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On the STSP Normal Distribution

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Abstract

We introduce the standard two-sided power normal distribution and consider the relation between the probability in standard two-sided power distribution and the probability in standard two-sided power normal distribution and obtain the even moment of the special two-sided power normal distribution including the cases considered by Gupta and Nadarajah(2004)

Keywords : beta normal distribution, inverse triangular distribution, standard two-sided power distribution, standard two-sided power normal distribution, triangular distribution

1. Introduction

Van Dorp and Kotz(2002) introduced the standard two-sided power(STSP) distribution including the uniform, a triangular and power function distributions and pointed out that the flexibility of the STSP class is comparable to that of the beta family. Let X be a random variable with probability density function given by

$$f(x \mid \theta, n)=\begin{cases} n(\frac{x}{\theta})^{n-1}, & 0 < x \leq \theta \\ n(\frac{1-x}{1-\theta})^{n-1}, & \theta < x < 1 \end{cases} \tag{1}$$

Then we will be said to follow a standard two-sided power distribution with parameters $\theta,\ n$ and we will denote it by STSP(θ, n), $0 \leqq \theta \leqq 1,\ n > 0,$ where n is not necessarily an integer.

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Gupta and Nadarajah(2004) introduced the beta normal distribution with the probability density function given by

$$f(x \mid \alpha, \beta) = \frac{1}{\sigma B(\alpha, \beta)} \Phi(\frac{x-\mu}{\sigma})^{\alpha-1}(1-\Phi(\frac{x-\mu}{\sigma}))^{\beta-1}\phi(\frac{x-\mu}{\sigma}), \qquad (2)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the probability density function and the cumulative distribution function of the standard normal distribution respectively. They considered the moments in cases of $\alpha = 2, \beta = 1$ and $\alpha = 1, \beta = 2$.

If F denotes the cumulative distribution function of the normal distribution with parameters μ and σ^2, then we can define the cumulative distribution function of STSP normal distribution by

$$G(y) = \int_0^{F(y)} f(x \mid \theta, n)\, dx, \quad 0 \le \theta \le 1, \ n > 0, \qquad (3)$$

Substituting the normal cumulative distribution function into (3) and differentiating it with respect to y, we get the corresponding probability density function

$$g(y \mid \theta, n) = \begin{cases} \dfrac{2^{\frac{1}{2}-n} e^{-\frac{(y-\mu)^2}{2\sigma^2}} n\, \theta^{1-n} (1+Erf(\frac{y-\mu}{\sqrt{2}\sigma}))^{n-1}}{\sqrt{\pi}\sigma}, & \frac{1}{2}(1+Erf(\frac{y-\mu}{\sqrt{2}\sigma})) \le \theta \\ \dfrac{2^{\frac{1}{2}-n} e^{-\frac{(y-\mu)^2}{2\sigma^2}} n\, (1-\theta)^{1-n} (1+Erfc(\frac{y-\mu}{\sqrt{2}\sigma}))^{n-1}}{\sqrt{\pi}\sigma}, & \frac{1}{2}(1+Erf(\frac{y-\mu}{\sqrt{2}\sigma})) > \theta \end{cases} \qquad (4)$$

$$= \begin{cases} \frac{1}{\sigma} n\, \theta^{1-n} \phi(\frac{y-\mu}{\sigma})(\Phi(\frac{y-\mu}{\sigma}))^{n-1}, & \frac{1}{2}(1+Erf(\frac{y-\mu}{\sqrt{2}\sigma})) \le \theta \\ \frac{1}{\sigma} n\, (1-\theta)^{1-n} \phi(\frac{y-\mu}{\sigma})(1-\Phi(\frac{y-\mu}{\sigma}))^{n-1}, & \frac{1}{2}(1+Erf(\frac{y-\mu}{\sqrt{2}\sigma})) > \theta \end{cases} \qquad (5)$$

where the error function

$$Erf(x) = 1 - Erfc(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2)\, dt.$$

Remark $g(y \mid \theta = 1, n = 2)$ in (4) is equal to the probability density function of the case of $\alpha = 2, \beta = 1$ in (2) and $g(y \mid \theta = 0, n = 2)$ in (4) is equal to the probability density function of the case of $\alpha = 1, \beta = 2$ in (2).

2. Main Results

The calculations of the paper make use of the well known complementary error function defined by

$$Erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2)\, dt$$

(See Section 2.8 in volume 2 of Prudnikov et al.(1990). The properties of this function that we shall need are:

$$1 - \Phi(x) = \frac{1}{2}\, Erfc(\frac{x}{\sqrt{2}})$$

If $n = 2m$ and $c^2 + p > 0$ then

$$\int_0^{\infty} x^n \exp(-px^2) Erfc(cx)\, dx = \frac{(-1)^m}{\sqrt{\pi}} \frac{\partial^m}{\partial p^m} [\frac{1}{\sqrt{p}}\, Tan^{-1}(\frac{\sqrt{p}}{c})] \tag{6}$$

, and If $n = 2m+1$, m is even, and $c^2 + p > 0$, then

$$\int_0^{\infty} x^n \exp(-px^2) Erfc(cx)\, dx = \frac{(-1)^m m!}{2p^{m+1}} - \frac{(-1)^m c}{2} \frac{\partial^m}{\partial p^m} [\frac{1}{p\sqrt{p+c^2}}] \tag{7}$$

If $n = 2m+1$, m is **odd**, and $c^2 + p > 0$, then

$$\int_0^{\infty} x^n \exp(-px^2) Erfc(cx)\, dx = \frac{(-1)^{m+1} m!}{2p^{m+1}} - \frac{(-1)^m c}{2} \frac{\partial^m}{\partial p^m} [\frac{1}{p\sqrt{p+c^2}}] \tag{8}$$

Remark. We split the incorrect Equation 2.8.9 in volume 2 of Prudnikov et al.(1990) into the above correct equations (7) and (8) (See Choi(2005), Letter to the Editor. Communications in Statistics Theory and Methods. Vol. 34, No. 4: to appear)

Theorem 1. ① $\int_0^{\theta} f(x \mid \theta, n)\, dx = \theta$ for all θ, n,

② $$\int_{-\infty}^{\mu + \sqrt{2}\sigma Erf^{-1}(0, 2\theta - 1)} g(y \mid \theta, n)\, dy = \theta \text{ for all } \theta, n$$

where the inverse error function $Erf^{-1}(0, 2\theta - 1)$ is defined as the solution for z in the equation $Erf(z) = 2\theta - 1$.

Proof. It is clear that $\int_0^{\theta} f(x \mid \theta, n)\, dx = \theta$ for all θ, n.

Since

$$g(y \mid \theta, n) = \begin{cases} \frac{1}{\sigma} n\, \theta^{1-n} \phi(\frac{y-\mu}{\sigma})(\Phi(\frac{y-\mu}{\sigma}))^{n-1}, & \frac{1}{2}(1+Erf(\frac{y-\mu}{\sqrt{2}\sigma})) \leq \theta \\ \frac{1}{\sigma} n (1-\theta)^{1-n} \phi(\frac{y-\mu}{\sigma})(1-\Phi(\frac{y-\mu}{\sigma}))^{n-1}, & \frac{1}{2}(1+Erf(\frac{y-\mu}{\sqrt{2}\sigma})) > \theta \end{cases}$$

$\frac{1}{2}(1 + Erf(\frac{y-\mu}{\sqrt{2}\sigma})) \leqq \theta$ if and only if $y \leqq \mu + \sqrt{2}\sigma Erf^{-1}(0, 2\theta - 1)$, and

$$\frac{d}{dy}(2^{-n}\theta^{1-n}(1 + Erf(\frac{y-\mu}{\sqrt{2}\sigma}))^n = 2^{\frac{1}{2}-n} e^{-\frac{(y-\mu)^2}{2\sigma^2}} n\theta^{1-n}(1 + Erf(\frac{y-\mu}{\sqrt{2}\sigma}))^{n-1},$$

$$\int_{-\infty}^{\mu + \sqrt{2}\sigma Erf^{-1}(0, 2\theta-1)} g(y \mid \theta, n)\, dy =$$

$$\int_{-\infty}^{\mu + \sqrt{2}\sigma Erf^{-1}(0, 2\theta-1)} \frac{d}{dy}(2^{-n}\theta^{1-n}(1 + Erf(\frac{y-\mu}{\sqrt{2}\sigma}))^n dy,$$

moreover

$$(2^{-n}\theta^{1-n}(1 + Erf(\frac{y-\mu}{\sqrt{2}\sigma}))^n \rightarrow \theta \text{ as } y \rightarrow \mu + \sqrt{2}\sigma Erf^{-1}(0, 2\theta - 1),$$

and

$$(2^{-n}\theta^{1-n}(1 + Erf(\frac{y-\mu}{\sqrt{2}\sigma}))^n \rightarrow 0 \text{ as } y \rightarrow -\infty.$$

Therefore

$$\int_{-\infty}^{\mu + \sqrt{2}\sigma Erf^{-1}(0, 2\theta-1)} g(y \mid \theta, n)\, dy = \theta.$$ ‖

Corollary 2. If $\mu = 0, \theta = \frac{1}{2}$, then

① $f(x \mid \theta, n)$ is symmetric about $x = \frac{1}{2}$ for all $n > 0$,

② $g(y \mid \theta, n)$ is symmetric about $y = 0$ for all $n > 0, \sigma^2 > 0$, (9)

③ $\int_0^{\theta} f(x \mid \theta, n)\, dx = \frac{1}{2} = \int_{-\infty}^{\mu + \sqrt{2}\sigma Erf^{-1}(0, 2\theta-1)} g(y \mid \theta, n)\, dy$

Proof. Since $\theta = \frac{1}{2}$, it is clear that $f(x \mid \theta, n)$ is symmetric about $x = \frac{1}{2}$ for all $n > 0$. Since $1 + Erf(-\frac{y}{\sqrt{2}\sigma}) = Erf(\frac{y}{\sqrt{2}\sigma})$ for all $\sigma > 0$, it is clear that $g(y \mid \theta, n)$ is symmetric about $y = 0$ for all $n > 0, \sigma^2 > 0$.

Next, we consider the probability density function of the cases $\mu = 0, \sigma^2 = 1, \theta = \frac{1}{2}$ in (5). Then

$$g(y \mid \frac{1}{2}, n) = \begin{cases} \dfrac{n e^{-\frac{y^2}{2}} (1 + Erf[\frac{y}{\sqrt{2}}])^{n-1}}{\sqrt{2\pi}}, & y \leqq 0 \\ \dfrac{n e^{-\frac{y^2}{2}} (-1 + Erf[\frac{y}{\sqrt{2}}])^{n-1}}{\sqrt{2\pi}}, & y > 0 \end{cases} \tag{10}$$

Let $Y_{(1)} \leqq Y_{(2)} \leqq \cdots \leqq Y_{(n)}$ be the order statistics from standard normal distribution $N(0,1)$, Then we obtain the following theorem

Theorem 3. The negative part of (10) is equal to 2^{n-1} times the negative part of probability density function of $Y_{(n)}$ and The positive part of (10) is equal to 2^{n-1} times the positive part of probability density function of $Y_{(1)}$. ∥

If $n = 2, \theta = \frac{1}{2}$, then the graph of the probability density function in (1) is a symmetric triangular, and hence the probability density function in (10) is also symmetric about $y = 0$. Therefore we know that the $(2n+1)$th moments of Y are all 0. we can obtain the $2n$th moments of Y.

Theorem 4. Let Y be the random variable with the probability density function in (10).
If $n = 2$, then the $2n$th moment of Y s given by

$$\begin{aligned} E(Y^{2n}) &= 2\int_0^{\infty} y^{2n} e^{-\frac{y^2}{2}} Erfc(\frac{y}{\sqrt{2}})\, dy \\ &= 2\sqrt{\frac{2}{\pi}} \frac{(-1)^n}{\sqrt{\pi}} \frac{\partial^n}{\partial p^n}\left(\frac{1}{\sqrt{p}} Tan^{-1}(\sqrt{2p})\right) \Big|_{p=\frac{1}{2}}. \end{aligned}$$

Proof. It is clear from (6) that Theorem 4 holds. ∥

We considered the case that the probability density function in (1) is a symmetric triangular. Next, we consider the case that the probability density function is a symmetric inverse triangular, Let X be a random variable with the probability density function:

$$f(x) = \begin{cases} 2-4x, & 0 \leqq x \leqq \frac{1}{2} \\ -2+4x, & \frac{1}{2} < x \leqq 1 \end{cases}$$

Then, it is from (3) and (5) that the probability density function of Y is given by

$$g(y) = \begin{cases} -e^{-\frac{y^2}{2}}\sqrt{\frac{2}{\pi}}Erf(\frac{y}{\sqrt{2}}), & y < 0 \\ e^{-\frac{y^2}{2}}\sqrt{\frac{2}{\pi}}Erf(\frac{y}{\sqrt{2}}), & y \geqq 0 \end{cases} \tag{11}$$

Theorem 5. Let Y be the random variable with the probability density function in (11).
Then the odd moments of Y are all 0 and the even moment of Y is given by

$$\begin{aligned} E(Y^{2n}) &= 2\int_0^\infty y^{2n}e^{-\frac{y^2}{2}}\sqrt{\frac{2}{\pi}}Erf(\frac{y}{\sqrt{2}})\,dy \\ &= 2\sqrt{\frac{2}{\pi}}\frac{(-1)^n}{\sqrt{\pi}}\frac{\partial^n}{\partial p^n}\left(\frac{1}{\sqrt{p}}\tan^{-1}(\frac{1}{\sqrt{2p}})\right)\Big|_{p=\frac{1}{2}}. \end{aligned}$$ ∥

Remark 6. We can not obtain the closed form expressions for $E(Y^n)$ in cases of $\mu \neq 0$, or $\theta \neq \frac{1}{2}$.

References

1. Gupta, Arjun K. and Nadarajah, Saralees (2004), On the Moments of the Beta Normal Distributions. *Communications in Statistics Theory and Methods.* 31:497-512.
2. Jeen Kap Choi(2005), Letter to the Editor, *Communications in Statistics Theory and Methods.* 34(to appear).
3. Prudnikov, A. P., Brychkov, Y.A., and Marichev, O. I. (1990), *Integral and Series.* Vol.2 Amsterdam, Gordon and Breach Science Publishers.
4. Van Dorp, J. R. and Kotz, S. (2002), The Standard Two-Sided Power Distribution and its Properties : With Applications in Financial Engineering. *The American Statistician,* Vol. 56, No.2, 90-99.

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