Journal of Korean
Data & Information Science Society
2005, Vol. 16, No. 2, pp. 467～476

Multivariate Control Charts for Several Related Quality Characteristics

Duk-Joon Chang[1) ・ Jae-Kyoung Shin[2)

Abstract

Multivariate control charts for monitoring mean vector of several related quality variables with combine-accumulate approach and accumulate-combine apprach were investigated. Shewhart chart is also proposed to compare the performances of CUSUM and EWMA charts. Numerical comparisons show that CUSUM and EWMA charts are more efficient than Shewhart chart for small or moderate shifts, and multivariate charts based on accumulate- combine approach is more efficient than corresponding multivariate charts based on combine-accumulate approach.

Keywords : accumulate-combine approach, Multivariate control chart

1. Introduction

In many industrial quality control, there exist several related quality variables for defining the quality of the product. When the quality characteristics are correlated, one can obtain better sensitivity by using multivariate control chart than separate control charts for each of the process parameters. Therefore, the control procedure using separate charts for each individual parameter may not be optimal for detecting simultaneous changes in the process.

The ability of a control chart to detect process changes is determined by the length of time required for the chart to signal when the process is out-of-control state. Therefore a good control chart should quickly detect shifts in production process parameters while producing few false alarms. The expected time to signal is simply the product of the average run length(ARL) and the length of sampling

1) First Author : Professor, Dept. of Statistics, Changwon National University, Changwon, 641-773, Korea.
E-mail : djchang@sarim.changwon.ac.kr

2) Professor, Dept. of Statistics, Changwon National University, Changwon, 641-773, Korea.

interval. Therefore, the ARL can be thought of as the expected time to signal.

The original work on multivariate control chart was introduced by Hotelling(1947). Ghare and Togersen(1968) presented multivariate Shewhart control chart based on Hotelling's T^2 statistic to control means of two variables simultaneously. And Alt(1984) and Jackson(1985) reviewed much of the studies on multivariate control charts. Multivariate approach for control chart has become increasingly popular in recent years.

Crosier(1988) and Pignatiello and Runger(1990) considered new multivariate CUSUM control schemes that accumulate past sample information for each parameter and then form a univariate CUSUM statistic from the multivariate data for monitoring the mean vector of a multivariate normal process. Pignatiello and Runger(1990) showed that the chart based on accmulate-combine approach that accumulates past sample information for each parameter and then combines the seperate accumulations into a univariate statistic does better than combine-accumulate approach that combines the multivariate data into a univarite statistic and then accmualtes.

A multivariate EWMA control chart for monitoring mean vector of a multivariate normal process using accmulate-combine approach was presented by Lowry et al.(1992). By simulation, they showed that the performances of the multivariate EWMA procedure performs better than the multivariate CUSUM procedures of Crosier(1988) and Pignatiello and Runger(1990), and it performs roughly the same if small shift in the mean vector has occurred. Vargas et al.(2004) presented a comparative study of the performance of CUSUM and EWMA charts in order to detect small changes of process average.

In this paper, we investigate multivariate control charts for simultaneously controlling the shifts on the mean vector of multivariate normal process. By markov chain method and simulation, we found that multivariate EWMA and CUSUM charts based on accumulate-combine feature is more effective than those of the charts based on the combine-accumulate feature.

2. Multivariate Shewhart Chart

Assume that the process of interest has $p\,(p \geq 2)$ quality variables represented by the random vector $\underset{\sim}{X}' = (X_1, X_2, \cdots, X_p)$ and we take a sequence of random vectors $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \underset{\sim}{X}_3, \cdots$ where $\underset{\sim}{X}_i = (\underset{\sim}{X}'_{i1}, \cdots, \underset{\sim}{X}'_{in})'$ is a sample of observations at the sampling time $i\,(i = 1, 2, \cdots)$ and $\underset{\sim}{X}_{ij} = (X_{ij1}, \cdots, X_{ijp})'$. It will be also assumed that the successive observation vectors are distributed independent multivariate normal distribution with $N_p(\underset{\sim}{\mu}, \Sigma)$. Hence, the distribution of $\underset{\sim}{X}$ is indexed by a set of parameters $\underset{\sim}{\theta} = (\underset{\sim}{\mu}, \Sigma)$ where $\underset{\sim}{\mu}$ is the mean vector and Σ is the dispersion matrix of $\underset{\sim}{X}$. Let $\underset{\sim}{\theta}_0 = (\underset{\sim}{\mu}_0, \Sigma_0)$ be the known target values for θ.

Because of the equivalence between control chart and a significance test, we can obtain multivariate control statistic for monitoring μ by using the likelihood ratio test(LRT) statistic for testing $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ where Σ_0 is known.

Likelihood ratio λ_i at the ith sample can be expressed as

$$\lambda_i = \exp\left[-\frac{n}{2} (\overline{X}_i - \mu_0)' \Sigma_0^{-1} (\overline{X}_i - \mu_0) \right].$$

Let us define $Z_i^2 = -2 \ln \lambda_i$. Then

$$Z_i^2 = n(\overline{X}_i - \mu_0)' \Sigma_0^{-1} (\overline{X}_i - \mu_0). \tag{2.1}$$

Thus, the sample statistic Z_i^2 can be used as the control statistic for monitoring μ of p related quality variables. Alt(1984) described various types of multivariate Shewhart type T^2 charts based on Hotelling's $T^2 = n(\overline{X}_i - \mu_0)' S^{-1} (\overline{X}_i - \mu_0)$ statistic and provided recommendations for implementation where S is the covariance matrix of the sample.

If the process is in-control, the statistic Z_i^2 has a chi-squared distribution with p degrees of freedom when $\mu = \mu_0$ and $\Sigma = \Sigma_0$. When the process has shifted to μ from the target μ_0, Z_i^2 has a non-central chi-square distribution with p degrees of freedom and noncentrality parameter $\tau^2 = n(\mu - \mu_0)' \Sigma_0^{-1} (\mu - \mu_0)$.

Shewhart chart, one of the most widely used control chart, has a good ability to detect large changes in monitored parameter quickly and is easy to implement the process. A multivariate Shewhart chart for μ based on the sample statistics Z_i^2 in (2.1) signals whenever

$$Z_i^2 \geq h \tag{2.2}$$

The percentage point of Z_i^2 can be obtained from the chi-square distribution when the process are in-control or target mean vector has changed. UCL(upper control limit) h can be obtained from chi-square distributions to satisfy a desired ARL. And, ARL of this chart can be calculated as $1/P$ where P denotes the probability that χ^2 exceeds the UCL. The on-target value of P is determined from the probability that χ^2 exceeds the UCL under the central $\chi^2(p)$ distribution while the off-target value of P is determined from the probability that χ^2 exceeds the UCL under the noncentral $\chi^2(p)$ distribution

Result 2.1 Let $X_{ij} = (X_{ij1}, X_{ij2}, \cdots, X_{ijp})'$ be distributed according to $N_p(\mu_0, \Sigma_0)$

and X_{ij}'s be independent. If the process parameters of the distribution are changed as $N(\mu, c\Sigma_0)$ where c is a constant, then

$$ARL = \frac{1}{1-F(h/c)},$$

where h is the control limit of the chart in (2.2) and $F(\cdot)$ is a chi-squared distribution function with p degrees of freedom and noncentrality parameter $n(\mu-\mu_0)'\Sigma_0^{-1}(\mu-\mu_0)/c$.

3. Multivariate CUSUM Chart

The CUSUM chart is a good alternative to the Shewhart control chart when small shifts are important. The CUSUM chart directly incorporates all of the information in the sequence of sample values by plotting the cumulative sums of the deviation of the sample values from the target value.

3.1 Combine-Accumulate Approach

The most direct and obvious method of replacing the multivariate Shewhart chart by a multivariate CUSUM procedure for μ is to form a CUSUM of the scalars based on the statistics Z_i^2 stated above. A multivariate CUSUM for μ based on the statistic $Z_i^2 (i=1, 2, \cdots)$ at the ith sample is

$$Y_{Z^2,i} = \max\{Y_{Z^2,i-1}, 0\} + (Z_i^2 - k_{Z^2}), \tag{3.1}$$

where $Y_{Z^2,0} = \omega(\omega \geq 0)$ and $k_{Z^2} \geq 0$. This chart signals whenever $Y_{Z^2,i} \geq h_{Z^2}$.

When the process is in-control or mean vector μ has shifted, the performances of the multivariate CUSUM chart in (3.1) can be evaluated by the Markov chain or integral equation approach.

3.2 Accumulate-Combine Approach

Crosier(1988) proposed a multivariate CUSUM chart which accumulate the $\overline{X}_i$ vectors before producing the quadratic forms. This chart is based on the statistics

$$C_i = \{n(S_{i-1} - \overline{X}_i - \mu_0)'\Sigma_0^{-1}(S_{i-1} + \overline{X}_i - \mu_0)\}^{\frac{1}{2}} \tag{3.2}$$

and

$$\mathbf{S}_i = \begin{cases} \mathbf{0} & \text{if } C_i \le k_1 \\ (\mathbf{S}_{i-1} + \overline{\mathbf{X}}_i - \boldsymbol{\mu}_0)(1 - k_1/C_i) & \text{if } C_i > k_1, \end{cases}$$

where $i = 1, 2, \cdots, \mathbf{S}_0 = \mathbf{0}$ and $k_1 > 0$.

This multivariate CUSUM scheme signals when

$$Y_i = \{ n\mathbf{S}_i' \Sigma_0^{-1} \mathbf{S}_i \}^{1/2} > h_1, \tag{3.3}$$

where $h_1 > 0$.

Pignatiello and Runger(1990) also proposed a multivariate CUSUM chart based on accumulate-combine approach for controlling mean vector of the multivariate normal process. This chart, called MC1, is based on the following vectors of cumulative sum :

$$\mathbf{D}_i = \sum_{j=i-l_i+1}^{i} (\overline{\mathbf{X}}_j - \boldsymbol{\mu}_0)$$

and

$$MC1_i = \max\{0, (n\mathbf{D}_i' \Sigma_0^{-1} \mathbf{D}_i)^{1/2} - kl_i\}, \tag{3.4}$$

where reference value $k > 0$,

$$l_i = \begin{cases} l_{i-1} + 1 & \text{if } MC1_{i-1} > 0 \\ 1 & \text{otherwise} \end{cases}$$

and $i = 1, 2, \cdots$.

An out-of-control signal is given as soon as $MC1_i > h_2$ where $h_2 > 0$. Pignatiello and Runger (1990) found that an MC1 chart based on the accumulate-combine approach has a superior ARL performance than a multivariate CUSUM chart based on the combine-accumulate approach.

4. Multivariate EWMA Chart

Multivariate exponentially weighted moving average (MEWMA) chart also takes two ways to use the past sample information as multivariate CUSUM chart, such as combine-accumulate approach and accumulate-combine approach.

4.1 Combine-Accumulate Approach

Multivariate EWMA chart for $\boldsymbol{\mu}$ based on the sample statistic $Z_i^2 (i = 1, 2, \cdots)$ in (2.1) is given by

$$Y_{Z^2,i} = (1-\lambda) Y_{Z^2,i-1} + \lambda Z_i^2, \tag{4.1}$$

where $Y_{Z^2,0} = \omega\,(\omega \geq 0)$ and $\lambda\ (0 < \lambda \leq 1)$ is a smoothing constant. This chart signals whenever $Y_{Z^2,i} \geq h$. When the smoothing constant λ is 1, this chart changes to multivariate Shewhart chart.

When the parameters are on-target or mean vector $\underset{\sim}{\mu}$ has shifted, the performances of this chart can be evaluated by the Markov chain approach or integral equation approach. The parameter h can be obtained to satisfy a specified ARL by Markov chain approach or integral equation approach.

4.2 Accumulate-Combine Approach

Lowry et al.(1992) proposed a MEWMA chart for $\underset{\sim}{\mu}$ with accumulate-combine technique. They asserted that MEWMA chart for $\underset{\sim}{\mu}$ is a more straightforward generalization of the coresponding univariate procedure than the multivariate CUSUM statistics in (3.3) and (3.4).

MEWMA chart for mean vector $\underset{\sim}{\mu}$ is a multivariate extension of univariate EWMA chart. The vectors of EWMA's are defined as

$$\underset{\sim}{Y}_i = (I-\Lambda)\,\underset{\sim}{Y}_{i-1} + \Lambda\,\overline{\underset{\sim}{X}}_i \tag{4.2}$$

$i = 1, 2, \cdots$ where $\underset{\sim}{Y}_0 = \underset{\sim}{\mu}_0$ and $\Lambda = diag(\lambda_1, \cdots, \lambda_p)$, $0 < \lambda_j \leq 1\ (j = 1, 2, \cdots, p)$. Equation (4.2) can be rewritten by repeated substitution as

$$\underset{\sim}{Y}_i = \sum_{k=1}^{i} \Lambda (I-\Lambda)^{i-k}\, \overline{\underset{\sim}{X}}_k + (I-\Lambda)^i \underset{\sim}{\mu}_0 \tag{4.3}$$

Then the dispersion matrix of $\underset{\sim}{Y}_i$ can be obtained as

$$\begin{aligned} \Sigma_{\underset{\sim}{Y}_i} &= \sum_{k=1}^{i} Var(\Lambda (I-\Lambda)^{i-k}\, \overline{\underset{\sim}{X}}_k) \\ &= \sum_{k=1}^{i} [\Lambda (I-\Lambda)^{i-k} \Sigma (I-\Lambda)^{i-k} \Lambda] / n, \end{aligned} \tag{4.4}$$

where Σ is a known covariance matrix. This MEWMA chart for means signals as soon as

$$T_i^2 = (\underset{\sim}{Y}_i - \underset{\sim}{\mu}_0)' \Sigma_{\underset{\sim}{Y}_i}^{-1} (\underset{\sim}{Y}_i - \underset{\sim}{\mu}_0) > h_1,$$

where $h_1\,(>0)$ is chosen to achieve a specified in-control ARL. If there is no distinct reason to weight past observations differently for the p quality variables being monitored, we can let $\lambda_1 = \cdots = \lambda_p = \lambda$. Then, MEWMA vectors can be written as

$$\underline{Y}_i = (1-\lambda)\,\underline{Y}_{i-1} + \lambda\,\overline{\underline{X}}_i \quad (4.5)$$

$i=1,2,\cdots$, and the dispersion matrix of $\underline{Y}_i$ is given by

$$\Sigma_{\underline{Y}_i} = \frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]\frac{\Sigma}{n}\,. \quad (4.6)$$

<Table 1> ARL for multivariate charts for mean vector ($p=2$)

shifted scale of τ	Shewhart Chart	CUSUM Chart				EWMA Chart			
		C-A Approach		A-C Approach		C-A Approach		A-C Approach	
		$k_Z^2=2.2$	$k_Z^2=2.6$	$k_2=0.4$	$k_2=0.6$	$\lambda=0.1$	$\lambda=0.3$	$\lambda=0.1$	$\lambda=0.3$
in-control	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
$\tau=0.5$	115.5	79.8	87.4	27.8	35.1	89.4	98.5	25.1	42.8
$\tau=1.0$	41.9	22.6	22.5	9.4	9.5	28.3	28.4	7.7	10.6
$\tau=1.5$	15.8	10.1	8.8	5.5	5.0	13.2	10.5	4.0	4.8
$\tau=2.0$	6.9	5.8	4.8	4.0	3.5	7.8	5.4	2.6	3.0
$\tau=2.5$	3.5	3.9	3.1	3.2	2.7	5.3	3.4	1.9	2.1
$\tau=3.0$	2.2	2.8	2.3	2.6	2.3	3.9	2.4	1.5	1.6
$\tau=3.5$	1.5	2.2	1.8	2.3	2.0	3.0	1.9	1.3	1.3
$\tau=4.0$	1.2	1.8	1.5	2.1	1.8	2.4	1.5	1.1	1.2
$\tau=4.5$	1.1	1.5	1.2	1.9	1.6	2.0	1.3	1.0	1.1
$\tau=5.0$	1.0	1.3	1.1	1.8	1.4	1.7	1.1	1.0	1.0

<Table 2> ARL for multivariate charts for mean vector ($p=4$)

shifted scale of τ	Shewhart Chart	CUSUM Chart				EWMA Chart			
		C-A Approach		A-C Approach		C-A Approach		A-C Approach	
		$k_Z^2=4.2$	$k_Z^2=4.8$	$k_2=0.4$	$k_2=0.6$	$\lambda=0.1$	$\lambda=0.3$	$\lambda=0.1$	$\lambda=0.3$
in-control	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
$\tau=0.5$	138.1	99.2	108.2	31.8	41.7	110.9	120.1	31.5	58.3
$\tau=1.0$	61.0	32.2	32.5	10.8	10.9	40.6	41.6	9.5	14.1
$\tau=1.5$	24.6	14.5	12.3	6.5	5.8	19.6	15.6	4.9	6.0
$\tau=2.0$	10.6	8.3	6.5	4.7	4.1	11.8	7.7	3.1	3.6
$\tau=2.5$	5.2	5.4	4.1	3.8	3.2	8.1	4.7	2.2	2.5
$\tau=3.0$	2.9	3.9	2.9	3.2	2.6	5.9	3.3	1.7	1.9
$\tau=3.5$	1.9	3.0	2.2	2.8	2.3	4.6	2.5	1.4	1.5
$\tau=4.0$	1.4	2.4	1.8	2.4	2.1	3.7	2.0	1.2	1.3
$\tau=4.5$	1.2	2.0	1.5	2.2	1.9	3.0	1.7	1.1	1.1
$\tau=5.0$	1.1	1.7	1.3	2.0	1.8	2.6	1.4	1.0	1.1

Lowry et al.(1992) also showed that the distribution of T_i^2 depends on μ and Σ only through the noncentrality parameter τ as

$$\tau = [\, n(\mu - \mu_0)' \Sigma^{-1} (\mu - \mu_0)]^{1/2}$$

and smaller values of λ are more effective in detecting the smaller shifts in the mean vectors.

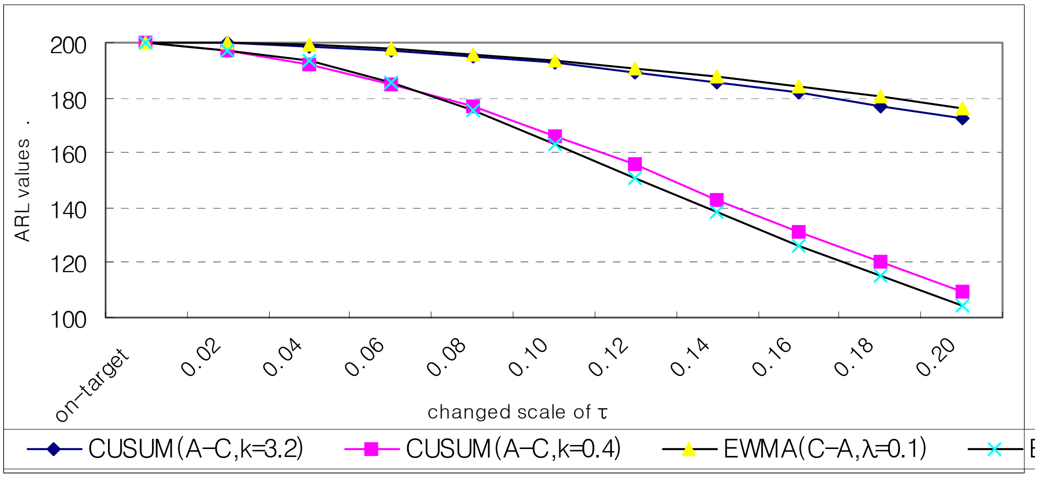
The multivariate EWMA chart based on accumulate-combine approach for monitoring means signals whenever $T_i^2 > h_1$. The false alarm probability of the chart based on T_i^2 is $(1-\alpha)$ where the signal probabilities of the chart for means is α. Since it is difficult to obtain the exact distribution of T_i^2, we obtain the parameters h_1 and performances of this chart by simulation.

5. Computational Results and Concluding Remarks

In order to evaluate the ARL performances and compare the proposed multivariate Shewhart, CUSUM and EWMA charts, some kinds of standards for comparison are necessary. For simplicity in our computation, we assume that $\mu_0 = 0'$, all diagonal and off-diagonal elements of Σ_0 are 1 and 0.3, respectively. And, the ARL of the proposed charts when the process is in-control were fixed to be 200 and the sample size for each characteristic was five for $p = 2 \sim 4$.

After the reference values and the smoothing constants of the proposed charts have been determined, the UCL $h's$ of the charts were calculated by Markov chains with the number of transient states $r = 100$ or simulation with 10,000 iterations. When the process is in-control or mean vector of the quality variables has changed, the ARL performances of the proposed charts are presented in Table 1~2 and Figure 1. The data in Table shows that accumulate-combine procedure yields small ARLs than combine-accumulate procedure both EWMA and CUSUM charts. When small or moderate changes in the process have occurred, CUSUM and EWMA charts are more efficient than Shewhart chart in our computation. In combine-accumulate approach, the performance of multivariate EWMA schemes is approximately equivalent to that of CUSUM schemes when small or moderate changes in the process have occurred. Numerical results for various reference values show that large reference values are efficient for large shifts and vice versa in multivariate CUSUM charts.

As illustrated in table, smaller values of λ are more effective in detecting small shifts in the mean vector in EWMA charts. Our numerical results also show that multivariate EWMA chart based on accumulate-combine approach can be recommended for any scale of shifts in mean vector of the process. The optimal selection of λ depends on the size of the shift in the mean vector to be detected quickly. And, it may be possible to improve the performance of the MC1 chart at selected off-target conditions with alternate choices of k.



<Figure 1> ARL performances for small shift($p=3$)

References

1. Alt, F.B. (1984). *Multivariate Quality Control in the Encyclopedia of Statistical Sciences*, eds. S. Kotz and Johnson Wiley, New York.
2. Crosier, R.B. (1988). Multivariate Generalization of Cumulative Sum Quality-Control Scheme, *Technometrics*, 30, 291-303.
3. Ghare, P.H. and Torgersen, P.E. (1968). The Multicharacteristic Control Chart, *Journal of Industrial Engineering*, 19, 269-272.
4. Hotelling H.(1947). *Multivariate Quality Control, Techniques of Statistical Analysis*, McGraw-Hill, New York, 111-184.
5. Jackson, J.S.(1985). Multivariate Quality Control, *Communications in Statistics - Theory and Method*, 14, 2657-2688.
6. Lowry, C.A., Woodall, W.H., Champ, C.W. and Rigdon, S.E. (1992). A Multivariate Exponentially Weighted Moving Average Control Charts, *Technometrics*, 34, 46-53.
7. Lucas, J.M. and Saccucci, M.S. (1990). Exponentially Weighted Moving Average Control Schemes : Properties and Enhancements, *Technometrics*, 32, 1-12.
8. Pignatiello, J.J., Jr. and Runger, G.C. (1990). Comparisons of Multivariate CUSUM Charts, *Journal of Quality Technology*, 22, 173-186.
9. Vargas, V.C.C., Lopes, L.F.D. and Sauza, A.M. (2004). Comparative study of

the performance of the CuSum and EWMA control charts, *Computer & Industrial Engineering*, 46, 707-724.
10. Woodall, W.H. and Ncube, M.M. (1985). Multivariate CUSUM Quality Control Procedure, *Technometrics*, 27, 285-292.

[received date : Apr. 2005, accepted date May. 2005]