-36	94907396	.0152
-34	131333450	.0211
-32	178269730	.0286
-30	237612339	.0382
-28	311289579	.05
-26	401180096	.0644
-24	509019217	.0817
-22	636298069	.1022
-20	784161289	.1259
-18	953309994	.1531
-16	1143917058	.1837
-14	1355561554	.2177
-12	1587188429	.2549
-10	1837098114	.295
-8	2102968914	.3377
-6	2381912814	.3825
-4	2670562957	.4289
-2	2965188705	.4762
0	3261832095	.5238
n=14		
-91	1	.0000
-89	14	.0000
-87	104	.0000
-85	545	.0000
-83	2260	.0000
-81	7889	.0000
-79	24087	.0000
-77	66013	.0000
-75	165425	.0000
-73	384320	.0000
-71	836604	.0000
-69	1720774	.0000
-67	3366951	.0000
-65	6301715	.0001
-63	11333950	.0001
-61	19664206	.0002
-59	33018845	.0004
-57	53808417	.0006
-55	85307324	.001
-53	131848959	.0015
-51	199027315	.0023
-49	293892785	.0034
-47	425126823	.0049
-45	603177658	.0069
-43	840337713	.0096
-41	1150743122	.0132
-39	1550277041	.0178
-37	2056361494	.0236
-35	2687627327	.0308
-33	3463458347	.0397
-31	4403413612	.0505
-29	5526540657	.0634
-27	6850601589	.0786
-25	8391242754	.0963
-23	10161146314	.1166
-21	12169207832	.1396
-19	14419787196	.1654
-17	16912080423	.194
-15	19639656789	.2253
-13	22590199305	.2591
-11	25745477052	.2953
-9	29081565821	.3336
-7	32569319638	.3736
-5	36175081035	.415
-3	39861603412	.4572
-1	43589145600	.5
n=15		
-105	1	.0000
-103	15	.0000
-101	119	.0000
-99	664	.0000
-97	2924	.0000
-95	10813	.0000
-93	34900	.0000
-91	100913	.0000
-89	266338	.0000
-87	650658	.0000
-85	1487262	.0000
-83	3208036	.0000
-81	6574987	.0000
-79	12876702	.0000
-77	24210652	.0000
-75	43874857	.0000
-73	76893688	.0001
-71	130702001	.0001
-69	216008780	.0002
-67	347855479	.0003
-65	546874905	.0004
-63	840743603	.0006
-61	1265804413	.001
-59	1868816646	.0014
-57	2708770039	.0021
-55	3858676557	.003
-53	5407232824	.0041
-51	7460227367	.0057
-49	10141552979	.0078
-47	13593677376	.0104
-45	17977426782	.0137
-43	23470948594	.0179
-41	30267741766	.0231
-39	38573677196	.0295
-37	48602974551	.0372
-35	60573155068	.0463
-33	74699049479	.0571
-31	91186003079	.0697
-29	110222482210	.0843
-27	131972343802	.1009
-25	156567077732	.1197
-23	184098366512	.1408
-21	214611324656	.1641
-19	248098778364	.1897
-17	284496923429	.2176
-15	323682655417	.2475
-13	365472802548	.2795
-11	409625411124	.3132
-9	455843139932	.3486
-7	503778718997	.3852
-5	553042325313	.4229
-3	603210630012	.4613
-1	653837184000	.5

Table 3: Kendall Table, n=16, 17

Σ	$f_{\Sigma,n}$	p
n=16		
−120	1	.0000
−118	16	.0000
−116	135	.0000
−114	799	.0000
−112	3723	.0000
−110	14536	.0000
−108	49436	.0000
−106	150349	.0000
−104	416687	.0000
−102	1067345	.0000
−100	2554607	.0000
−98	5762643	.0000
−96	12337630	.0000
−94	25214332	.0000
−92	49424984	.0000
−90	93299841	.0000
−88	170193528	.0000
−86	300895514	.0000
−84	516904175	.0000
−82	864758990	.0000
−80	1411630971	.0001
−78	2252363761	.0001
−76	3518133274	.0002
−74	5386849007	.0003
−72	8095352708	.0004
−70	11953378607	.0006
−68	17359124169	.0008
−66	24816143500	.0012
−64	34951121492	.0017
−62	48531922166	.0023
−60	66485138296	.0032
−58	89912212033	.0043
−56	120103060111	.0057
−54	158546035306	.0076
−52	206933001077	.0099
−50	267158300666	.0128
−48	341310475240	.0163
−46	431655734716	.0206
−44	540612412513	.0258
−42	670715939669	.0321
−40	824574247362	.0394
−38	1004813937317	.048
−36	1214018029149	.058
−34	1454656580146	.0695
−32	1729011950596	.0826
−30	2039100928637	.0975
−28	2386596304403	.1141
−26	2772750766933	.1325
−24	3198326165099	.1529
−22	3663531206900	.1751
−20	4167970557662	.1992
−18	4710608032606	.2251
−16	5289746167127	.2528
−14	5903023902036	.2821
−12	6547433462513	.3129
−10	7219356767714	.345
−8	7914620918050	.3783
−6	8628571508414	.4124
−4	9356161749210	.4472
−2	10092054683429	.4823
0	10830735204571	.5177
n=17		
−136	1	.0000
−134	17	.0000
−132	152	.0000
−130	951	.0000
−128	4674	.0000
−126	19210	.0000
−124	68646	.0000
−122	218995	.0000
−120	635682	.0000
−118	1703027	.0000
−116	4257634	.0000
−114	10020277	.0000
−112	22357907	.0000
−110	47572239	.0000
−108	96997223	.0000
−106	190297064	.0000
−104	360490592	.0000
−102	661386105	.0000
−100	1178290264	.0000
−98	2043049119	.0000
−96	3454679291	.0000
−94	5707039329	.0000
−92	9225158067	.0000
−90	14611957638	.0000
−88	22707159997	.0001
−86	34660121917	.0001
−84	52018178741	.0001
−82	76831767634	.0002
−80	111777126483	.0003
−78	160296711019	.0005
−76	226756634983	.0006
−74	316619422032	.0009
−72	436629182302	.0012
−70	595005024080	.0017
−68	801637129643	.0023

Table 4: Kendall Table, n=17, 18

Σ	$f_{\Sigma, n}$	p
−66	1068278526134	.003
−64	1408724242384	.004
−62	1838968346129	.0052
−60	2377328394881	.0067
−58	3044526201276	.0086
−56	3863713599631	.0109
−54	4860432184240	.0137
−52	6062496834782	.017
−50	7499794290759	.0211
−48	9203990097855	.0259
−46	11208139905000	.0315
−44	13546204287237	.0381
−42	16252469915874	.0457
−40	19360883868940	.0544
−38	22904312015729	.0644
−36	26913736538085	.0757
−34	31417411569614	.0883
−32	36439999436075	.1024
−30	42001712862871	.1181
−28	48117490590668	.1353
−26	54796234945869	.1541
−24	62040139924250	.1744
−22	69844137185302	.1964
−20	78195484997195	.2198
−18	87073521651475	.2448
−16	96449600275900	.2712
−14	106287216464094	.2988
−12	116542333915043	.3277
−10	127163906580590	.3575
−8	138094588933943	.3882
−6	149271619194331	.4197
−4	160627853973395	.4516
−2	172092927136606	.483
0	183594500959394	.5162
n=18		
−153	1	.0000
−151	18	.0000
−149	170	.0000
−147	1121	.0000
−145	5795	.0000
−143	25005	.0000
−141	93651	.0000
−139	312646	.0000
−137	948328	.0000
−135	2651355	.0000
−133	6908989	.0000
−131	16929266	.0000
−129	39287173	.0000
−127	86859412	.0000
−125	183856635	.0000
−123	374153699	.0000
−121	734644291	.0000
−119	1396030396	.0000
−117	2574320659	.0000
−115	4617369761	.0000
−113	8072048900	.0000
−111	13779087278	.0000
−109	23004240671	.0000
−107	37616179099	.0000
−105	60323270450	.0000
−103	94983173372	.0000
−101	147000716431	.0000
−99	223830781038	.0000
−97	335603649887	.0001
−95	495890340629	.0001
−93	722624617705	.0001
−91	1039196467498	.0002
−89	1475728652577	.0002
−87	2070543379593	.0003
−85	2871820018644	.0004
−83	3939437158673	.0006
−81	5346983110793	.0008
−79	7183908407803	.0011
−77	9557782123393	.0015
−75	12596601285340	.002
−73	16451089726904	.0026
−71	21296909953506	.0033
−69	27336699628291	.0043
−67	34801833797133	.0054
−65	43953805716247	.0069
−63	55085113853613	.0086
−61	68519541014367	.0107
−59	84611714219222	.0132
−57	103745841453179	.0162
−55	126333534046876	.0197
−53	152810641402659	.0239
−51	183633047948193	.0287
−49	219271410254625	.0342
−47	260204844591362	.0406
−45	306913610939646	.0479
−43	359870877539386	.0562
−41	419533689068755	.0655
−39	486333300052781	.076
−37	560665071450345	.0876
−35	642878160917580	.1004
−33	733265264358698	.1145
−31	832052686532033	.13
−29	939391030349221	.1467

Table 5: Kendall Table, n=28, 19

Σ	$f_{\Sigma,n}$	p
−27	1055346797024811	.1648
−25	1179895181671517	.1843
−23	1312914330949974	.2051
−21	1454181301054429	.2271
−19	1603369916175306	.2504
−17	1760050680596615	.2749
−15	1923692843149606	.3005
−13	2093668652615200	.327
−11	2269259778914386	.3544
−9	2449665809839128	.3826
−7	2634014669074216	.4114
−5	2821374740781872	.4407
−3	3010768431416670	.4703
−1	3201186852864000	.5
n=19		
−171	1	.0000
−169	19	.0000
−167	189	.0000
−165	1310	.0000
−163	7105	.0000
−161	32110	.0000
−159	125761	.0000
−157	438407	.0000
−155	1386735	.0000
−153	4038090	.0000
−151	10947079	.0000
−149	27876345	.0000
−147	67163518	.0000
−145	154022930	.0000
−143	337879565	.0000
−141	712033264	.0000
−139	1446677555	.0000
−137	2842707951	.0000
−135	5417028610	.0000
−133	10034398370	.0000
−131	18106447252	.0000
−129	31885534360	.0000
−127	54889773910	.0000
−125	92505947214	.0000
−123	152829192659	.0000
−121	247812272380	.0000
−119	394812676165	.0000
−117	618642508875	.0000
−115	954243507407	.0000
−113	1450126939047	.0000
−111	2172734627486	.0000
−109	3211891807811	.0000
−107	4687533600976	.0000
−105	6757893123934	.0001
−103	9629338988879	.0001
−101	13568041503261	.0001
−99	18913628583658	.0002
−97	26094962670802	.0002
−95	35648127424434	.0003
−93	48236656660874	.0004
−91	64673967300500	.0005
−89	85947873013335	.0007
−87	113246956462527	.0009
−85	147988466989210	.0012
−83	191847289532085	.0016
−81	246785402669267	.002
−79	315081112902596	.0026
−77	399357223471931	.0033
−75	502607174584481	.0041
−73	628218084013652	.0052
−71	779989528948813	.0064
−69	962146848244429	.0079
−67	1179347715119461	.0097
−65	1436680739692179	.0118
−63	1739654913473152	.0143
−61	2094178807901745	.0172
−59	2506528588562697	.0206
−57	2983304106492085	.0245
−55	3531372576657090	.029
−53	4157799647847766	.0342
−51	4869768002252958	.04
−49	5674483989156700	.0466
−47	6579073185708788	.0541
−45	7590466177017352	.0624
−43	8715276244835256	.0716
−41	9959671034770864	.0819
−39	11329240621606072	.0931
−37	12828864696328200	.1055
−35	14462581842877940	.1189
−33	16233464044624888	.1334
−31	18143499649291896	.1492
−29	20193488017951656	.166
−27	22382948983199424	.184
−25	24710050041333992	.2031
−23	27171553904576480	.2234
−21	29762788646924396	.2447
−19	32477642199735616	.267
−17	35308582402596600	.2903
−15	38246703206625152	.3144
−13	41281796978920240	.3394
−11	44402452188277080	.365
−9	47596175084741472	.3913
−7	50849533340829464	.418

Table 6: Kendall Table, n=19, 20

Σ	$f_{\Sigma, n}$	p
−5	54148319021736344	.4451
−3	57477727715917752	.4725
−1	60822550204416016	.5
n=20		
−190	1	.0000
−188	20	.0000
−186	209	.0000
−184	1519	.0000
−182	8624	.0000
−180	40734	.0000
−178	166495	.0000
−176	604902	.0000
−174	1991637	.0000
−172	6029727	.0000
−170	16976806	.0000
−168	44853151	.0000
−166	112016669	.0000
−164	266039599	.0000
−162	603919164	.0000
−160	1315952428	.0000
−158	2762629983	.0000
−156	5605337934	.0000
−154	11022366544	.0000
−152	21056764914	.0000
−150	39163212165	.0000
−148	71048746506	.0000
−146	125938520227	.0000
−144	218444466131	.0000
−142	371273651685	.0000
−140	619085891955	.0000
−138	1013898442359	.0000
−136	1632540512827	.0000
−134	2586782633499	.0000
−132	4036905534456	.0000
−130	6209629214863	.0000
−128	9421493146329	.0000
−126	14108959583787	.0000
−124	20866698684791	.0000
−122	30495699794105	.0000
−120	44063029264102	.0000
−118	62975211170205	.0000
−116	89067331133056	.0000
−114	124710041528880	.0001
−112	172936663791384	.0001
−110	237592524644632	.0001
−108	323508512123607	.0001
−106	436700578812224	.0002
−104	584596539854220	.0002
−102	776291000193646	.0003
−100	1022828590590533	.0004
−98	1337514890816964	.0005
−96	1736253471780020	.0007
−94	2237906402857094	.0009
−92	2864674359931699	.0012
−90	3642491154253026	.0015
−88	4601426110689644	.0019
−86	5776086292208129	.0024
−84	7206009138776374	.003
−82	8936034713260647	.0037
−80	11016645479659132	.0045
−78	13504260439638172	.0056
−76	16461469583459456	.0068
−74	19957194032692112	.0082
−72	24066757023879004	.0099
−70	28871851058831464	.0119
−68	34460387174974828	.0142
−66	40926213404221088	.0168
−64	48368691114249232	.0199
−62	56892120069552400	.0234
−60	66605005701654000	.0274
−58	77619165210357472	.0319
−56	90048672683213744	.037
−54	104008647351507200	.0428
−52	119613893312118432	.0492
−50	136977403432461504	.0563
−48	156208744602168736	.0642
−46	177412345870248704	.0729
−44	200685715171890496	.0825
−42	226117614162993824	.0929
−40	253786224002016480	.1043
−38	283757337613189376	.1166
−36	316082615909293888	.1299
−34	350797946539261952	.1442
−32	387921943870334400	.1594
−30	427454628056358528	.1757
−28	469376319151943296	.1929
−26	513646779307063936	.2111
−24	560204632151782912	.2303
−22	608967083622865408	.2503
−20	659829962792510592	.2712
−18	712668094863818752	.2929
−16	767336011554586240	.3154
−14	823668996779710848	.3386
−12	881484458059176448	.3623
−10	940583606630439424	.3866
−8	1000753422042399488	.4113
−6	1061768870261406976	.4364
−4	1123395338226308352	.4618
−2	1185391242530828288	.4872
0	1247510765645811456	.5128

Table 7: Kendall Table, n=21

Σ	$f_{\Sigma,n}$	p
	n=21	
-210	1	.0000
-208	21	.0000
-206	230	.0000
-204	1749	.0000
-202	10373	.0000
-200	51107	.0000
-198	217602	.0000
-196	822504	.0000
-194	2814141	.0000
-192	8843868	.0000
-190	25820674	.0000
-188	70673825	.0000
-186	182690494	.0000
-184	448730093	.0000
-182	1052649257	.0000
-180	2368601685	.0000
-178	5131231668	.0000
-176	10736569602	.0000
-174	21758936146	.0000
-172	42815701060	.0000
-170	81978913225	.0000
-168	153027659730	.0000
-166	278966179937	.0000
-164	497410645859	.0000
-162	868684296025	.0000
-160	1487770179356	.0000
-158	2501668580981	.0000
-156	4134208927313	.0000
-154	6720990955910	.0000
-152	10757894498729	.0000
-150	16967517683865	.0000
-148	26388993853388	.0000
-146	40497908584024	.0000
-144	61364495252146	.0000
-142	91859929006652	.0000
-140	135922354351590	.0000
-138	198896249569367	.0000
-136	287960818072440	.0000
-134	412665254263386	.0000
-132	585590895688226	.0000
-130	823162363567944	.0000
-128	1146631712479386	.0000
-126	1583261242545104	.0000
-124	2167731843879097	.0000
-122	2943804399606612	.0001
-120	3966261716545460	.0001
-118	5303157521470469	.0001
-116	7038397094808130	.0001
-114	9274670957152396	.0002
-112	12136758534450596	.0002
-110	15775212783169166	.0003
-108	20370429264643948	.0004
-106	26137094063705748	.0005
-104	33328994242898336	.0007
-102	42244162257474192	.0008
-100	53230312037339216	.001
-98	66690509447713280	.0013
-96	83089003820002528	.0016
-94	102957130521561584	.002
-92	126899177503911712	.0025
-90	155598091898951808	.003
-88	189820886549282016	.0037
-86	230423591441379488	.0045
-84	278355581976816512	.0054
-82	334663105506514688	.0066
-80	400491820207975040	.0078
-78	477088156827741952	.0093
-76	565799314620138752	.0111
-74	668071708499865856	.0131
-72	785447695409127168	.0154
-70	919560424481656960	.018
-68	1072126677929572608	.021
-66	1244937597689131776	.0244
-64	1439847226568814080	.0282
-62	1658758831593031424	.0325
-60	1903609020881787392	.0373
-58	2176349713015317504	.0426
-56	2478928068484973056	.0485
-54	2813264545440775680	.0551
-52	3181229295278417920	.0623
-50	3584617166310897664	.0702
-48	4025121634404009472	.0788
-46	4504308026536098816	.0882
-44	5023586445283660800	.0983
-42	5584184837792276480	.1093
-40	6187122680515234816	.1211
-38	6833185769677400064	.1337
-36	7522902616021628928	.1472
-34	8256522940118126592	.1616
-32	9033998750825795584	.1768
-30	9854968464144115712	.1929
-28	10718744482754054144	.2098
-26	11624304608413292544	.2275
-24	12570287600769351680	.246
-22	13554993128128288768	.2653
-20	14576386279611107328	.2853
-18	15632106725559422976	.306
-16	16719482525861466112	.3272
-14	17835548496086413312	.3491
-12	18977068951093350400	.3714
-10	20140564557340479488	.3942

Table 8: Kendall Table, n=21, 22

Σ	$f_{\Sigma, n}$	p
−8	21322342940681052160	.4173
−6	22518532618151161856	.4408
−4	23725119752156917760	.4644
−2	24937987165389262848	.4881
0	26152955006320173056	.5119
n=22		
−231	1	.0000
−229	22	.0000
−227	252	.0000
−225	2001	.0000
−223	12374	.0000
−221	63481	.0000
−219	281083	.0000
−217	1103587	.0000
−215	3917728	.0000
−213	12761596	.0000
−211	38582270	.0000
−209	109256095	.0000
−207	291946589	.0000
−205	740676682	.0000
−203	1793325939	.0000
−201	4161927624	.0000
−199	9293159292	.0000
−197	20029728894	.0000
−195	41788665040	.0000
−193	84604366100	.0000
−191	166583279325	.0000
−189	319610939055	.0000
−187	598577118991	.0000
−185	1095987764829	.0000
−183	1964672060624	.0000
−181	3452442238231	.0000
−179	5954110808839	.0000
−177	10088319685045	.0000
−175	16809310423353	.0000
−173	27567204099578	.0000
−171	44534718969302	.0000
−169	70923703978822	.0000
−167	111421586742172	.0000
−165	172786011320493	.0000
−163	264645757636651	.0000
−161	400567663258148	.0000
−159	599462860178258	.0000
−157	887421309649013	.0000
−155	1300081432680731	.0000
−153	1885661591799355	.0000
−151	2708802196431153	.0000
−149	3855391093209479	.0000
−147	5438570356841358	.0000
−145	7606149173060725	.0000
−143	10549674606487400	.0000
−141	14515438912387000	.0000
−139	19817727749561444	.0000
−137	26854637074190216	.0000
−135	36126806362761632	.0000
−133	48259430688284912	.0000
−131	64027922480498168	.0001
−129	84387593850643392	.0001
−127	110507720396665280	.0001
−125	143810325645710224	.0001
−123	186013989994600384	.0002
−121	239182937536687456	.0002
−119	305781587055394048	.0003
−117	388734668521044992	.0003
−115	491492902793037184	.0004
−113	618104119478876416	.0005
−111	773289546123564800	.0007
−109	962524841777158528	.0009
−107	1192125270854970112	.0011
−105	1469334221119307264	.0013
−103	1802414065383276800	.0016
−101	2200738153747372800	.002
−99	2674882506175507968	.0024
−97	3236715559079101440	.0029
−95	3899484110057496576	.0035
−93	4677893408371815424	.0042
−91	5588179161896320000	.005
−89	6648169081291442176	.0059
−87	7877331466197404672	.007
−85	9296808263501574144	.0083
−83	10929430001030899712	.0097
−81	12799710027669788672	.0114
−79	14933815578427631616	.0133
−77	17359513334875265024	.0154
−75	20106087370868326400	.0179
−73	23204227662326743040	.0206
−71	26685887698116079616	.0237
−69	30584110155016175616	.0272
−67	34932820089653321728	.0311
−65	39766585648387702784	.0354
−63	45120346894738604032	.0401
−61	51029113993277022208	.0454
−59	57527636657447911424	.0512
−57	64650047453261561856	.0575
−55	72429482236551946240	.0644
−53	80897681672757608448	.072
−51	90084578428401860608	.0801
−49	100017875215746793472	.089
−47	110722619399678427136	.0985
−45	122220780322518204416	.1087

Table 9: Kendall Table, n=22, 23

Σ	$f_{\Sigma, n}$	p
−43	134530835852957368320	.1197
−41	147667374905999654912	.1314
−39	161640722799966060544	.1438
−37	176456596304945741824	.157
−35	192115795088016834560	.1709
−33	208613935970625224704	.1856
−31	225941235982524940288	.201
−29	244082349627927560192	.2172
−27	263016265079767826432	.234
−25	282716263197520723968	.2515
−23	303149942336373915648	.2697
−21	324279310897410408448	.2885
−19	346060948479170641920	.3079
−17	368446235352213684224	.3278
−15	391381648813564690432	.3482
−13	414809123811911991296	.369
−11	438666474092409913344	.3903
−9	462887869017207144448	.4118
−7	487404360198911033344	.4336
−5	512144451162306969600	.4556
−3	537034702445992017920	.4778
−1	562000363888803840000	.5
n=23		
−253	1	.0000
−251	23	.0000
−249	275	.0000
−247	2276	.0000
−245	14650	.0000
−243	78131	.0000
−241	359214	.0000
−239	1462801	.0000
−237	5380529	.0000
−235	18142125	.0000
−233	56724395	.0000
−231	165980490	.0000
−229	457927079	.0000
−227	1198603761	.0000
−225	2991929700	.0000
−223	7153857324	.0000
−221	16447016616	.0000
−219	36476745510	.0000
−217	78265410550	.0000
−215	162869776650	.0000
−213	329453055975	.0000
−211	649063995030	.0000
−209	1247641114021	.0000
−207	2343628878849	.0000
−205	4308300939451	.0000
−203	7760743177430	.0000
−201	13714853984268	.0000
−199	23803173656939	.0000
−197	40612484016811	.0000
−195	68179687835306	.0000
−193	112714405701021	.0000
−191	183638105762115	.0000
−189	295059679742691	.0000
−187	467845652480914	.0000
−185	732491300861470	.0000
−183	1133058672173029	.0000
−181	1732520791674605	.0000
−179	2619940307997679	.0000
−177	3920017578750786	.0000
−175	5805669877390849	.0000
−173	8514452044093108	.0000
−171	12369801348637548	.0000
−169	17808287101112806	.0000
−167	25414269690894208	.0000
−165	35963624686442552	.0000
−163	50478465021710560	.0000
−161	70295096783507176	.0000
−159	97147769185636768	.0000
−157	133271123106160160	.0000
−155	181524599683636224	.0000
−153	245542433844449344	.0000
−151	329913218384669376	.0000
−149	440393371577235072	.0000
−147	584159162503975936	.0000
−145	770102228794597504	.0000
−143	1009173744744542848	.0000
−141	1314782545788616448	.0001
−139	1703252568552024832	.0001
−137	2194344903681803776	.0001
−135	2811849560300502016	.0001
−133	3584251685114418176	.0001
−131	4545476445458895872	.0002
−129	5735716054722066432	.0002
−127	7202341473644942336	.0003
−125	9000900147935009792	.0003
−123	11196199731325540352	.0004
−121	13863476088327987200	.0005
−119	17089641972800602112	.0007
−117	20974610643945709568	.0008
−115	25632686324567965696	.001
−113	31194010849390096384	.0012
−111	37806053124318773248	.0015
−109	45635125159827890176	.0018
−107	54867905500848963584	.0021
−105	65712947908029218816	.0025
−103	78402150215302348800	.003
−101	93192155468084264960	.0036

Table 10: Kendall Table, n=23, 24

Σ	$f_{\Sigma, n}$	p	Σ	$f_{\Sigma, n}$	p
−99	110365654812964929536	.0043	n=24		
−97	130232559246296580096	.005	−276	1	.0000
−95	153131005321567928320	.0059	−274	24	.0000
−93	179428158351162966016	.0069	−272	299	.0000
−91	209520775603386122240	.0081	−270	2575	.0000
−89	243835491573560573952	.0094	−268	17225	.0000
−87	282828787675824717824	.0109	−266	95356	.0000
−85	326986609728786137088	.0126	−264	454570	.0000
−83	376823598451208159232	.0146	−262	1917371	.0000
−81	432881900887536762880	.0167	−260	7297900	.0000
−79	495729534275415048192	.0192	−258	25440025	.0000
−77	565958278358219620352	.0219	−256	82164420	.0000
−75	644181077524801650688	.0249	−254	248144910	.0000
−73	731028940394124410880	.0283	−252	706071989	.0000
−71	827147331499813765120	.032	−250	1904675750	.0000
−69	933192057491120324608	.0361	−248	4896605450	.0000
−67	1049824658651742208000	.0406	−246	12050462774	.0000
−65	1177707325423408119808	.0456	−244	28497479390	.0000
−63	1317497368863210471424	.051	−242	64974224900	.0000
−61	1469841283399675019264	.0569	−240	143239635450	.0000
−59	1635368449703589969920	.0633	−238	306109412100	.0000
−57	1814684534763937071104	.0702	−236	635562468075	.0000
−55	2008364655156134674432	.0777	−234	1284626463105	.0000
−53	2216946377803784257536	.0858	−232	2532267577126	.0000
−51	2440922640060843556864	.0944	−230	4875896455975	.0000
−49	2680734677478284460032	.1037	−228	9184197395425	.0000
−47	2936765052977688870912	.1136	−226	16944940572832	.0000
−45	3209330885159046610944	.1241	−224	30659794556825	.0000
−43	3498677375966803656704	.1353	−222	54462968211488	.0000
−41	3804971738797586776064	.1472	−220	95075452213649	.0000
−39	4128297627255061872640	.1597	−218	163255139970824	.0000
−37	4468650162075349286912	.1729	−216	275969545312631	.0000
−35	4825931649229813448704	.1867	−214	459607649611945	.0000
−33	5199948075868961636352	.2011	−212	754667323974107	.0000
−31	5590406462649617350656	.2162	−210	1222512958312896	.0000
−29	5996913141175771201536	.232	−208	1955004202449971	.0000
−27	6418973013909676490752	.2483	−206	3088062708642510	.0000
−25	6855989841139921321984	.2652	−204	4820583042390036	.0000
−23	7307267585629046702080	.2827	−202	7440522151783954	.0000
−21	7772012830638143963136	.3006	−200	11360536738605040	.0000
−19	8249338271400487550976	.3191	−198	17166199462138564	.0000
−17	8738267264073184837632	.338	−196	25680635059215056	.0000
−15	9237739400033619410944	.3573	−194	38050399931107088	.0000
−13	9746617057413871697920	.377	−192	55858608766809344	.0000
−11	10263692866291551436800	.397	−190	81272715587926896	.0000
−9	10787698009284970283008	.4173	−188	117236010821313472	.0000
−7	11317311265727840256000	.4378	−186	167713826779028992	.0000
−5	11851168695398348881920	.4584	−184	238007675921422144	.0000
−3	12387873847198779179008	.4792	−182	335153101478180096	.0000
−1	12926008369442493300736	.5	−180	468419916283400832	.0000

Table 11: Kendall Table, n=24

Σ	$f_{\Sigma, n}$	p
−178	649936755223859712	.0000
−176	895465474214324736	.0000
−174	1225354889425337344	.0000
−172	1665707648518555648	.0000
−170	2249798631334696448	.0000
−168	3019788145723593216	.0000
−166	4028778252362374144	.0000
−164	5343265738471247872	.0000
−162	7046050461370791936	.0000
−160	9239662873751734272	.0000
−158	12050379375380064256	.0000
−156	15632898539702808576	.0000
−154	20175755044853706752	.0000
−152	25907551081997021184	.0000
−150	33104086885764571136	.0001
−148	42096472581655486464	.0001
−146	53280302511632392192	.0001
−144	67125970312859271168	.0001
−142	84190198015968968704	.0001
−140	105128845035228233728	.0002
−138	130711052894774493184	.0002
−136	161834768647381090304	.0003
−134	199543674002514247680	.0003
−132	245045528039235977216	.0004
−130	299731908940401278976	.0005
−128	365199314414586036224	.0006
−126	443271551411503693824	.0007
−124	536023313508010688512	.0009
−122	645804809158471712768	.001
−120	775267266175973720064	.0012
−118	927389097752797118464	.0015
−116	1105502473558171451392	.0018
−114	1313319996593005395968	.0021
−112	1554961143262884134912	.0025
−110	1834978081378408333312	.003
−108	2158380439422080122880	.0035
−106	2530658561427829161984	.0041
−104	2957804746260644102144	.0048
−102	3446331939062414311424	.0056
−100	4003289317272699011072	.0065
−98	4636274195066175094784	.0075
−96	5353439659371971739648	.0086
−94	6163497348898985345024	.0099
−92	7075714795746160017408	.0114
−90	8099906768073334456320	.0131
−88	9246420082647352999936	.0149
−86	10526111398386244517888	.017
−84	11950317556626091933696	.0193
−82	13530818100828833841152	.0218
−80	15279789687684741464064	.0246
−78	17209752192625573625856	.0277
−76	19333506414961271767040	.0312
−74	21664063400209148805120	.0349
−72	24214565518441137045504	.039
−70	26998199566097256022016	.0435
−68	30028102292905138323456	.0484
−66	33317258893268556447744	.0537
−64	36878395140492582977536	.0594
−62	40723863980071815479296	.0656
−60	44865527532418381119488	.0723
−58	49314635583196988702720	.0795
−56	54081701758178414297088	.0872
−54	59176378686552615682048	.0954
−52	64607333549370167525376	.1041
−50	70382125485755040006144	.1134
−48	76507086386500831543296	.1233
−46	82987206640630065266688	.1338
−44	89826027413777095327744	.1448
−42	97025541026525837524992	.1564
−40	104586100965175612932096	.1686
−38	112506342996346012696576	.1813
−36	120783118770360209637376	.1947
−34	129411443186948171104256	.2086
−32	138384456661469207986176	.223
−30	147693403272040912257024	.238
−28	157327625589635466395648	.2536
−26	167274576796773408309248	.2696
−24	177519850488737625538560	.2861
−22	188047228327446131507200	.3031
−20	198838745485773706887168	.3205
−18	209874773582964041187328	.3383
−16	221134120573766467584000	.3564
−14	232594146819104818855936	.3749
−12	244230896338500581326848	.3936
−10	256019242028122136117248	.4126
−8	267933043427064978341888	.4318
−6	279945315431899851653120	.4512
−4	292028406198970898972672	.4707
−2	304154182338317169721344	.4902
0	316294219394922307387392	.5098
n=25		
−300	1	.0000
−298	25	.0000
−296	324	.0000
−294	2899	.0000
−292	20124	.0000
−290	115480	.0000
−288	570050	.0000
−286	2487421	.0000
−284	9785321	.0000

Table 12: Kendall Table, n=24, 25

Σ	$f_{\Sigma, n}$	p	Σ	$f_{\Sigma, n}$	p
−282	35225346	.0000	−182	48985457040508542976	.0000
−280	117389766	.0000	−180	64615267517502701568	.0000
−278	365534676	.0000	−178	84786201979314012160	.0000
−276	1071606665	.0000	−176	110686312539159248896	.0000
−274	2976282415	.0000	−174	143779038888185217024	.0000
−272	7872887865	.0000	−172	185858345270378561536	.0000
−270	19923350639	.0000	−170	239112967146951737344	.0000
−268	48420830029	.0000	−168	306200887059879886848	.0000
−266	113395054929	.0000	−166	390335226467082043392	.0000
−264	256634690379	.0000	−164	495382798786722398208	.0000
−262	562744102479	.0000	−162	625976615670675537920	.0000
−260	1198306570554	.0000	−160	787643670491277557760	.0001
−258	2482933033659	.0000	−158	986949336817870307328	.0001
−256	5015200610785	.0000	−156	1231659711755628052480	.0001
−254	9891097066760	.0000	−154	1530923200779746017280	.0001
−252	19075294462185	.0000	−152	1895472578439108100096	.0001
−250	36020235035016	.0000	−150	2337848664376397529088	.0002
−248	66680029591817	.0000	−148	2872646622994983026688	.0002
−246	121142997803006	.0000	−146	3516785724504936022016	.0002
−244	216218450014080	.0000	−144	4289803192049574871040	.0003
−242	379473589967679	.0000	−142	5214172501656647761920	.0003
−240	655443135184954	.0000	−140	6315646196962457288704	.0004
−238	1115050784342329	.0000	−138	7623622927816991965184	.0005
−236	1869718106399065	.0000	−136	9171538020618505027584	.0006
−234	3092231057414061	.0000	−134	10997276439123161251840	.0007
−232	5047235234424007	.0000	−132	13143606499169861107712	.0008
−230	8135297860902097	.0000	−130	15658632162057987817472	.001
−228	12955880655147224	.0000	−128	18596261153273777487872	.0012
−226	20396402100859188	.0000	−126	22016685541254196363264	.0014
−224	31756936934788480	.0000	−124	25986870771641133039616	.0017
−222	48923131500321600	.0000	−122	30581048494125653950464	.002
−220	74603754509073888	.0000	−120	35881207850985995632640	.0023
−218	112654125942701600	.0000	−118	41977579229572122017792	.0027
−216	168512669735286048	.0000	−116	48969103827302315196416	.0032
−214	249785242083577504	.0000	−114	56963881750340420239360	.0037
−212	367020946795478912	.0000	−112	66079590780092995862528	.0043
−210	534734138012039808	.0000	−110	76443867409831855915008	.0049
−208	772740529306998784	.0000	−108	88194641292455432945664	.0057
−206	1107891098517601792	.0000	−106	101480413865245038084096	.0065
−204	1576306138904546560	.0000	−104	116460471643989369421824	.0075
−202	2226233709931010816	.0000	−102	133305024522200346001408	.0086
−200	3121682239204762624	.0000	−100	152195259385750122135552	.0098
−198	4347006468835543040	.0000	−98	173323299472451248848896	.0112
−196	6012659654385887232	.0000	−96	196892060181733933318144	.0127
−194	8262363210268369920	.0000	−94	223114992481655230300160	.0144
−192	11281988100851992576	.0000	−92	252215705676807571243008	.0163
−190	15310490383669053440	.0000	−90	284427462096517955846144	.0183
−188	20653296514490687488	.0000	−88	319992537240417551253504	.0206
−186	27698592308537507840	.0000	−86	359161440077226507763712	.0232
−184	36937032669330931712	.0000	−84	402191989528266520133632	.0259

Table 13: Kendall Table, n=25

Σ	$f_{\Sigma, n}$	p	Σ	$f_{\Sigma, n}$	p
−82	449348244672041455714304	.029	−40	2839757609880907065327616	.1831
−80	500899287868792037703680	.0323	−38	3035473361559519506726912	.1957
−78	557117861809084019245056	.0359	−36	3238980393917948406792192	.2088
−76	618278863419391778750464	.0399	−34	3450134108413652153401344	.2224
−74	684657699587874099298304	.0441	−32	3668752516257520294035456	.2365
−72	756528511779308813942784	.0488	−30	3894616129931241681059840	.2511
−70	834162278760566866575360	.0538	−28	4127468157443659821744128	.2661
−68	917824808825444940906496	.0592	−26	4367015006232606991712256	.2815
−66	1007774635056224610025472	.065	−24	4612927100141774267482112	.2974
−64	1104260829253326929395712	.0712	−22	4864840009289542349619200	.3136
−62	1207520752167025664589824	.0778	−20	5122355888950252161466368	.3302
−60	1317777759538999535337472	.085	−18	5385045219842649665568768	.3472
−58	1435238885169321635479552	.0925	−16	5652448838521240632164352	.3644
−56	1560092523729962030596096	.1006	−14	5924080242950803730464768	.3819
−54	1692506137314318266400768	.1091	−12	6199428154868592459907072	.3997
−52	1832624010711328247250944	.1181	−10	6477959317257704756477952	.4176
−50	1980565081093140392181760	.1277	−8	6759121502221031668973568	.4358
−48	2136420868181668927635456	.1377	−6	7042346701807028147322880	.454
−46	2300253530990673829822464	.1483	−4	7327054471940780412895232	.4724
−44	2472094076910350083555328	.1594	−2	7612655397595646764515328	.4908
−42	2651940748200409179357184	.171	0	7898554645735339290787840	.5092

Table 14: Appendix C: max errors

n	error	n	error
4	.1265	18	.016
5	.0962	19	.0148
6	.0745	20	.0138
7	.0602	21	.0129
8	.0501	22	.0121
9	.0424	23	.0113
10	.0367	24	.0107
11	.032	25	.0101
12	.0284	26	.0095
13	.0253	27	.0091
14	.0228	28	.0086
15	.0207	29	.0082
16	.0189	30	.0078
17	.0174		

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