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A Note on Renewal Reward Process with Fuzzy Rewards[1)]

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Abstract

In recently, Popova and Wu(1999) proved a theorem which presents the long-run average fuzzy reward per unit time. In this note, we improve this result. Indeed we will show uniform convergence of a renewal reward processes with respect to the level α modeled as a fuzzy random variables.

Keywords : Fuzzy random variables, Fuzzy renewal reward processes

1. Introduction

The theory of fuzzy sets introduced by Zadeh(1975) has been extensively studied and applied in statistics and probability areas in recent years. Since Kwakernaak(1978) and Puri and Ralescu(1986) introduced the concept of fuzzy random variable, there has been increasing interests for fuzzy random variable. Among others, strong law of large numbers for independent fuzzy random variables have been studied by several researchers. But there are only few papers investigating the renewal process in fuzzy environments. Popova and Wu(1999) considered a renewal rewards process with random inter-arrival times and fuzzy random rewards. Hwang(2000) considered a renewal process having inter-arrival

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times which are fuzzy random variables and proved a theorem for the rate of a renewal process having inter-arrival times which are fuzzy random variables. In this note, we improve Popova and Wu's results. Indeed we prove a uniform convergence of a renewal reward processes with respect to the level α which presents the long-run average fuzzy reward per unit time using Hong's(2003) strong law of large numbers for fuzzy random variables.

2. Preliminaries

Throughout this paper, we follow the notations in Popova and Wu(1978). If X is a universal set, a fuzzy subset $\widetilde{A}$ of X is defined by its membership function $\xi_{\widetilde{A}}: X \to [0,1]$. We can also write the fuzzy set $\widetilde{A}$ as $\{(x, \xi_{\widetilde{A}}(x)) \} x \in X\}$. The α-level set of $\widetilde{A}$ is denoted by $\widetilde{A}_\alpha = \{x : \xi_{\widetilde{A}}(x) \ge \alpha\}$, where $\widetilde{A}_0$ is the closure of the set $\{x : \xi_{\widetilde{A}}(x) \neq 0\}$. We will call $\widetilde{A}$ a normal fuzzy set if there exists x such that $\xi_{\widetilde{A}}(x) = 1$, and a convex fuzzy sets if $\xi_{\widetilde{A}}[\lambda x + (1-\lambda)y] \ge \min\{\xi_{\widetilde{A}}(x), \xi_{\widetilde{A}}(y)\}$ for $\lambda \in [0, 1]$ (That is, $\xi_{\widetilde{A}}$ is a quasi-concave function.)

Definition 2.1. (i) $\tilde{a}$ is called a fuzzy number if $\tilde{a}$ is a normal convex fuzzy set and the α-level set, $\tilde{a}_\alpha$, is bounded $\forall \alpha \neq 0$.

(ii) $\tilde{a}$ is called a closed fuzzy number if $\tilde{a}$ is a fuzzy number and its membership function $\xi_{\tilde{a}}$ is upper semicontinuous.

(iii) $\tilde{a}$ is called a bounded fuzzy number if $\tilde{a}$ is a fuzzy number and its membership function $\xi_{\tilde{a}}$ has compact support.

Let $\tilde{a}$ be a fuzzy number. We regard, $\tilde{a}_0$, the 0 level set of $\tilde{a}$ as the closure of the set $\{x : \xi_{\tilde{a}}(x) \neq 0\}$. If $\tilde{a}$ is a bounded fuzzy number then $\tilde{a}_0$ is a compact set.

Now we consider the binary operations of fuzzy number.

- The membership function of $\tilde{a} \odot \tilde{b}$ is defined by

$$\xi_{\tilde{a} \odot \tilde{b}} = \sup_{x \circ y = z} \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\} \tag{1}$$

for $\odot = \oplus, \ominus$ or $\otimes$ and $\circ = +, -$ or $\times$, where $\oplus \leftrightarrow +$, $\ominus \leftrightarrow -$ and

$\otimes \leftrightarrow \times$.

- The membership function of the inverse of $\tilde{a}$ is defined by

$$\xi_{\frac{1}{\tilde{a}}}(z) = \sup_{z=1/x, x\neq 0} \min \xi_{\tilde{a}}(x) = \xi_{\tilde{a}}\left(\frac{1}{z}\right). \tag{2}$$

- The quotient of $\tilde{a}$ and $\tilde{b}$ is defined by

$$\tilde{a} \oslash \tilde{b} = \tilde{a} \otimes \left(\frac{1}{\tilde{b}}\right). \tag{3}$$

Let $\tilde{a}$ be a fuzzy number. $\tilde{a}$ is called nonnegative if $\xi_{\tilde{a}}(x)=0$ for all $x<0$, called non-positive if $\xi_{\tilde{a}}(x)=0$ for all $x>0$, called positive if $\xi_{\tilde{a}}(x)=0$, for all $x\leq 0$ or called negative if $\xi_{\tilde{a}}(x) = 0$ for all $x\geq 0$.

Let "$\odot_{\text{int}}$" be any binary operation $\oplus_{\text{int}}$, $\ominus_{\text{int}}$, $\otimes_{\text{int}}$ or $\oslash_{\text{int}}$ between two closed intervals $\widetilde{a_\alpha} = [\widetilde{a_\alpha^L}, \widetilde{a_\alpha^U}]$ and $\widetilde{b_\alpha} = [\widetilde{b_\alpha^L}, \widetilde{b_\alpha^U}]$. Then $\widetilde{a_\alpha} \odot_{\text{int}} \widetilde{b_\alpha}$ is defined by

$$\widetilde{a_\alpha} \odot_{\text{int}} \widetilde{b_\alpha} \equiv \{z\in R \mid z = x \circ y, \ \forall x \in \widetilde{a_\alpha}, \forall \ y \in \widetilde{b_\alpha},$$

where "$\circ$" is an usual binary operation $+$, $-$, $\times$ or }.

Note that $\widetilde{a_\alpha} \oslash_{\text{int}} \widetilde{b_\alpha}$ is well-defined when $\widetilde{b_\alpha}$ does not contain zero.

Let "$\odot$" be any binary operation $\oplus$, $\ominus$ or $\otimes$ between two fuzzy numbers. It is well-known that if $\tilde{a}$ and $\tilde{b}$ are two closed fuzzy numbers, then $(\tilde{a}\odot\tilde{b})_\alpha = \widetilde{a_\alpha} \odot_{\text{int}} \widetilde{b_\alpha}$. If $\tilde{b}$ is also positive or negative, then $(\tilde{a}\oslash\tilde{b})_\alpha = \widetilde{a_\alpha} \oslash_{\text{int}} \widetilde{b_\alpha}$.

Therefore we can assure that:

- If $\tilde{a}$ and $\tilde{b}$ are two closed fuzzy numbers then

$$(\tilde{a} \oplus \tilde{b})_\alpha = [\widetilde{a_\alpha^L} + \widetilde{b_\alpha^L}, \ \widetilde{a_\alpha^U} + \widetilde{b_\alpha^U}], \tag{4}$$

$$(\tilde{a} \ominus \tilde{b})_\alpha = [\widetilde{a_\alpha^L} - \widetilde{b_\alpha^U}, \ \widetilde{a_\alpha^U} - \widetilde{b_\alpha^L}], \tag{5}$$

- If $\tilde{a}$ and $\tilde{b}$ are two closed fuzzy numbers then

$$(\tilde{a}\otimes\tilde{b})_\alpha=[\min\{\widetilde{a}^L_\alpha\widetilde{b}^L_\alpha,\ \widetilde{a}^L_\alpha\widetilde{b}^U_\alpha,\ \widetilde{a}^U_\alpha\widetilde{b}^L_\alpha,\ \widetilde{a}^U_\alpha\widetilde{b}^U_\alpha\},\ \max\{\widetilde{a}^L_\alpha\widetilde{b}^L_\alpha,\ \widetilde{a}^L_\alpha\widetilde{b}^U_\alpha,\ \widetilde{a}^U_\alpha\widetilde{b}^L_\alpha,\ \widetilde{a}^U_\alpha\widetilde{b}^U_\alpha\}]. \tag{6}$$

- If $\tilde{a}$ and $\tilde{b}$ are two nonnegative closed fuzzy numbers then

$$(\tilde{a}\otimes\tilde{b})_\alpha=[\widetilde{a}^L_\alpha\widetilde{b}^L_\alpha,\ \ \widetilde{a}^U_\alpha\widetilde{b}^U_\alpha]. \tag{7}$$

- If $\tilde{a}$ is a nonnegative closed fuzzy number and $\tilde{b}$ is a positive closed fuzzy number then

$$(\tilde{a}\oslash\tilde{b})_\alpha=[\widetilde{a}^L_\alpha\widetilde{b}^U_\alpha,\ \ \widetilde{a}^U_\alpha\widetilde{b}^L_\alpha]. \tag{8}$$

The next result will be useful in this paper. It is called Resolution Identity introduced by Zadeh(1983).

Let $\widetilde{A}$ be a fuzzy set with membership function $\xi_{\widetilde{A}}$ and $A_\alpha = x{:}\ \xi_{\widetilde{A}}(x)\geq\alpha$. Then

$$\xi_{\widetilde{A}}(x)=\sup_{\alpha\in[0,1]}\alpha 1_{A_\alpha}(x), \tag{9}$$

where

$$1_{A_\alpha}(x)=\begin{cases}1 & \text{if} \quad x\in A_\alpha,\\ 0 & \text{otherwise}.\end{cases}$$

We defined the metric d_∞ between two closed fuzzy numbers $\tilde{a}$ and $\tilde{b}$ by $d_\infty(\tilde{a},\tilde{b}) = \sup_{0\leq\alpha\leq 1}\max(|\widetilde{a}^L_\alpha-\widetilde{b}^L_\alpha|,\ |\widetilde{a}^U_\alpha-\widetilde{b}^U_\alpha|)$. Let F_R be a set of all fuzzy real numbers induced by the real number system R and let $(\Omega,\ A,\ P)$ be a probability space.

Definition 2.2. Let $\widetilde{X}:\Omega\rightarrow F_R$ be a closed fuzzy number valued function. $\widetilde{X}$ is called fuzzy random variable if $\widetilde{X}^L_\alpha$ and $\widetilde{X}^U_\alpha$ are random variables for all $\alpha\in[0,1]$.

We shall say that two fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$ are independent if and only if each random variable in the set $\{\widetilde{X}^L_\alpha,\ \widetilde{X}^U_\alpha : 0\leq\alpha\leq 1\}$ is independent of each random variable in the set $\{\widetilde{Y}^L_\alpha,\ \widetilde{Y}^U_\alpha : 0\leq\alpha\leq 1\}$, and

$\widetilde{X}$ and $\widetilde{Y}$ are identically distributed if and only if $\widetilde{X}_\alpha^L$ and $\widetilde{Y}_\alpha^L$ are identically distributed for all $\alpha \in [0,1]$ and $\widetilde{X}_\alpha^U$ and $\widetilde{Y}_\alpha^U$ are identically distributed for all $\alpha \in [0,1]$.

3. Fuzzy renewal reward processes

Consider a renewal process $\{N(t),\ t \geq 0\}$ having inter-arrival times X_n for $n \geq 1$, and suppose that each time a renewal occurs we receive a reward. We denote by $\widetilde{R}_n$, the fuzzy reward earned at the time of the nth renewal, where $\widetilde{R}_n$ is a fuzzy random variable. We shall assume that the $\widetilde{R}_n$ for $n \geq 1$ are independent and identically distributed. If we let

$$\widetilde{R}(t) = \oplus_{n=1}^{N(t)} \widetilde{R}_n$$

then $\widetilde{R}(t)$ represents the total fuzzy reward earned by time t. By Proposition 3.4, $(\widetilde{R}_n)_\alpha^L$ and $(\widetilde{R}_n)_\alpha^U$ are random variables for all α. We thus can adopt the following notations.

$$E[\widetilde{R}_\alpha^L] = E[(\widetilde{R}_n)_\alpha^L],\ \ E[\widetilde{R}_\alpha^U] = E[(\widetilde{R}_n)_\alpha^U],\ \ E[X] = E[X_n].$$

Let $\{\widetilde{X}_n\}$ and $\widetilde{X}$ be fuzzy random variables defined on the same probability space $(\Omega,\ A,\ P)$. Suppose that, for all α, $(\widetilde{X}_n)_\alpha^L \rightarrow \widetilde{X}_\alpha^L$ with probability one and $(\widetilde{X}_n)_\alpha^U \rightarrow \widetilde{X}_\alpha^U$ with probability one.

The following result is due to Popova and Wu(2000).

Theorem 3.1. Let $a_\alpha^L = E[\widetilde{R}_\alpha^L]/E[X]$ and $a_\alpha^U = E[\widetilde{R}_\alpha^U]/E[X]$. Let $\tilde{a}$ be a fuzzy number with membership function

$$\xi_{\tilde{a}}(r) = \sup_{0 \leq \alpha \leq 1} \alpha 1_{[a_\alpha^L,\, a_\alpha^U]}(r).$$

Suppose that $E[\widetilde{R}_\alpha^L] < \infty$, $E[\widetilde{R}_\alpha^U] < \infty$ and $E[X] < \infty$ for all α. Then we have $\widetilde{R}(t) \oslash t \rightarrow \tilde{a}$ with probability on level-wise if a_α^L and a_α^U are left-continuous with respect to α. Furthermore, we have $\widetilde{a}_\alpha^L = a_\alpha^L$ and

$\widetilde{a}_\alpha^U = a_\alpha^U$.

Recently, Hong(2003) proved the following theorem.

Theorem 3.2. Let $\{\widetilde{R}_n\}$ be a sequence of independent and identically distributed fuzzy random variables with $E|\widetilde{R}_\alpha^L| < \infty$ and $E|\widetilde{R}_\alpha^U| < \infty$. Then we have

$$d_\infty\left(\frac{1}{n}\oplus_{i=1}^{n}\widetilde{R}_i,\ E\widetilde{R}_1\right) \to 0,$$

with probability one as $t \to \infty$, where $(E\widetilde{R}_1)_\alpha = [E\widetilde{R}_\alpha^L,\ E\widetilde{R}_\alpha^U]$, $\alpha \in (0,1]$.

We now prove our main result which generalizes Theorem 3.1.

Theorem 3.3. Let $a_\alpha^L = E[\widetilde{R}_\alpha^L]/E[X]$ and $a_\alpha^U = E[\widetilde{R}_\alpha^U]/E[X]$. Let $\widetilde{a}$ be a fuzzy number with membership function

$$\xi_{\widetilde{a}}(r) = \sup_{0\le\alpha\le1}\alpha 1_{[a_\alpha^L,\ a_\alpha^U]}(r).$$

Suppose that $E[\widetilde{R}_\alpha^L] < \infty$, $E[\widetilde{R}_\alpha^U] < \infty$ and $E[X] < \infty$ for all α. Then we have with probability 1 and $t \to \infty$,

$$d_\infty(\widetilde{R}(t)\oslash t,\ \widetilde{a}) \to 0.$$

Proof. From Eq. (4) and Eq. (8), we have

$$\begin{aligned}
&|(\widetilde{R}(t)\oslash t)_\alpha^L - a_\alpha^L| \\
&= \left|\frac{\sum_{n=1}^{N(t)}(\widetilde{R}_n)_\alpha^L}{N(t)}\frac{N(t)}{t} - E[\widetilde{R}_\alpha^L]\frac{1}{E[X]}\right| \\
&\le \left|\frac{\sum_{n=1}^{N(t)}(\widetilde{R}_n)_\alpha^L}{N(t)}\frac{N(t)}{t} - \frac{N(t)}{t}E[\widetilde{R}_\alpha^L] + \frac{N(t)}{t}E[\widetilde{R}_\alpha^L] - E[\widetilde{R}_\alpha^L]\frac{1}{E[X]}\right| \\
&\le \left|\frac{N(t)}{t}\right|\left|\frac{\sum_{n=1}^{N(t)}(\widetilde{R}_n)_\alpha^L}{N(t)} - E[\widetilde{R}_\alpha^L]\right| + E|\widetilde{R}_1^L|\left|\frac{N(t)}{t} - \frac{1}{E[X]}\right|.
\end{aligned}$$

Similarly, we have

$$\left|(\widetilde{R}(t)\oslash t)_\alpha^U - a_\alpha^U\right| \le \left|\frac{N(t)}{t}\right|\left|\frac{\sum_{n=1}^{N(t)}(\widetilde{R}_n)_\alpha^U}{N(t)} - E[\widetilde{R}_\alpha^U]\right| + E\left|\widetilde{R}_0^U\right|\left|\frac{N(t)}{t} - \frac{1}{E[X]}\right|$$

Then we have

$$\begin{aligned} d_\infty(\widetilde{R}(t)\oslash t, a) &\le \left|\frac{N(t)}{t}\right| d_\infty\left(\frac{\oplus_{n=1}^{N(t)}\widetilde{R}_n}{N(t)}, E[\widetilde{R}_1]\right) \\ &+ \left(E\left|\widetilde{R}_1^L\right| + E\left|\widetilde{R}_0^U\right|\right)\left(\frac{N(t)}{t} - \frac{1}{E[X]}\right). \end{aligned}$$

By Theorem 3.2, we obtain that

$$d_\infty\left(\frac{\oplus_{n=1}^{N(t)}\widetilde{R}_n}{N(t)}, E[\widetilde{R}_1]\right) \to 0$$

with probability on as $t \to \infty$, since $N(t) \to \infty$ with probability one as $t \to \infty$ from Ross(1983), p. 57. Applying the fact that

$$\frac{N(t)}{t} \to \frac{1}{E[X]} \quad \text{with probability one as } t \to \infty$$

(see Ross(1983), p. 58), we get the desired result.

Remark. By above theorem, we have stronger result about T-age replacement policy in Section 5 of Popova and Wu(1999) as with probability 1,

$$d_\infty\left(R(t)\oslash t, \frac{\widetilde{C}_1 \oplus \widetilde{C}_2 F(T)}{T(1-F(T)) + \int_0^T xf(x)dx}\right) \to 0 \quad \text{as } t \to \infty.$$

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