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Bayes Estimation of P(Y<X) of the Burr-Type X Model under Asymmetric Loss Function

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Abstract

The present paper obtains Bayes estimators for the probability $P(Y< X)$ when X and Y are two independent but not identically distributed the Burr-type X random variables and an asymmetric loss functions are used. An approximation based on the Laplace approximation method (Tierney and Kadane(1986)) is used for obtaining the Bayes estimators of $P(Y< X)$. In order to compare the Bayes estimators with respect to squared error loss(SEL), linex loss and squarex loss functions respectively and MLE of the system reliability, Monte Carlo simulations are used.

Keywords : Burr-type X model, Laplace approximation, Linex loss, Squarex loss

1. Introduction

The one parameter Burr-type X distribution as a member of Burr family of distribution was first introduced by Burr (1942). The probability density function and the distribution function of the Burr-type X random variable with parameter θ, which is denoted by Burr(θ), are given respectively by

$$f(x|\theta) = 2\theta x e^{-x^2}(1-e^{-x^2})^{\theta-1},\ x>0,\ \theta>0 \tag{1.1}$$

and

$$F(x|\theta) = (1-e^{-x^2})^{\theta} \tag{1.2}$$

Under the Burr-type X model, Jaheen (1995) provided the Bayesian prediction bounds for future observations in the presence of outliers. Jaheen (1997) also obtained Bayes and Empirical Bayes estimates of the reliability and failure rate functions of the Burr-type X model under squared error loss function. Kim and Chung (2005) considered the Bayesian estimation of $P(Y<X)$ form the Burr-type X model containing spurious observations under

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SEL. But the symmetric nature of SEL gives equal weight to overestimation as well as underestimation, while in the estimation of parameter of lifetime model overestimation may be more serious than underestimation or vice-versa. Inappropriateness of SEL has been noticed by Zellner (1986). A number of asymmetric loss functions were proposed for their use. Among these, one of the most popular asymmetric loss function is the linex loss function. It is introduced by Varian (1975) and is expressed by

$$L(\Delta)=b(e^{a\Delta}-a\Delta-1),\quad a\neq 0,\quad b>0 \tag{1.3}$$

where $\Delta=\hat{\theta}-\theta$ denotes the scalar estimation error in using $\hat{\theta}$ to estimate θ and b and a are scale and shape parameters of the loss function respectively. The linex loss function may be considered as nature extension of SEL because for small values of a(near to zero) the linex loss function is same as SEL. For the choice of negative or positive values of a, it gives more weight to overestimation or underestimation.(For other details, see Zellner (1986))

Thompson and Basu (1996) introduced a further generalization of the linex loss function, the squarex loss function, in the context of system reliability estimation. The squarex loss function is the following form;

$$L(\Delta)=b(e^{a\Delta}+d\Delta^2-a\Delta-1) \tag{1.4}$$

where $|a|\neq 0, b, d\geq 0$. If $d=0$, squarex loss and linex loss are identical. Hence, squarex loss represents a generalization of linex loss and it is richer family of asymmetric loss functions that are appropriate when estimating system reliability. Chaturvedi et al. (2000) obtained a Bayes estimator of the disturbance variance in a linear regression model under these loss functions. And they showed that a Bayes estimator under squarex loss function is a weighted average of a Bayes estimator under linex loss function and a Bayes estimator under squared error loss function.

In this paper, we deal with the problem of estimating $R=P(Y<X)$ when X and Y are independent Burr-type **X** random variables under asymmetric loss functions. The prior distribution for the parameters of the model has been taken as a natural conjugate prior. The estimators obtained will be compared for their performance on the basis of their root mean squared error(RMSE) under asymmetric loss functions.

In the next section, the estimation of MLE and Bayes estimators under asymmetric loss function and SEL are obtained. In section 3, in oder to compare MLE and Bayes estimators, these estimators are computed via Monte Carlo simulation study.

2. Estimation of $R=P(Y<X)$

Let X be a random sample of size n from a Burr-type **X** with parameter θ and let Y be another Burr-type **X** random variable with parameter β. Then

$$R=P(Y<X)=\frac{\theta}{\theta+\beta}. \tag{2.1}$$

2.1. MLE of R

Let $X_1, \cdots, X_n$ be a random sample of size n from a Burr-type **X** with parameter θ and $Y_1, \cdots, Y_m$ be a random sample size m from a Burr-type **X** with parameter β. The maximum likelihood estimator $\hat{R}_{ML}$ of parameter R has been obtained by Ahmad et al. (1997) and gives as the following forms:

$$\hat{R}_{ML} = \frac{\hat{\theta}_{ML}}{\hat{\theta}_{ML} + \hat{\beta}_{ML}} \tag{2.2}$$

where

$$\hat{\theta}_{ML} = \frac{n}{T_1} \text{ with } \mathrm{T}_1 = \sum_{i=1}^{n} -\log(1 - e^{x_i^2}) \tag{2.3}$$

and

$$\hat{\beta}_{ML} = \frac{m}{T_2} \text{ with } \mathrm{T}_2 = \sum_{i=1}^{m} -\log(1 - e^{y_i^2}). \tag{2.4}$$

2.2. Bayes estimators under Linex loss(LL) and Squarex loss(SL) functions

The parameters θ and β are assumed to be random variables. The gamma conjugate prior densities of both θ and β are used for obtaining the Bayes estimate of R under asymmetric loss functions. In this case, the random variable θ is assumed to be Gamma(d_1, c_1) and β is Gamma(d_2, c_2).

The joint posterior of θ and β is obtained as

$$\pi(\theta, \beta | X, Y) = \frac{(c_1 + T_1)^{n+d_1}(c_2 + T_2)^{m+d_2}}{\Gamma(n+d_1)\Gamma(m+d_2)} \theta^{n+d_1-1}\beta^{m+d_2-1}\exp(-(c_1+T_1)\theta - (c_2+T_2)\beta). \tag{2.5}$$

Since R is a function of the parameters θ and β, by using usual transformation of random variables, one can obtain the posterior density of R in the form

$$\pi(R|X, Y) = \frac{\Gamma(n+m+d_1+d_2)}{\Gamma(n+d_1)\Gamma(m+d_2)}(c_1+T_1)^{n+d_1}(c_2+T_2)^{m+d_2}$$

$$\times \frac{R^{n+d_1-1}(1-R)^{m+d_2-1}}{((c_1+T_1)R + (c_2+T_2)(1-R))^{n+m+d_1+d_2}}$$

where $0 < R < 1$.

The Bayes estimators $\hat{R}_{SE}$, $\hat{R}_{LL}$ and $\hat{R}_{SL}$ under SEL, linex loss and squarex loss functions are given as respectively

$$\hat{R}_{SE} = E(R|X, Y),$$

$$\hat{R}_{LL} = -\frac{1}{a}\log\left(E[e^{-aR}|X, Y]\right)$$

and

$$\hat{R}_{SL} = \hat{R}_{LL} + \frac{1}{a}\log\left(1 + \frac{2d}{a}(\hat{R}_{SE} - \hat{R}_{SL})\right) \tag{2.6}$$

where $E(\cdot)$ denotes the posterior expectation. Therefore, this expected value contains an integral which is usually not obtainable in a simple closed form.

In such a situation, we can use numerical integration technique, which can be computationally intensive, especially in a high dimensional parameter space. We can also use approximation methods such as the approximate form due to Lindley (1980) or that of Tierney and Kadane (1986). Here, we adopt the Tierney and Kadane (1986) approximation.

Note that for small values of a, the squarex loss function in (1.4) is almost symmetric and not far from a SEL. On expanding $e^{a\Delta} = 1 + a\Delta + \frac{a^2\Delta^2}{2}$, the squarex loss function is approximated to a SEL, that is, $L(\Delta) = b\left(\frac{a^2}{2} + d\right)\Delta^2$. Therefore, the Bayes estimates of $R = P(Y < X)$ under linex and squarex loss functions are not far different from those obtained with SEL.

Also, the above equation $\hat{R}_{SL}$ in (2.6) can be solved numerically but the explicit expression for this estimate can not be obtained. However, if d is small in comparing to a so that the terms of order $O\left(\frac{d}{a}\right)$ is negligible, then (2.6) can be approximately rewritten as

$$\hat{R}_{AP} = \frac{a^2\hat{R}_{LL} + 2d\hat{R}_{SE}}{a^2 + 2d}. \tag{2.7}$$

The approximate estimator in (2.7) is a weighted average of the Bayes estimator under linex loss and the Bayes estimator under SEL.

2.3. Laplace approximation

Let $\zeta = (\theta, \beta)$ and $l(\zeta; x)$ be the likelihood function of ζ based on the n observations and $\pi(\zeta|x)$ denotes the posterior distribution of ζ. Then the posterior mean of a function $\phi(\zeta)$ can be written as

$$E(\phi(\zeta)|X) = \int\phi(\zeta)\pi(\zeta|X)d\zeta = \frac{\int e^{nL^*}}{\int e^{nL}} \tag{2.8}$$

where

$$L(\zeta) = \frac{1}{n}\log\pi(\zeta|x) \text{ and } L^*(\zeta) = L(\zeta) + \frac{1}{n}\log\phi(\zeta).$$

Following Tierney and Kadane (1986), the equation (2.8) can be approximated as following forms;

$$E(\phi(\zeta)|X)=\left(\frac{|\Sigma^*|}{|\Sigma|}\right)^{\frac{1}{2}}\exp\left(n\left(L^*(\hat{\zeta}^*)-L(\hat{\zeta})\right)\right)$$

$$=\left(\frac{|\Sigma^*|}{|\Sigma|}\right)^{\frac{1}{2}}\frac{\phi(\hat{\zeta}^*)\pi(\hat{\zeta}^*|x)}{\pi(\hat{\zeta}|x)} \quad (2.9)$$

where $\hat{\zeta}^*$ and $\hat{\zeta}$ maximizes $L^*(\zeta)$ and $L(\zeta)$ respectively and Σ^* and Σ are minus the inverse Hassians of $L^*(\zeta)$ and $L(\zeta)$ at $\hat{\zeta}^*$ and $\hat{\zeta}$ respectively.

We apply this approximation to obtain the Bayes estimators of the $R=P(Y<X)$ given by (2.1) under SEL, linex loss and squarex loss functions. In this case, the functions L^* and L are given by respectively

$$L(\theta,\beta)\propto\frac{1}{n}((n+d_1-1)\log\theta+(m+d_2-1)\log\beta-(c_1+T_1)\theta-(c_2+T_2)\beta)$$

and

$$L^*(\theta,\beta)=L(\theta,\beta)+\frac{1}{n}\phi(\theta,\beta). \quad (2.10)$$

Let $L_1=\frac{\partial L(\theta,\beta)}{\partial\theta}$, $L_2=\frac{\partial L(\theta,\beta)}{\partial\beta}$, $L_{11}=\frac{\partial^2 L(\theta,\beta)}{\partial\theta^2}$, $L_{12}=\frac{\partial^2 L(\theta,\beta)}{\partial\theta\partial\beta}$ and $L_{22}=\frac{\partial^2 L(\theta,\beta)}{\partial\beta^2}$ be the first and second derivatives of $L(\theta,\beta)$. Then, the posterior mode $(\hat{\theta},\hat{\beta})$ is obtained by equating L_1 and L_2 to zero and solving the resulting equations in θ and β. Therefore, $\hat{\theta}=\frac{n+d_1-1}{c_1+T_1}$ and $\hat{\beta}=\frac{m+d_2-1}{c_2+T_2}$. Using L_{11}, L_{12} and L_{22}, one can see that

$$|\Sigma|=\frac{1}{L_{11}L_{22}-L_{12}^2} \quad (2.11)$$

which is evaluated at the posterior mode $(\hat{\theta},\hat{\beta})$.

Similar derivatives are needed to determine the mode $(\hat{\theta}^*,\hat{\beta}^*)$ and $|\Sigma^*|$ of $L^*(\theta,\beta)$. Let L_1^*, L_2^*, L_{11}^*, L_{12}^* and L_{22}^* be the first and second derivatives of $L^*(\theta,\beta)$. Differentiate (2.10) with respect to θ and β and equate each result L_1^* and L_2^* to zero. Therefore, the mode $(\hat{\theta}^*,\hat{\beta}^*)$ of θ and β can be obtained iteratively by solving the two resulting equations. Using L_{11}^*, L_{12}^* and L_{22}^*, we get

$$|\Sigma^*|=\frac{1}{L_{11}^*L_{22}^*-L_{12}^{*2}} \quad (2.12)$$

which is evaluated at the posterior mode $(\hat{\theta}^*,\hat{\beta}^*)$.

Note that all of these values of $L^*(\theta,\beta)$ and the first and the second derivatives of $L^*(\theta,\beta)$ in (2.10) can be found if $\phi(\theta,\beta)$ has an explicit functional form in the parameters θ and β.

Substituting from (2.11) and (2.12) in (2.9), the Bayes estimators $\hat{R}_{SE}$ and $\hat{R}_{LL}$ of a function $\phi(\theta,\beta)$ under SEL, linex loss take of the forms respectively

$$\hat{R}_{SE}=E(\phi(\theta,\beta)|X,Y)=\left(\frac{L_{11}L_{22}-L_{12}^*}{L_{11}^*L_{22}^*-L_{12}^*}\right)^{\frac{1}{2}}\frac{\phi(\hat{\theta}^*,\hat{\beta}^*)\pi(\hat{\theta}^*,\hat{\beta}^*|x,y)}{\pi(\hat{\theta},\hat{\beta}|x,y)} \tag{2.13}$$

and

$$\hat{R}_{LL}=-\frac{1}{a}\log(E[e^{-a\phi(\theta,\beta)}|X,Y])=-\frac{1}{a}\log\left\{\left(\frac{L_{11}L_{22}-L_{12}^*}{L_{11}^*L_{22}^*-L_{12}^*}\right)^{\frac{1}{2}}\frac{\phi(\hat{\theta}^*,\hat{\beta}^*)\pi(\hat{\theta}^*,\hat{\beta}^*|x,y)}{\pi(\hat{\theta},\hat{\beta}|x,y)}\right\} \tag{2.14}$$

where $\pi(\theta,\beta|x,y)$ is the posterior density function given by (2.5) evaluated at the modes $(\hat{\theta},\hat{\beta})$ and $(\hat{\theta}^*,\hat{\beta}^*)$ of the functions $L(\theta,\beta)$ and $L^*(\theta,\beta)$ respectively.

Under squarex loss function, the Bayes estimator $\hat{R}_{SL}$ of a function $\phi(\theta,\beta)$ given by (2.14) can be solved by Newton-Raphson iteration scheme using the nonlinear equation as following forms;

$$\exp(a(R-\hat{R}_{LL}))+\frac{2d}{a}(R-\hat{R}_{SE})-1=0. \tag{2.15}$$

The solution of the above equation given by (2.15) yields the Bayes estimator $\hat{R}_{SL}$ under squarex loss function.

In our case, $\phi(\theta,\beta)=\dfrac{\theta}{\theta+\beta}$ and the Bayes estimates of R with respect to SEL and Linex loss function are obtained as follows:

$$\hat{R}_{SE}=E\left(\frac{\theta}{\theta+\beta}|X,Y\right)$$

$$=\left(\frac{\dfrac{(n-d_1-1)(m+d_2-1)}{\widehat{\theta^2}\widehat{\beta^2}}}{\dfrac{(n_1+d_1)(m+d_2)}{\hat{\theta}^{*2}\hat{\beta}^{*2}}-\dfrac{n+d_1}{\hat{\theta}^{*2}(\hat{\theta}^*+\hat{\beta}^*)^2}-\dfrac{m+d_2}{\hat{\beta}^{*2}(\hat{\theta}^*+\hat{\beta}^*)^2}}\right)^{\frac{1}{2}}\frac{\hat{\theta}^*}{\hat{\theta}^*+\hat{\beta}^*}\left(\frac{\hat{\theta}^*}{\hat{\theta}}\right)^{n+d_1-1}$$

$$\times\left(\frac{\hat{\beta}^*}{\hat{\beta}}\right)^{m+d_2-1}\exp(-(c_1+T_1)(\hat{\theta}^*-\hat{\theta})-(c_2+T_2)(\hat{\beta}^*-\hat{\beta}))$$

and

$$\hat{R}_{LL} = -\frac{1}{a}\log\left(E\left(e^{-a\frac{\theta}{\theta+\beta}}|X, Y\right)\right)$$

$$= -\frac{1}{a}\log\left(K(\hat{\theta},\hat{\beta},\hat{\theta}^*,\hat{\beta}^*)^{\frac{1}{2}}\left(\frac{\hat{\theta}^*}{\hat{\theta}}\right)^{n+d_1-1}\left(\frac{\hat{\beta}^*}{\hat{\beta}}\right)^{m+d_2-1}\exp\left(-\frac{a\hat{\theta}^*}{\hat{\theta}^*+\hat{\beta}^*}\right)\right.$$
$$\left.\times\exp\left(-(c_1+T_1)(\hat{\theta}^*-\hat{\theta}) - (c_2+T_2)(\hat{\beta}^*-\hat{\beta})\right)\right)$$

where $\hat{\theta} = \frac{n+d_1-1}{c_1+T_1}$, $\hat{\beta} = \frac{m+d_2-1}{c_2+T_2}$, $\hat{\theta}^*$ and $\hat{\beta}^2$ are the posterior mode of $L^*(\theta,\beta)$, T_1 and T_2 are given in (2.3) and (2.4) and

$$K(\hat{\theta},\hat{\beta},\hat{\theta}^*,\hat{\beta}^*) = \frac{\dfrac{(n+d_1-1)(m+d_2-1)}{\hat{\theta}^2\hat{\beta}^2}}{\left(-\dfrac{n+d_1-1}{\hat{\theta}^{*2}} + \dfrac{2a\hat{\beta}^{*2}}{(\hat{\theta}^{*2}+\hat{\beta}^{*2})}\right)\left(-\dfrac{m+d_2-1}{\hat{\beta}^{*2}} - \dfrac{2a\hat{\theta}^*}{(\hat{\theta}^*+\hat{\beta}^*)^3}\right) - \left(\dfrac{a(\hat{\beta}^*-\hat{\theta}^*)}{(\hat{\theta}^*+\hat{\beta}^*)^3}\right)^2}$$

Finally, the Bayes estimator relative to squarex loss function, $\hat{R}_{SL}$ is obtained by using (2.15).

3. Simulation Study and Conclusions

In order to compare MLE and Bayes estimators of the parameter R under SEL, linex and squarex loss functions, Monte Carlo simulation was performed. The following steps summarize the simulations:

(1) For a given values $\theta = 9$ and $\beta = 3$.

(2) Using the results for θ and β, the sample size n and m are generated from the Burr-type **X** density using the inverse cumulative distribution function respectively.

(3) The MLE $\hat{R}_{ML}$ and Bayes estimates $\hat{R}_{SE}$, $\hat{R}_{LL}$ and $\hat{R}_{SL}$ are computed respectively.

(4) Above steps were repeated 1000 times

(5) The average values and RMSE of the estimates were calculated over the 1000 samples.

$$\text{RMSE} = \sqrt{\frac{1}{1000}\sum_{i=1}^{1000}(\hat{R}_i - R)^2}$$

where R is true value and $\hat{R}_i$ is an estimate of R at i th sample.

The IMSL subroutines were used for pseudo-random number generation. The computation results for various estimates are shown in Table 1. Entries within parentheses represent the corresponding RMSE. Table 1 represents the MLE and Bayes estimates of $R = P(Y < X)$ relative to symmetric and asymmetric losses for different values of a and d. If a becomes a negative value, then it tends to give more weight to overestimation. Otherwise, it gives more weight to underestimation.

It is seen that the Baye estimates are better than that of MLE in the sense of comparing the RMSE of the estimates. For small value of d in comparing to a(for example, a=-5, d =0.01) the approximation estimates $\hat{R}_{AP}$ in (2.7) is approximately 0.7502 and it is seen that the Bayes estimate $\hat{R}_{SL}$ and its approximation estimate $\hat{R}_{AP}$ are almost same values. While for small value of a(for example a=0.1 and d=5), the RMSE relative to asymmetric loss functions are not far different form those obtained with SEL. One would expect that as the sample size increases, the RMSE decreases.

Table 1. MLE and Bayes estimates of $R=P(Y<X)$ and RMSE for $n=m=50$

a	d	$\hat{R}_{ML}$	$\hat{R}_{SE}$	$\hat{R}_{LL}$	$\hat{R}_{SL}$
5	5	0.7485 (0.0377)	0.7461 (0.0341)	0.7438 (0.0348)	0.7445 (0.0346)
1	5			0.7464 (0.0339)	0.7461 (0.0341)
-1	5			0.7477 (0.0337)	0.7463 (0.0341)
-5	5			0.7502 (0.0331)	0.7490 (0.0334)
-5	0.01			0.7502 (0.0331)	0.7502 (0.0334)

Table 1. MLE and Bayes estimates of $R=P(Y<X)$ and RMSE for $n=m=100$

a	d	$\hat{R}_{ML}$	$\hat{R}_{SE}$	$\hat{R}_{LL}$	$\hat{R}_{SL}$
5	5	0.7490 (0.0272)	0.7478 (0.0258)	0.7465 (0.0261)	0.7469 (0.0260)
1	5			0.7479 (0.0258)	0.7478 (0.0258)
0.1	5			0.7481 (0.0257)	0.7478 (0.0258)

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