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On the Functional Central Limit Theorem of Negatively Associated Processes[1)]

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Abstract

A functional central limit theorem is obtained for a stationary linear process of the form $X_t = \sum_{j=0}^{\infty} a_j x_{t-j}$, where $\{x_t\}$ is a strictly stationary sequence of negatively associated random variables with suitable conditions and $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty} |a_j| < \infty$.

Keywords : Functional central limit theorem, Linear process, Negatively associated random variables.

1. Introduction

The sequence of negatively associated random variables(NA) are widely used in time series, reliability theory and multivariate analysis of statistics, so it is very important to investigate the limit properties of those sequences and there has been increased interest in the study of NA random variables(see, Su et al.(1997), Liang and Su(1999), Yuan et al(2003), Baek et al(2003), etc).

Definition 1.(Joag-Dev and Proschan(1983)). A finite family of random variables $\{X_i \mid 1 \le i \le n\}$ is said to be negatively associated(NA) if for every pair of disjoint subsets A and B of $\{1, 2, \cdots, n\}$,

$$Cov(f_1(X_i, i \in A),\ f_2(X_j, j \in B)) \le 0,$$

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whenever f_1 and f_2 are coordinatewise increasing the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

Consider a stationary ergodic linear process

$$X_t = \sum_{j=0}^{\infty} a_j x_{t-j}, \quad \sum_{j=0}^{\infty} |a_j| < \infty \tag{1}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\{x_t | t \in Z^+\}$ is a strictly stationary sequence of martingale differences

$$E(x_t | \mathcal{F}_{t-1}) = 0, \quad E(x_t^2 | \mathcal{F}_{t-1}) = \sigma^2 < \infty, \quad a.s. \tag{2}$$

Here $\mathcal{F}_t$ is the sub-σ algebra generated by x_s, $s \le t$, and σ^2 is the prediction variance.

The linear process are of special importance in time series analysis and they arise in a wide variety of contexts (see, e.q., Hannan, 1970 Chapter 6).

Applications to business, economics, engineering and physical sciences are extremely broad and a vast amount of literature is devoted to the study of theorems for linear process under various conditions. Fakhre-Zakeri and Farshidi(1993) established central limit theorem(CLT) under the independent identically distribution assumption on x_t and Fakhre-Zakeri and Lee(1997) proved a functional central limit theorem($FCLT$) under the strong mixing condition on x_t as follows.

Let $S_n = \sum_{t=1}^{n} X_t$, $\tau^2 = \left(\sum_{j=0}^{\infty} a_j\right)^2 \sigma^2 < \infty$. Define for $n \ge 1$, the stochastic process

$$\xi_n(u) = n^{-1/2}\tau^{-1}(S_r + (un - r)X_{r+1}), \quad r/n \le u \le (r+1)/n, \tag{3}$$

where $r = 0, \cdots, n-1$.

Theorem A.(Fakhre-Zakeri and Lee(1997)). Assume that (1) and (2) hold. Then for all fixed k and $B \in \mathcal{F}_k$, $P(B) > 0$,

$$\lim_{n\to\infty} P(n^{-1/2}\tau S_n \le x | B) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-1/2y^2} dy, \quad \text{for all } x.$$

Theorem B.(____________). Assume that (1) and (2) hold. Let $\{N_n; n \in \mathrm{N}\}$ be a sequence of positive integer-valued random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. If $N_n/n \to N$ in probability where N is a real-valued random variable with $P(0 < N < \infty) = 1$, then the process $\{\xi_{N_n}(u); 0 \le u \le 1\}$ converges weakly to the Wiener measure.

The main purpose of this paper is to extend Theorems A and B to NA process and we establish a $CLT(FCLT)$ for a strictly stationary linear process of the form (1) generated by NA process.

The following lemma is needed to prove the Main result.

Lemma 1.1.(Su et al(1997)). Let $\{X_i \mid i \geq 1\}$ be a sequence of NA random variables with $EX_i = 0$. For $p \geq 2$ If $\beta_p = \sup E|X_i|^p < \infty$, then there exists some constant $k_p > 0$ depending only on p such that

$$E\left(\max_{1 \leq k \leq n} \left|\sum_{i=1}^{k} X_i\right|^p\right) \leq k_p\left(n\beta_p + (n\beta_2)^{p/2}\right) \tag{4}$$

for any integer n, where $\beta_2 = \sup EX_i^2$.

2. Main result

The following lemma is needed to prove Theorems 2.2 and 2.3 and we established by modifying the proof of Lemma 3 in Fakhre-Zakeri and Lee(1997). Doob's maximal inequality has played important role in their proof. However, in our case, Doob's maximal inequality cannot be used, instead, NA case of Su et al(1997) will be used.

Lemma 2.1. Let $\{x_t\}$ be a strictly stationary NA process with $Ex_t = 0$, $0 < Ex_t^2 < \infty$. Let $X_t = \sum_{j=0}^{\infty} a_j x_{t-j}$, $S_k = \sum_{t=1}^{k} X_t$, $\tilde{X}_t = (\sum_{j=0}^{\infty} a_j) x_t$, and let $\tilde{S}_k = \sum_{t=1}^{k} \tilde{X}_t$, where $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty} |a_j| < \infty$. Then $(n^{-1/2}) \max_{1 \leq k \leq n} |\tilde{S}_k - S_k| \to 0$ in probability as $n \to \infty$.

Proof. As in the proof of Lemma 3 of Fakhre-Zakeri and Lee(1997), we have

$$\begin{aligned} \tilde{S}_k &= \sum_{t=1}^{k}\left(\sum_{j=0}^{k-t} a_j\right) x_t + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} a_j\right) x_t \\ &= \sum_{t=1}^{k}\left(\sum_{j=0}^{t-1} a_j x_{t-j}\right) + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} a_j\right) x_t \end{aligned}$$

so that

$$\begin{aligned} \tilde{S}_k - S_k &= -\sum_{t=1}^{k}\left(\sum_{j=t}^{\infty} a_j x_{t-j}\right) + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} a_j\right) x_t \\ &= I_1 + I_2 \quad \text{(say)}. \end{aligned}$$

It suffices to show that

$$n^{-1/2} \max_{1 \leq k \leq n} |I_1| \to 0 \text{ in probability as } n \to \infty, \tag{5}$$

and

$$n^{-1/2} \max_{1 \leq k \leq n} |I_2| \to 0 \text{ in probability as } n \to \infty. \tag{6}$$

To prove (5), by Markov inequality, for $r > 2$,

$$n^{-r/2}\max_{1\le k\le n}\left|\sum_{t=1}^{k}\sum_{j=t}^{\infty}a_j x_{t-j}\right|^r$$

$$= n^{-r/2}E\max_{1\le k\le n}\left|\sum_{j=1}^{\infty}\sum_{t=1}^{j\wedge k}a_j x_{t-j}\right|^r$$

$$\le n^{-r/2}\sum_{j=1}^{\infty}|a_j|^r k_r\beta_r(j\wedge k) + n^{-r/2}\sum_{j=1}^{\infty}|a_j|^r k_r(\beta_2(j\wedge k))^{r/2}$$

by Lemma 1.1

$$\le n^{-r/2}\sum_{j=1}^{\infty}|a_j|^r(k_r\beta_r(j\wedge k))^{r/2} + n^{-r/2}\sum_{j=1}^{\infty}|a_j|^r(k_r\beta_2(j\wedge k))^{r/2}$$

$$= \left(\sum_{j=1}^{\infty}|a_j|(k_r\beta_r(j\wedge k)/n)^{1/2}\right)^r + \left(\sum_{j=1}^{\infty}|a_j|(k_r\beta_2(j\wedge k)/n)^{1/2}\right)^r$$

by the dominated convergence theorem

$$= o(1).$$

To prove (6), define

$$I_2 = a_1x_k + a_2(x_k + x_{k-1}) + a_3(x_k + x_{k-1} + x_{k-2}) + \cdots + a_k(x_k + \cdots + x_1)$$
$$+ (a_{k+1} + a_{k+2} + \cdots)(x_k + x_{k-1} + x_{k-2} + \cdots + x_1)$$
$$= I_3 + I_4 \quad \text{(say)}.$$

Let $\{p_n\}$ be a sequence of positive integers such that

$$p_n \to \infty \text{ and } p_n/n \to 0 \text{ as } n\to\infty. \tag{7}$$

Then

$$n^{-1/2}\max_{1\le k\le n}|I_4|$$
$$\le \left(\sum_{j=0}^{\infty}|a_j|\right)n^{-1/2}\max_{1\le k\le p_n}|x_1 + \cdots + x_{kj}|\left(\sum_{j>p_n}|a_j|\right)n^{-1/2}\max_{1\le k\le n}|x_1 + \cdots + x_k|$$
$$= I_5 + I_6 \quad \text{(say)}$$

It follows from Markov inequality, (4) and (7) that for $r>2$,

$$I_5 \le \left(\sum_{j=0}^{\infty}|a_j|\right)^r n^{-r/2}E\max_{1\le k\le p_n}|x_1 + x_2 + \cdots + x_k|^r$$
$$\le k_r\beta_r\sum_{j=0}^{\infty}|a_j|^r 1/(p_n)^{r/2}(p_n/n)^{r/2} + k_r\beta_2^{r/2}\sum_{j=0}^{\infty}|a_j|^r(\beta_2^{r/2}p_n/n)^{r/2}$$

by Lemma 1.1

$$= o(1).$$

Similarly, by assumption $\sum_{j=0}^{\infty}|a_j|<\infty$, for $r>2$, we can obtain that $I_6\to 0$ in probability as $n\to\infty$. Hence $n^{-1/2}\max_{1\le k\le n}|I_4|\to 0$ in probability as $n\to\infty$. It remains to show that $k_n = n^{-1/2}\max_{1\le k\le n}|I_3|\to 0$ in probability as $n\to\infty$. For each $m\ge 1$, $I_{3,m} = b_1x_t + b_2($

$x_t+x_{t-1})+\cdots+b_k(x_t+\cdots+x_1)$, where $b_x=a_k$ for $k\le n$ and $b_k=0$ otherwise, and let $k_{n,m}=n^{-1/2}\max_{1\le k\le n}|I_{3,m}|$.

Then

$$k_{n,m}\le n^{-1/2}(|a_1|+\cdots+|a_n|)(|x_1|+\cdots+|x_m|)$$
$$\to 0 \text{ in probability as } n\to\infty \tag{8}$$

for each m, and note that

$$|k_{n,m}-k_n|\le n^{-1/2}\max_{1\le k\le n}\left|\sum_{i=1}^{k}(a_i-b_i)(x_k+\cdots+x_{k-i+1})\right| \tag{9}$$
$$\le 2n^{-1/2}\sum_{j>m}|a_i|\max_{1\le k\le n}|x_1+\cdots+x_j|. \tag{10}$$

Therefore, it follows from Markov inequality, (4), (8) and (9) that for any $\varepsilon>0$,

$$\lim_{m\to\infty}\lim_{n\to\infty}\sup P(|k_{n,m}-k_n|>\varepsilon)$$
$$\le \lim_{m\to\infty}2^r\varepsilon^{-r}\Big(\sum_{j>m}|a_j|\Big)^r\lim_{n\to\infty}\sup n^{-r/2}E\max_{1\le k\le n}|x_1+\cdots+x_j|^r$$
$$\le k_r\beta_r\lim_{m\to\infty}\varepsilon^{-r}2^r\Big(\sum_{j>m}|a_j|\Big)^r\lim_{n\to\infty}\sup n^{1-r/2}+k_r\beta_2\lim_{m\to\infty}\varepsilon^{-r}2^r\sum_{j>m}|a_j|^r$$

by Lemma 1.1

$$\to 0 \text{ by } \sum_{j=0}^{\infty}|a_j|<\infty). \tag{11}$$

In view of (10) and (11), it follows from Theorem 4.2 of Billingsley(1968, p.25) that $k_n\to 0$ in probability as $n\to\infty$ and thus (6) is proved.

Theorem 2.2. Let $\{X_t\}$ be a stationary linear process of the form (1), where $\{a_j\}$ is a sequence of constants with $\sum_{j=0}^{\infty}|a_j|<\infty$ and $\{x_t\}$ is a strictly stationary NA process with $Ex_t=0$, $0<Ex_t^2<\infty$. Assume $\sum_{t=2}^{\infty}\mathrm{Cov}\,(x_1,x_t)<\infty$, then the linear process $\{X_t\}$ fulfill the central limit theorem, where $0<\sigma^2=Ex_1^2+2\sum_{t=2}^{\infty}Ex_1x_t<\infty$.

Proof. Let $\mathfrak{X}_t=\left(\sum_{j=0}^{\infty}a_j\right)x_t$ and $\mathfrak{S}_n=\sum_{t=1}^{n}\mathfrak{X}_t=\left(\sum_{j=0}^{\infty}a_j\right)\sum_{t=1}^{n}x_t$.

Then,

$$\sum_{t=2}^{\infty}\mathrm{Cov}(\mathfrak{X}_1,\mathfrak{X}_t)=\left(\sum_{j=0}^{\infty}a_j\right)^2\sum_{t=2}^{\infty}\mathrm{Cov}\,(x_1,x_t)<\infty,\ \left(\text{by } \sum_{j=0}^{\infty}|a_j|<\infty\right).$$

Since $\{\mathfrak{X}_t \mid t\in Z^+\}$ is a stationary NA process, $\{\mathfrak{X}_t \mid t\in Z^+\}$ satisfies the CLT by Theorem 17 of Newman(1984); that is,

$$n^{-1/2}\mathfrak{S}_n\to N(0,\tau^2) \text{ in distribution as } n\to\infty. \tag{12}$$

According to Lemma 2.1, we also have

$$n^{-1/2}|\tilde{S}_n - S_n| \to 0 \quad \text{in probability as} \quad n \to \infty. \tag{13}$$

Hence from (12) and (13) the desired conclusion follows.

Theorem 2.3. Let $\{X_t\}$, $\{a_j\}$ and $\{x_t\}$ be defined as Theorem 2.2. Assume $0 < \sigma^2 = Ex_1^2 + 2\sum_{t=2}^{\infty} Ex_1x_t < \infty$, then the linear process $\{\xi_n\}$ fulfills the $FCLT$; that is, the linear process $\{\xi_n\}$ converges weakly to Wiener measure W on the space of all functions on $C[0, T]$.

Proof. Let $\tilde{X}_t = \left(\sum_{j=0}^{\infty} a_j\right)x_t$, then

$$\begin{aligned} &E\tilde{X}_1^2 + 2\sum_{t=2}^{\infty} E(\tilde{X}_1\tilde{X}_t) \\ &= \left(\sum_{j=0}^{\infty} a_j\right)^2 Ex_1^2 + 2\left(\sum_{j=0}^{\infty} a_j\right)^2 \sum_{t=2}^{\infty} E(x_1x_t) \\ &= \left(\sum_{j=0}^{\infty} a_j\right)^2 \sigma^2 = \tau^2 < \infty. \end{aligned}$$

Since $\{\tilde{X}_t \mid t \in Z^+\}$ is a stationary NA process, $\{\tilde{X}_t \mid t \in Z^+\}$ satisfies the condition of Theorem 3 in Su, Zhao, and Wang(1997). This implies that Theorem 2.3 holds for the sequence $\{\tilde{\xi}_n\}$, where we define $\tilde{\xi}_n$ as in (3), with $\tilde{S}_r$ replacing S_r. So, by Lemma 2.1, we conclude that $|\tilde{\xi}_n(u) - \xi_n(u)| \to 0$ in probability as $n \to \infty$, for all $0 \le u \le 1$. Hence the desired conclusion follows.

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