Journal of Korean
Data & Information Science Society
2005, Vol. 16, No. 1, pp. 145 ~156

Estimation for the Half-Logistic Distribution Based on Multiply Type-II Censored Samples

Suk-Bok Kang[1] · Young Kou Park[2]

Abstract

In this paper, we derive the approximate maximum likelihood estimators (AMLEs) of the scale parameter of the half-logistic distribution based on multiply Type-II censored samples. We compare the proposed estimators in the sense of the mean squared error (MSE) for various censored samples.

Keywords : Approximate maximum likelihood estimator, Half-Logistic distribution, Multiply Type-II censored sample

1. Introduction

A random variable X is said to have a half-logistic distribution with the probability density function (pdf)

$$f(x;\theta,\sigma)=\frac{2\exp\left(-\frac{x-\theta}{\sigma}\right)}{\sigma\left\{1+\exp\left(-\frac{x-\theta}{\sigma}\right)\right\}^2}\,,\quad x\geq\theta,\ \ \sigma>0 \tag{1.1}$$

and the cumulative distribution function (cdf)

$$F(x;\theta,\sigma)=\frac{1-\exp\left(-\frac{x-\theta}{\sigma}\right)}{1+\exp\left(-\frac{x-\theta}{\sigma}\right)}\,,\quad x\geq\theta,\ \ \sigma>0 \tag{1.2}$$

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1) First Author : Professor, Department of Statistics, Yeungnam University, Gyongsan, 712-749, Korea
E-mail : sbkang@yu.ac.kr

2) Associate Professor, Department of Mathematics, Yeungnam University, Gyongsan, 712-749,
E-mail : ykpark@yu.ac.kr

where θ and σ are the location and scale parameters, respectively.

The approximate maximum likelihood estimation method was first developed by Balakrishnan (1989) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution. Kang (1996) obtained the AMLE for the scale parameter of the double exponential distribution based on Type-II censored samples and he showed that the proposed estimator is generally more efficient than the best linear unbiased estimator and the optimum unbiased absolute estimator.

Multiply Type-II censored sampling arises in a life-testing experiment whenever the experimenter does not observe the failure times of some units placed on a life-test. Another situation where multiply censored samples arise naturally is when some units failed between two points of observation with exact times of failure of these units unobserved.

Kong and Fei (1996) discussed the limit theorems for the maximum likelihood estimator under general multiply Type-II censoring. Recently, Kang (2003) proposed the AMLEs of the location and the scale parameters of the two-parameter exponential distribution with multiply Type-II censoring.

Application of the half-logistic distribution to life-testing had been well demonstrated by Balakrishnan (1985) who derived several recurrence relations satisfied by the single and the product moments of order statistics and applied them in a simple recursive process to compute the means, variances, and covariances of order statistics for sample sizes up to 15.

Balakrishnan and Joshi (1983) have obtained several recurrence relations for the moments and product moments of order statistics from a symmetrically truncated logistic distribution and applied them to tabulate the means, variances and covariances.

Balakrishnan and Puthenpura (1986) tabulated the coefficients of the best linear unbiased estimators of the location and scale parameters of the half-logistic distribution based on complete samples.

In this paper, we derive the AMLEs of the scale parameter σ and the location parameter θ based on multiply Type-II censored sample. We also compare the proposed estimators in the sense of the MSE for various censored samples.

2. Approximate Maximum Likelihood Estimators

Let us assume that the following multiply Type-II censored sample from a sample of size n is

$$X_{a_1:n} < X_{a_2:n} < \cdots < X_{a_s:n} \tag{2.1}$$

where $1 \le a_1 < a_2 < \cdots < a_s \le n$, and

$$a_0 = 0\,,\ a_{s+1} = n+1\,,\ F(x_{a_0:n}) = 0\,,\ F(x_{a_{s+1}:n}) = 1\,. \tag{2.2}$$

The likelihood function based on the multiply Type-II censored sample (2.1) can be written as

$$L = n!\prod_{j=1}^{s} f(x_{a_j:n})\prod_{j=1}^{s+1}\frac{[F(x_{a_j:n}) - F(x_{a_{j-1}:n})]^{a_j - a_{j-1}-1}}{(a_j - a_{j-1}-1)!}\,. \tag{2.3}$$

The random variable $Z_{i:n} = (X_{i:n} - \theta)/\sigma$ has a standard half-logistic distribution with the pdf and cdf;

$$f(z) = \frac{2e^{-z}}{[1+e^{-z}]^2}\,,\quad F(z) = \frac{1-e^{-z}}{1+e^{-z}}\,,\qquad 0 \le z < \infty.$$

The $f(z)$, $f'(z)$, and $F(z)$ satisfy as

$$f'(z) = -F(z)f(z)$$
$$f(z) = [1-F(z)][1+F(z)]/2\,.$$

From the equation (2.3), the likelihood function is a monotonically increasing function of θ. Thus the MLE of θ is $\widehat{\theta} = X_{a_1:n}$.

On differentiating the log-likelihood function with respect to σ in turn and equation to zero, we obtain the estimating equation as

$$\begin{aligned}\frac{\partial \ln L}{\partial\sigma} = -\frac{1}{2\sigma}\Bigg[&2s + (a_1-1)\frac{Z_{a_1:n}}{F(Z_{a_1:n})} - (a_1-1)F(Z_{a_1:n})Z_{a_1:n}\\ &-(n-a_s)Z_{a_s:n} - (n-a_s)F(Z_{a_s:n})Z_{a_s:n} - 2\sum_{j=1}^{s}F(Z_{a_j:n})Z_{a_j:n}\\ &+2\sum_{j=2}^{s}(a_j - a_{j-1}-1)\frac{f(Z_{a_j:n})Z_{a_j:n} - f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})}\Bigg] = 0\,.\end{aligned} \tag{2.4}$$

Since the likelihood equation is very complicated, the equation (2.4) does not admit an explicit solution for σ.

Let

$$\xi_i = F^{-1}(p_i) = -\ln\left(\frac{1-p_i}{1+p_i}\right),\quad \text{where}\ \ p_i = \frac{i}{n+1}\,,\ \ q_i = 1-p_i.$$

We may approximate the following functions in Taylor series around the points ξ_{a_1}, ξ_{a_s}, ξ_{a_j}, and $(\xi_{a_j}, \xi_{a_{j-1}})$, respectively.

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} \simeq \alpha_{11} + \beta_{11} Z_{a_1:n} \tag{2.5}$$

$$F(Z_{a_j:n}) Z_{a_j:n} \simeq \delta_{1j} + \varkappa_{1j} Z_{a_j:n} \tag{2.6}$$

$$\frac{f(Z_{a_j:n}) Z_{a_j:n} - f(Z_{a_{j-1}:n}) Z_{a_{j-1}:n}}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \simeq \alpha_j + \beta_j Z_{a_j:n} + \gamma_j Z_{a_{j-1}:n} \tag{2.7}$$

$$\frac{1}{F(Z_{a_1:n})} \simeq \alpha_{21} + \beta_{21} Z_{a_1:n} \tag{2.8}$$

$$F(Z_{a_j:n}) \simeq \delta_{2j} + \varkappa_{2j} Z_{a_j:n} \tag{2.9}$$

$$\frac{f(Z_{a_j:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \simeq \alpha_{1j} + \beta_{1j} Z_{a_j:n} + \gamma_{1j} Z_{a_{j-1}:n} \tag{2.10}$$

$$\frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \simeq \alpha_{2j} + \beta_{2j} Z_{a_j:n} + \gamma_{2j} Z_{a_{j-1}:n} \tag{2.11}$$

where

$$\alpha_{11} = f(\xi_{a_1}) \left[\frac{\xi_{a_1}}{p_{a_1}} \right]^2$$

$$\beta_{11} = \frac{1}{p_{a_1}} \left[1 - \frac{f(\xi_{a_1})}{p_{a_1}} \xi_{a_1} \right]$$

$$\delta_{1j} = -f(\xi_{a_j}) \xi_{a_j}{}^2$$

$$\varkappa_{1j} = f(\xi_{a_j}) \xi_{a_j} + p_{a_j}$$

$$\alpha_j = \frac{\xi_{a_j}{}^2 f(\xi_{a_j}) p_{a_j} - \xi_{a_{j-1}}{}^2 f(\xi_{a_{j-1}}) p_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}} + \left[\frac{\xi_{a_j} f(\xi_{a_j}) - \xi_{a_{j-1}} f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right]^2$$

$$\beta_j = \frac{f(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}} \left[1 - p_{a_j}\xi_{a_j} - \frac{\xi_{a_j} f(\xi_{a_j}) - \xi_{a_{j-1}} f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right]$$

$$\gamma_j = -\frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \left[1 - p_{a_{j-1}}\xi_{a_{j-1}} - \frac{\xi_{a_j} f(\xi_{a_j}) - \xi_{a_{j-1}} f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right]$$

$$\alpha_{21} = \frac{1}{p_{a_1}} \left[1 + \frac{f(\xi_{a_1})}{p_{a_1}} \xi_{a_1} \right]$$

$$\beta_{21} = -\frac{f(\xi_{a_1})}{p_{a_1}{}^2}$$

$$\delta_{2j} = p_{a_j} - f(\xi_{a_j})\xi_{a_j}$$

$$\kappa_{2j} = f(\xi_{a_j})$$

$$\alpha_{1j} = \frac{f(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}} \left[1 + p_{a_j}\xi_{a_j} + \frac{\xi_{a_j} f(\xi_{a_j}) - \xi_{a_{j-1}} f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right]$$

$$\beta_{1j} = -\frac{f(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}} \left[p_{a_j} + \frac{f(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}} \right]$$

$$\gamma_{1j} = \frac{f(\xi_{a_j}) f(\xi_{a_{j-1}})}{[p_{a_j} - p_{a_{j-1}}]^2}$$

$$\alpha_{2j} = \frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \left[1 + p_{a_{j-1}}\xi_{a_{j-1}} + \frac{\xi_{a_j} f(\xi_{a_j}) - \xi_{a_{j-1}} f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right]$$

$$\beta_{2j} = -\frac{f(\xi_{a_j}) f(\xi_{a_{j-1}})}{[p_{a_j} - p_{a_{j-1}}]^2} = -\gamma_{1j}$$

$$\gamma_{2j} = -\frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \left[p_{a_{j-1}} - \frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right].$$

By substituting the equations (2.5), (2.6), and (2.7), into the equation (2.4), we can derive an estimator of σ as follows;

$$\widehat{\sigma_1} = \frac{B_1 + C_1\widehat{\theta}}{A_1} \tag{2.12}$$

where

$$A_1 = 2s + (a_1 - 1)(\alpha_{11} - \delta_{11}) - (n - a_s)\delta_{1s} - 2\sum_{j=1}^{s}\delta_{1j} + 2\sum_{j=2}^{s}(a_j - a_{j-1} - 1)\alpha_j$$

$$B_1 = (a_1 - 1)(\varkappa_{11} - \beta_{11})X_{a_1:n} - (n - a_s)(1 + \varkappa_{1s})X_{a_s:n} - 2\sum_{j=1}^{s}\varkappa_{1j}X_{a_j:n}$$
$$- 2\sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\beta_j X_{a_j:n} + \gamma_j X_{a_{j-1}:n})$$

$$C_1 = (a_1 - 1)(\beta_{11} - \varkappa_{11}) - (n - a_s)(1 + \varkappa_{1s}) - 2\sum_{j=1}^{s}\varkappa_{1j} + 2\sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\beta_j + \gamma_j).$$

By substituting the equations (2.7), (2.8), and (2.9), into the equation (2.4), we can derive an estimator of σ as follows;

$$\widehat{\sigma_2} = \frac{-B_2 + \sqrt{B_2{}^2 + 4A_2C_2}}{2A_2} \tag{2.13}$$

where

$$A_2 = 2\left[s + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\alpha_j\right]$$

$$B_2 = (a_1 - 1)(\alpha_{21} - \delta_{21})X_{a_1:n} - (n - a_s)(1 + \delta_{2s})X_{a_s:n} - 2\sum_{j=1}^{s}\delta_{2j}X_{a_j:n}$$
$$+ 2\sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\beta_j X_{a_j:n} + \gamma_j X_{a_{j-1}:n})$$
$$-\left[(a_1 - 1)(\alpha_{21} - \delta_{21}) - (n - a_s)(1 + \delta_{2s}) - 2\sum_{j=1}^{s}\delta_{2j} + 2\sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\beta_j + \gamma_j)\right]\widehat{\theta}$$

$$C_2 = (a_1 - 1)(\beta_{21} - \varkappa_{21})(X_{a_1:n} - \widehat{\theta})^2 - (n - a_s)\varkappa_{2s}(X_{a_s:n} - \widehat{\theta})^2 - 2\sum_{j=1}^{s}\varkappa_{2j}(X_{a_j:n} - \widehat{\theta})^2.$$

By substituting the equations (2.5), (2.6), (2.10), and (2.11) into the equation (2.4), we can derive an estimator of σ as follows;

$$\widehat{\sigma_3} = \frac{-B_3 + \sqrt{B_3{}^2 + 4A_3C_3}}{2A_3} \tag{2.14}$$

where

$$A_3 = 2s + (a_1 - 1)(\alpha_{11} - \delta_{11}) - (n - a_s) - 2\sum_{j=2}^{s}\delta_{1j}$$

$$\begin{aligned} B_3 = \; & (a_1 - 1)(\beta_{11} - \kappa_{11}) X_{a_1:n} - (n - a_s)(1 + \kappa_{1s}) X_{a_s:n} - 2\sum_{j=1}^{s}\kappa_{1j} X_{a_j:n} \\ & + 2\sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\alpha_{1j} X_{a_j:n} - \alpha_{2j} X_{a_{j-1}:n}) \\ & - \Big[(a_1 - 1)(\beta_{11} - \kappa_{11}) - (n - a_s)(1 + \kappa_{1s}) - 2\sum_{j=1}^{s}\kappa_{1j} + 2\sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\alpha_{1j} - \alpha_{2j}) \Big]\theta \end{aligned}$$

$$\begin{aligned} C_3 = \; & 2\sum_{j=1}^{s}(a_j - a_{j-1} - 1)\{\beta_{1j}(X_{a_j:n} - \widehat{\theta})^2 + 2\gamma_{1j}(X_{a_j:n} - \widehat{\theta})(X_{a_{j-1}:n} - \widehat{\theta}) \\ & - \gamma_{2j}(X_{a_{j-1}:n} - \widehat{\theta})^2\}. \end{aligned}$$

By substituting the equations (2.8), (2.9), (2.10), and (2.11) into the equation (2.4), we can derive an estimator of σ as follows;

$$\widehat{\sigma_4} = \frac{-B_4 + \sqrt{B_4^{\,2} + 8sC_4}}{4s} \qquad (2.15)$$

where

$$\begin{aligned} B_4 = \; & (a_1 - 1)(\alpha_{21} - \delta_{21}) X_{a_1:n} - (n - a_s)(1 + \delta_{2s}) X_{a_s:n} - 2\sum_{j=1}^{s}\delta_{2j} X_{a_j:n} \\ & + 2\sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\alpha_{1j} X_{a_j:n} - \alpha_{2j} X_{a_{j-1}:n}) \\ & - \Big[(a_1 - 1)(\alpha_{21} - \delta_{21}) - (n - a_s)(1 + \delta_{2s}) - 2\sum_{j=1}^{s}\delta_{2j} + 2\sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\alpha_{1j} - \alpha_{2j}) \Big]\theta \end{aligned}$$

$$C_4 = C_2 + C_3.$$

For the half-logistic distribution the maximum likelihood method does not provide an explicit estimator for the scale parameter based on either complete or multiply Type-II censored samples. But we provided several explicit estimators by approximating the likelihood function.

From the equations (2.12) to (2.15), the MSEs of these estimators are simulated by Monte Carlo method for sample size $n = 20, 50$ and various choices of censoring. These values are given in Table 1.

From Table 1, the estimators $\widehat{\sigma_2}$ and $\widehat{\sigma_4}$ are generally more efficient than $\widehat{\sigma_1}$ and $\widehat{\sigma_3}$ in the sense of the MSE. But $\widehat{\sigma_1}$ is simpler than the other estimators.

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Table 1. The relative mean squared errors for the estimators of the scale parameter σ. (Location parameter θ is known.)

n	k	a_j	MSE			
			$\widehat{\sigma_1}$	$\widehat{\sigma_2}$	$\widehat{\sigma_3}$	$\widehat{\sigma_4}$
20	0	1～20	0.034212	0.035070	0.034212	0.035070
	1	1～19	0.035608	0.036274	0.035608	0.036274
		2～20	0.035530	0.036597	0.035530	0.036597
	2	1～18	0.037029	0.037601	0.037029	0.037601
		3～20	0.037577	0.037052	0.037577	0.037052
		2～19	0.037001	0.037898	0.037001	0.037898
	3	1～17	0.039245	0.039775	0.039245	0.039775
		4～20	0.040438	0.037125	0.040438	0.037125
		2～18	0.038526	0.039307	0.038526	0.039307
		3～19	0.039191	0.038330	0.039191	0.038330
	4	2～17	0.040872	0.041603	0.040872	0.041603
		4～19	0.042277	0.038397	0.042277	0.038397
		3～18	0.040909	0.039732	0.040909	0.039732
		2～4 7～14 16～20	0.035613	0.036521	0.040563	0.041743
	5	3～17	0.043495	0.042052	0.043495	0.042052
		4～18	0.044292	0.039809	0.044292	0.039809
		2～6 10～19	0.037082	0.037813	0.039986	0.040835
	6	4～17	0.047251	0.042108	0.047251	0.042108
		1 2 6～9 12～15 17～20	0.034387	0.035051	0.042026	0.043754

Table 1. (continued)

n	k	a_j	MSE			
			$\widehat{\sigma_1}$	$\widehat{\sigma_2}$	$\widehat{\sigma_3}$	$\widehat{\sigma_4}$
50	0	1～50	0.013799	0.013948	0.013799	0.013948
	1	1～49	0.013994	0.014123	0.013994	0.014123
		2～50	0.014017	0.014327	0.014017	0.014327
	2	1～48	0.014209	0.014332	0.014209	0.014332
		3～50	0.014339	0.014420	0.014339	0.014420
		2～49	0.014213	0.014502	0.014213	0.014502
	3	1～47	0.014423	0.014536	0.014423	0.014536
		4～50	0.014771	0.014427	0.014771	0.014427
		2～48	0.014434	0.014721	0.014434	0.014721
		3～49	0.014540	0.014593	0.014540	0.014593
	4	2～47	0.014649	0.014927	0.014649	0.014927
		4～49	0.014979	0.014600	0.014979	0.014600
		3～48	0.014771	0.014818	0.014771	0.014818
		2～4 7～14 16～50	0.014017	0.014318	0.014993	0.015376
	5	3～47	0.014991	0.015025	0.014991	0.015025
		4～48	0.015224	0.014823	0.015224	0.014823
		2～6 10～19 21～50	0.014019	0.014316	0.015205	0.015591
	6	4～47	0.015454	0.015030	0.015454	0.015030
		1 2 6~9 12～15 17～50	0.013801	0.013939	0.015259	0.015599

Table 2. The relative mean squared errors for the estimators of the location parameter θ and scale parameter σ.

n	k	a_j	MSE				
			$\widehat{\theta}$	$\widehat{\sigma_1}$	$\widehat{\sigma_2}$	$\widehat{\sigma_3}$	$\widehat{\sigma_4}$
20	0	1～20	0.017038	0.036775	0.036415	0.036775	0.036415
	1	1～19	0.017038	0.038505	0.038057	0.038505	0.038057
		2～20	0.052820	0.042767	0.040060	0.042767	0.040060
	2	1～18	0.017038	0.040198	0.039690	0.040198	0.039690
		3～20	0.106883	0.051513	0.044131	0.051513	0.044131
		2～19	0.052820	0.045116	0.042150	0.045116	0.042150
	3	1～17	0.017038	0.042834	0.042314	0.042834	0.042314
		4～20	0.182402	0.064041	0.049026	0.064041	0.049026
		2～18	0.052820	0.047561	0.044327	0.047561	0.044327
		3～19	0.106883	0.054764	0.046772	0.054764	0.046772
	4	2～17	0.052820	0.051061	0.047568	0.051061	0.047568
		4～19	0.182402	0.068543	0.052318	0.068543	0.052318
		3～18	0.106883	0.058264	0.049552	0.058264	0.049552
		2～4 7～14 16～20	0.052820	0.042762	0.040189	0.040777	0.040546
	5	3～17	0.106883	0.063079	0.053571	0.063079	0.053571
		4～18	0.182402	0.073537	0.055838	0.073537	0.055838
		2～6 10～19	0.052820	0.045074	0.042302	0.043471	0.042031
	6	4～17	0.182402	0.080169	0.060797	0.080169	0.060797
		1 2 6～9 12～15 17～20	0.017038	0.036832	0.036568	0.039347	0.040356

Table 2. (continued)

n	k	a_j	MSE				
			$\widehat{\theta}$	$\widehat{\sigma}_1$	$\widehat{\sigma}_2$	$\widehat{\sigma}_3$	$\widehat{\sigma}_4$
50	0	1~50	0.002957	0.014256	0.014201	0.014256	0.014201
	1	1~49	0.002957	0.014492	0.014428	0.014492	0.014428
		2~50	0.009145	0.015244	0.014812	0.015244	0.014812
	2	1~48	0.002957	0.014735	0.014669	0.014735	0.014669
		3~50	0.018347	0.016737	0.015575	0.016737	0.015575
		2~49	0.009145	0.015517	0.015064	0.015517	0.015064
	3	1~47	0.002957	0.014986	0.014914	0.014986	0.014914
		4~50	0.030479	0.018755	0.016483	0.018755	0.016483
		2~48	0.009145	0.015797	0.015332	0.015797	0.015332
		3~49	0.018347	0.017070	0.015865	0.017070	0.015865
	4	2~47	0.009145	0.016101	0.015612	0.016101	0.015612
		4~49	0.030479	0.019157	0.016808	0.019157	0.016808
		3~48	0.018347	0.017403	0.016164	0.017403	0.016164
		2~4 7~14 16~50	0.009145	0.015243	0.014816	0.015010	0.014940
	5	3~47	0.018347	0.017774	0.016481	0.017774	0.016481
		4~48	0.030479	0.019565	0.017148	0.019565	0.017148
		2~6 10~19 21~50	0.009145	0.015243	0.014819	0.015116	0.015080
	6	4~47	0.030479	0.020025	0.017509	0.020025	0.017509
		1 2 6~9 12~15 17~50	0.002957	0.014256	0.014205	0.014774	0.014938

[received date : Nov. 2004, accepted date : Jan. 2005]