LARGE DEVIATION PRINCIPLE FOR DIFFUSION PROCESSES IN A CONUCLEAR SPACE

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ABSTRACT. We consider a type of large deviation principle obtained by Freidlin and Wentzell for the solution of Stochastic differential equations in a conuclear space. We are using exponential tail estimates and exit probability of a Ito process. The nuclear structure of the state space is also used.

1. Introduction

The large deviation principle(LDP) for stochastic differential equations(SDEs) in infinite dimensions has been considered by many authors. Xiong [13] and Kallinapur and Xiong [7] consider the random fields driven by a Gaussian white noise in space-time and derive the large deviation results. Peszat [11] considers an LDP in a Hilbert space and Marquez-Carreras and Sarra [8] deal an LDP for mild solutions to a perturbed stochastic heat equation with an worthy martingale measure.

We are going to consider an LDP for a strong solution to an SDE in $\mathcal{S}'(R^d)$. Here $\mathcal{S}'(R^d)$ is the dual of Schwartz space $\mathcal{S}(R^d)$ which is the space of infinitely differentiable functions with compact support. We leave it in our next paper to set an LDP for a mild solution in this conuclear space. Since $\mathcal{S}'(R^d)$ is neither a Hilbert space nor a metric space, we need to reduce that space onto a Hilbert space.

Recall that if $\{\mu_\epsilon\}$ is a family of probability measure on a Polish space $\mathcal{P}$ with metric ρ, then μ_ϵ is said to have an LDP with rate functional

$I : \mathcal{P} \to [0, \infty]$, which is a lower semi-continuous function such that the sets

$$\mathcal{K}(r) = \{x \in \mathcal{P} : I(x) \leq r\}, \quad r \geq 0$$

are compact and I satisfies the following:

(L1) for each open set $O \subset \mathcal{P}$

$$\liminf_{\epsilon \to 0} 2\epsilon^2 \log \mu_\epsilon(O) \geq - \inf\{I(x) : x \in O\},$$

(L2) for each closed set $C \subset \mathcal{P}$

$$\limsup_{\epsilon \to 0} 2\epsilon^2 \log \mu_\epsilon(C) \leq - \inf\{I(x) : x \in C\}.$$

Instead of directly trying to show the above inequalities, we will follow a reformulated equivalent form of the so-called Freidlin-Wentzell exponential estimates.(see [2])

(L1') $\forall x \in \mathcal{P}$, $\forall \delta > 0$ and $\forall \gamma > 0$ $\exists \epsilon_0$ such that $\forall \epsilon \in (0, \epsilon_0]$

$$\mu_\epsilon\{y : \rho(y, x) \leq \delta\} \geq \exp\{-\frac{I(x) + \gamma}{2\epsilon^2}\},$$

(L1') $\forall r$, $\forall \delta > 0$ and $\forall \gamma > 0$ $\exists \epsilon_0$ such that $\forall \epsilon \in (0, \epsilon_0]$

$$\mu_\epsilon\{y : \rho(y, \mathcal{K}(r)) \geq \delta\} \leq \exp\{-\frac{r - \gamma}{2\epsilon^2}\}.$$

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with a rigid-continuous increasing family $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$ of sub-σ-fields of $\mathcal{F}$ each containing $\mathbf{P}-$null sets.

Let $W(t)$ be a standard $\mathcal{S}'(R^d)-$valued Wiener process, i.e., it is a Wiener process with

(1.1) $E \exp(iW(t)[\xi]) = \exp -t(|\xi|_0^2/2)$, for all $\xi \in \mathcal{S}(R^d)$,

where $|\cdot|_0$ is the usual L_2-norm in R^d. Thus the covariance functional Q of $W(t)$ is given by $Q = I$.

We consider the following perturbed stochastic differential equations:

(1.2) $dX^\epsilon(t) = B(t, X^\epsilon(t))dt + \epsilon G(t, X^\epsilon(t))dW(t), \quad X^\epsilon(0) = x,$

where

$$B : R^+ \times \mathcal{S}'(R^d) \to \mathcal{S}'(R^d)$$

and
$$G : R^+ \times \mathcal{S}'(R^d) \to L(\mathcal{S}'(R^d), \mathcal{S}'(R^d))$$
are two measurable functions, and $L(\mathcal{B}, \mathcal{C})$ denotes the space of operators from $\mathcal{B}$ into $\mathcal{C}$.

The case of Gaussian perturbations with G being constant was solved by Smolenski, et al.[9]. You may refer a short history on this subject to [11].

Under different sets of conditions, (1.1) has strong or mild solution. We define a strong solution of (1.1) as an $\mathcal{S}'(R^d)$-valued process $X(t)$, $t \in [0, T]$, which takes values in $D(A) \cap D(G)$ and $\mathbf{P}$ a.s.,

(1.3)
$$X^\epsilon(t) = x + \int_0^t B(s, X^\epsilon(s))ds + \epsilon \int_0^t G(s, X^\epsilon(s))dW(s),$$
$$X^\epsilon(0) = x \in \mathcal{S}'(R^d).$$

SDEs of the type (1.3) have been studied by several authors. For instance, Kallianpur, Mitoma and Wolpert [3], Kallianpur, et al.[6], and Walsh [12] have researched the linear and quasilinear equations.

The existence and uniqueness of solution to equation(1.2) is well known, which is stated in the next section. It is also known that under some weak conditions there exist some Hilbert spaces which are embedded in $\mathcal{S}'(R^d)$ such that $X(t)$ has regular versions in these Hilbert spaces for $0 \le t \le T$.

Now, we consider the following deterministic problem.

(1.4)
$$\frac{dz}{dt} = B(t, z) + G(t, z)\psi, \quad z(0) = x.$$

Let H be a Hilbert space with a norm $\| \cdot \|_H$. We denote $\mathcal{H}_T = L^2(0, T; H), T > 0; \{\psi \in H : \int_0^T \|\psi\|^2 ds < \infty\}$ and $|\psi|_{\mathcal{H}_T}^2 = \int_0^T \|\psi\|_H^2 ds$. For $\psi \in \mathcal{H}_T$, the strong solution z^ψ satisfies the following deterministic equaion:

$$z^\psi(t, x) = x + \int_0^t B(s, z^\psi)ds + \int_0^t G(s, z^\psi)\psi(s)ds.$$

Also, let $I : C([0, T]; H) \longrightarrow [0, \infty]$ be given by
$$I(z) = \inf\{|\psi|_{\mathcal{H}_T}^2 : z = z^\psi(\cdot, x)\},$$
where $I(z)$ depends on x and T. And set
$$\mathcal{K}(r) = \{z \in C([0, T]; H) : I(z) \le r\}, \quad r \ge 0.$$
This $I(z)$ is our desired rate function.

2. Preliminaries

Let $\mathcal{S}(R^d)$ be the Schwartz space on R^d consisting of C^∞ functions which together with all their derivatives vanish at infinity faster than any power of $|x|$. $\mathcal{S}(R^d)$ is a separable Frechet space whose topology is given by the following increasing sequence $\{|\cdot|_p; p = 1, 2, ...\}$ of Hilbertian norms;

$$|\phi|_p^2 = \int (S^p \phi)^2 ds, \text{ where } S\phi = |x|^2 \phi - \Delta\phi.$$

Let $\mathcal{S}_p$ denote the completion of $\mathcal{S}(R^d)$ with respect to the norm $|\cdot|_p$. For each p there exists $q > p$ (in fact, $q > p + \frac{d}{2}$) such that the canonical inclusion from $\mathcal{S}_q$ into $\mathcal{S}_p$ is a Hilbert-Schmidt operator. That is $|\cdot|_p <_{\text{HS}} |\cdot|_q$.

Let $\mathcal{S}'(R^d)$ and $\mathcal{S}_{-p}$ denote the dual spaces of $\mathcal{S}(R^d)$ and $\mathcal{S}_p$, respectively. Let $|\cdot|_{-p}$ denote the norm on $\mathcal{S}_{-p}$. In fact we have $\mathcal{S}_0 = L_2(R^d)$ and the following continuous inclusions

$$\cap_{p=1}^\infty \mathcal{S}_p = \mathcal{S}(R^d) \subset \mathcal{S}_q \subset \mathcal{S}_p \subset \mathcal{S}_0 \subset \mathcal{S}_{-p} \subset \mathcal{S}_{-q} \subset \mathcal{S}'(R^d) = \cup_{p=1}^\infty \mathcal{S}_{-p}.$$

It is known that (e.g. see [1] or [12]) there exists a sequence $\{\xi_j; j = 1, 2....\}$ in $\mathcal{S}(R^d)$ such that $\{\xi_j\}$ is a CONS for $\mathcal{S}_0$ and is a complete orthogonal system in $\mathcal{S}_p$ for any p, $p = 0, 1, 2....$ For each positive integer p, let $\xi_j^{(p)} \equiv |\xi_j|_p^{-1}\xi_j$. Then $\{\xi_j^{(p)}; j \geq 1\}$ is a CONS for $\mathcal{S}_p$.

Define θ_p to be the isometric linear operator

$$\theta_p : \mathcal{S}_{-p} \to \mathcal{S}_p$$

such that $\theta_p \xi_j^{(-p)} = \xi_j^{(p)}$ for all $j \geq 1$. For each p the restriction of θ_p to $\mathcal{S}(R^d)$ is a continuous linear operator from $\mathcal{S}(R^d)$ into itself.

Now, we introduce a set of assumptions and state a theorem from [13] on the existence and uniqueness of solution of SDE (1.2).

CONDITION (S). *For any $T > 0$, there exists $p_0 = p_0(T) \geq 0$ such that for any $p \geq p_0$ we can find $q = q(p) \geq p$ and $K = K(p,q) \in L^1([0,T])$ satisfying the following conditions;*
  S1) *(Continuity)* $B(t, \cdot)|_{\mathcal{S}_{-p}} : \mathcal{S}_{-p} \to \mathcal{S}_{-q}$ *and* $G(t, \cdot)|_{\mathcal{S}_{-p}} : \mathcal{S}_{-p} \to L(\mathcal{S}_{-p}, \mathcal{S}_{-p})$ *are continuous for all $t \in [0, T]$.*
  S2) *(Coercivity) For all $t \in [0, T]$ and $\xi \in \mathcal{S}_{-p}$, we have*

$$2B(t, \xi)[\theta_p \xi] \leq K(t)(1 + |\xi|_{-p}^2).$$

S3) (Growth) *For all $t \in [0, T]$ and $\xi \in \mathcal{S}_{-p}$, we have*

$$|B(t, \xi)|^2_{-q} + \|G(t, \xi)\|^2_{L(\mathcal{S}_{-p}, \mathcal{S}_{-p})} \le K(t)(1 + |\xi|^2_{-p}).$$

S4) (Monotonicity) *For all $t \in [0, T]$ and $\xi_1, \xi_2 \in \mathcal{S}_{-p}$, we have*

$$2\langle B(t, \xi_1) - B(t, \xi_2), \xi_1 - \xi_2 \rangle_{-q}$$
$$+ \|G(t, \xi_1) - G(t, \xi_2)\|^2_{L(\mathcal{S}_{-q}, \mathcal{S}_{-q})}$$
$$\le K(t)(|\xi_1 - \xi_2|^2_{-q}).$$

S5) *For all $t \in [0, T]$ and $\xi \in \mathcal{S}_{-p}$, we have*

$$\|G(t, \xi)\|^2_{L(\mathcal{S}_{-p}, \mathcal{S}_{-p})} \le K(t).$$

THEOREM 2.1. [13] *Suppose B and G satisfy Condition S1)-S4). Then the $\mathcal{S}'(R^d)$-valued Stochastic differential equation (1.2) has a unique solution X. Moreover, if $X(0) \in \mathcal{S}_{-r_0}$ and $p_1 \ge p_0 \vee r_0$ such that the inclusion map from $\mathcal{S}_{(-p_0 \vee r_0)}$ into $\mathcal{S}_{-p_1}$ is a Hilbert-Schmidt operator, then $X|_{[0,T]} \in \mathcal{S}_{-p_1}$ a.s. and*

$$E[\sup_{0 \le t \le T} |X(t)|^2_{-p_1}] < \infty.$$

This theorem implies that we can regard the solution process as a $\mathcal{S}_{-p}$-valued process for sufficiently large p. This kind way of regularization is also used in [3].

3. Large deviation theorem

Let B and G satisfy Condition (S) and we consider the following strong solution of (1.2):

$$(3.1) \quad X^\epsilon(t) = X(0) + \int_0^t B(s, X^\epsilon(s))ds + \epsilon \int_0^t G(s, X^\epsilon(s))dW(s).$$

For $\psi \in \mathcal{H}_T$, let z^ψ satisfy the following deterministic equaion:

$$z^\psi(t) = x + \int_0^t B(s, z^\psi(s))ds + \int_0^t G(s, z^\psi(s))\psi(s)ds.$$

The following theorem is an extension of the exit probability estimated by Chow and Menaldi [1] for Ito process in $\mathcal{S}'(R^d)$.

THEOREM 3.1. *Assume that β is a predictable $L(\mathcal{S}_{-p}, \mathcal{S}_{-p})-$ valued process and there exists a constant $l < \infty$ such that*

$$\int_0^T \|\beta(s)\|^2_{L(\mathcal{S}_{-p}, \mathcal{S}_{-p})} ds \le l, \; P - \text{almost surely.}$$

Then for all $r > 0$ one has

$$(3.2) \qquad \mathbf{P}\left\{ \sup_{0 \le t \le T} \left| \int_0^t \beta(s) dW(s) \right|_{-p} \ge r \right\} \le 3 \cdot \exp\left\{ -\frac{r^2}{4l} \right\}.$$

PROOF. Let $Z(t) = \int_0^t \beta(s) dW(s)$ and $\{\xi_j : j \ge 1\}$ be a CONS for $\mathcal{S}_0$. Let $\xi_j^{(p)} \equiv |\xi_j|_p^{-1} \xi_j$. Then $\{\xi_j^{(p)} : j \ge 1\}$ forms a CONS for $\mathcal{S}_p$. For any $j \ge 1$, we have

$$Z(t)[\xi_j^{(p)}] = \sum_{k=1}^\infty \int_0^t (\beta(s)\xi_k)[\xi_j^{(p)}] dW(s)[\xi_k].$$

Apply the Ito formula to get

$$(Z(t)[\xi_j^{(p)}])^2 = \sum_{k=1}^\infty \int_0^t 2Z(s)[\xi_j^{(p)}](\beta(s)\xi_k)[\xi_j^{(p)}] dW(s)[\xi_k]$$

$$(3.3) \qquad\qquad + \int_0^t |\beta(s)^*[\xi_j^{(p)}]|_0^2 ds,$$

where β^* denotes the adjoint of β. (Note that $\theta_p Z(s) = \sum_{j=1}^\infty Z(s)[\xi_j^{(p)}] \xi_j^{(p)}$)

Hence if we sum up the equation (3.3) over j we obtain

$$|Z(t)|^2_{-p} = 2 \sum_{k=1}^\infty \int_0^t (\beta(s)\xi_k)[\theta_p Z(s)] dW(s)[\xi_k]$$

$$+ \int_0^t \|\beta(s)\|^2_{L(\mathcal{S}_{-p}, \mathcal{S}_{-p})} ds.$$

For the following estimations let

$$\phi_\lambda(x) = (1 + \lambda x)^{\frac{1}{2}}, \quad \text{for any } \lambda, \, x \in R_+.$$

Also for convenience, we abuse some notations as $|\cdot|_{-p} = \|\cdot\|$ and $\|\cdot\|^2_{L(\mathcal{S}_{-p},\mathcal{S}_{-p})} = \|\cdot\|_L$.

We again apply the Ito formula to (3.3), then

$$
(3.4) \quad \phi_\lambda(\|Z(t)\|^2)
$$

$$
= 1 + \sum_{k=1}^\infty \int_0^t \phi_\lambda^{-1}(\|Z(s)\|^2)(\frac{\lambda}{2}\|\beta(s)\|^2_L)ds
$$

$$
+ \lambda \sum_{k=1}^\infty \int_0^t \phi_\lambda^{-1}(\|Z(s)\|^2)(\beta(s)\xi_k)[\theta_p(Z(s))]dW(s)[\xi_k]
$$

$$
- \frac{1}{2}\lambda^2 \int_0^t \phi_\lambda^{-1}(\|Z(s)\|^2)|\beta(s)^*(\theta_p Z(s))|_0^2 ds
$$

$$
+ \frac{1}{2}\lambda^2 \int_0^t \phi_\lambda^{-1}(\|Z(s)\|^2)|\beta(s)^*(\theta_p Z(s))|_0^2 ds
$$

$$
- \lambda^2 \int_0^t \phi_\lambda^{-3}(\|Z(s)\|^2)|\beta(s)^*\theta_p(Z(s))|_0^2 ds).
$$

Note that $\phi_\lambda^{-1}(x) \le 1$ and θ_p is an isometry from $\mathcal{S}_{-p}$ into $\mathcal{S}_p$.

For every $\lambda \in R$, let

$$
(3.5)
$$

$$
\eta_t^\lambda \equiv \lambda \sum_{k=1}^\infty \int_0^t \phi_\lambda^{-1}(\|Z(s)\|^2)(\beta(s)\xi_k)[\theta_p(Z(s))]dW(s)[\xi_k]
$$

$$
- \frac{1}{2}\lambda^2 \int_0^t \phi_\lambda^{-2}(\|Z(s)\|^2)|\beta(s)^*\theta_p(Z(s))|_0^2 ds.
$$

Then

$$
\phi_\lambda(\|Z(t)\|^2)
$$

$$
(3.6) \quad \le 1 + \frac{\lambda}{2}\int_0^t \frac{\|\beta(s)\|^2_L \cdot \lambda\|Z(s)\|^2}{1+\lambda\|Z(s)\|^2}ds + \frac{\lambda}{2}l + \eta_t^\lambda \quad \text{a.s. } \forall t \in [0,T]
$$

$$
\le 1 + \frac{\lambda}{2}\int_0^t \|\beta(s)\|^2_L ds + \frac{\lambda}{2}l + \eta_t^\lambda \le 1 + \eta_t^\lambda + \lambda l.
$$

Let $Z_t^\lambda \equiv \exp\{\eta_t^\lambda\}$ for every real λ. Then Z_t^λ is a local martingale and it follows easily from the Ito formula that $EZ_t^\lambda = 1$. Hence using Doob's

martingale inequality, we have

$$\mathbf{P}\{ \sup_{0 \le s \le t} |Z(s)|_{-p} \ge r \}$$

(3.7)
$$\le \mathbf{P}\{ \sup_{0 \le s \le t} \phi_\lambda(|Z(s)|^2_{-p}) \ge (1 + \lambda r^2)^{\frac{1}{2}} \}$$

$$\le \mathbf{P}\{ \sup_{0 \le s \le t} Z^\lambda_s \ge \exp\{(1 + \lambda r^2)^{\frac{1}{2}} - 1 - \lambda l\} \}$$

$$\le \exp\{-(1 + \lambda r^2)^{\frac{1}{2}} + 1 + \lambda l\}.$$

If $r^2 > 2l$ then taking $\lambda = \frac{r^4 - 4l^2}{4l^2 r^2}$ we obtain the desired bound (3.1). If $r^2 < 2l$, then $3\exp(\frac{-r^2}{4l}) \ge 1$. Thus (3.1) holds for any $r > 0$. $\qquad \square$

From now on, $\mathcal{H}_T$ is $L^2(0, T : \mathcal{S}_{-p})$ and $|\psi|^2_{\mathcal{H}_T} = \int_0^T |\psi|^2_{-p} ds$.

THEOREM 3.2. (Lower bound) *Under the condition* (S), $\forall T > 0$, $\forall l > 0$, $\forall \delta > 0$, $\forall \gamma > 0$, *there exists* $\epsilon_0 > 0$ *such that* $\forall \epsilon \in [0, \epsilon_0]$ *and* $\forall \psi \in \mathcal{H}_T$ *satisfying* $|\psi|^2_{\mathcal{H}_T} \le l$, *we have for any sufficiently large* p

$$(3.8) \quad \mathbf{P}\left\{ \sup_{0 \le t \le T} |X^\epsilon(t, x) - z^\psi(t, x)|_{-p} \le \delta \right\} \ge \exp\left\{ -\frac{|\psi|^2_{\mathcal{H}_T} + \gamma}{2\epsilon^2} \right\}.$$

PROOF. Let $l > 0$, $r > 0$, $\delta > 0$ and $\psi \in \mathcal{H}_T$ satisfying $|\psi|_{\mathcal{H}_T} \le l$. Define

$$W^\epsilon(t) = W(t) - \frac{1}{\epsilon} \int_0^t \psi(s) ds, \ t \le T.$$

According to the Girsanov theorem, W^ϵ is a cylindrical Wiener process on the probability space $(\Omega, \mathcal{F}, P^\epsilon)$, where

$$\frac{dP}{dP^\epsilon} = \exp\left\{ -\frac{1}{\epsilon} \int_0^T \psi(s) W(ds) - \frac{1}{2\epsilon^2} \int_0^T |\psi(s)|^2_{-p} ds \right\}.$$

Note that $P \ll P^\epsilon$.

Now, consider the following:

$$X^\epsilon(t) - z^\psi(t) \le X^\epsilon(0) - z^\psi(0)$$

$$+ \int_0^t B(s, X^\epsilon(s)) - B(s, z^\psi(s)) ds$$

$$+ \epsilon \int_0^t G(s, X^\epsilon(s)) dW^\epsilon(s)$$

$$+ \int_0^t G(s, X^\epsilon(s)) \psi(s) ds - \int_0^t G(s, z^\psi(s)) \psi(s) ds,$$

$$|X^\epsilon(t) - z^\psi(t)|_{-p} \leq K_1 \int_0^t |X^\epsilon(s) - z^\psi(s)|_{-p}\, ds$$

$$+ \epsilon \left| \int_0^t G(s, X^\epsilon(s)) dW^\epsilon(s) \right|_{-p},$$

where K_1 is a constant. By Condition S and the Gronwall inequality, there exists a constant K_2 such that

$$\sup_{0 \leq t \leq T} |X^\epsilon(t) - z^\psi(t)|_{-p} \leq K_2 \sup_{0 \leq t \leq T} \left| \epsilon \int_0^t G(s, X^\epsilon(s)) dW^\epsilon(s) \right|_{-p}.$$

Let $\tilde{l} \in (0, l)$, and

$$(3.9) \qquad \mathcal{G}(\epsilon, t) = \int_0^t G(s, X^\epsilon(s)) dW^\epsilon(s)$$

$$\mathcal{A}(\epsilon) = \left\{ \sup_{0 \leq t \leq T} |X^\epsilon(t, x) - z^\psi(t, x)|_{-p} \leq \delta \right\}$$

$$\mathcal{B}(\epsilon) = \left\{ \left| \epsilon \int_0^T \psi(t) dW^\epsilon(t) \right| \leq \frac{\tilde{l}}{2} \right\}.$$

Then

$$\mathbf{P}\{\mathcal{A}(\epsilon)\} = E^\epsilon \left[\frac{dP}{dP^\epsilon} : \mathcal{A}(\epsilon) \right] \geq \exp \left\{ -\frac{|\psi|^2_{\mathcal{H}} + \tilde{l}}{2\epsilon^2} \right\} P^\epsilon \{\mathcal{A}(\epsilon) \cap \mathcal{B}(\epsilon)\}.$$

We also want to show that $P^\epsilon \{\mathcal{A}(\epsilon) \cap \mathcal{B}(\epsilon)\} \to 1$ uniformly with respect to x and ψ on bounded sets. Now,

$$P^\epsilon \{\mathcal{A}(\epsilon) \cap \mathcal{B}(\epsilon)\} \geq 1 - P^\epsilon \left\{ \sup_{0 \leq t \leq T} |\mathcal{G}(\epsilon, t)|_{-p} > \frac{\delta}{K_2 \epsilon} \right\} - P^\epsilon \{\mathcal{B}^c(\epsilon)\}.$$

Note that by Condition S(5), there exists a constant l_0 such that

$$(3.10) \qquad \int_0^T \|G(s, X^\epsilon(s))\|_L^2\, ds \leq l_0.$$

Now Theorem 3.1 shows that

$$P^\epsilon \left\{ \sup_{0 \leq t \leq T} |\mathcal{G}(\epsilon, t)|_{-p} > \frac{\delta}{K_2 \epsilon} \right\} \leq 3 \exp \left\{ -\frac{\delta^2}{4K_2^2 \epsilon^2 l_0} \right\} \to 0$$

and $P^\epsilon\{\mathcal{B}^c(\epsilon)\} \to 0$ as $\epsilon \to 0$. Hence, we get the lower bound. $\square$

Next, we are going to consider the upper bound. We adapt the following lemma and proposition for the strong solution to (1.2) from the originals for a mild solution in a Hilbert space [11]. The proof for the Theorem 3.5(Upper Bound) is intrinsically same with that of Theorem 1.3 in [11].

LEMMA 3.3. $\forall r >, \exists M > 0$ such that $\forall \epsilon \in (0, 1]$ one has

$$(3.11) \qquad \mathbf{P}\left\{\sup_{0 \le t \le T} |X^\epsilon(t)|_{-p} \ge M\right\} \le \exp\left\{-\frac{r}{\epsilon^2}\right\}$$

PROOF.

$$|X^\epsilon(t)|^2_{-p}$$

$$\le k_1 + k_2 \left(\int_0^t |B(s, X^\epsilon(s))|^2_{-p} ds\right) + \epsilon \left|\int_0^t G(s, X^\epsilon(s)) dW(s)\right|^2_{-p}$$

$$\le k_1 + k_2 \int_0^t K(t)(1 + |X^\epsilon(s)|^2_{-p}) ds + \epsilon \left|\int_0^t G(s, X^\epsilon(s)) dW(s)\right|^2_{-p}$$

$$\le k_1 + k_3 \left(\int_0^t K(t)|X^\epsilon(s)|^2_{-p} ds\right) + \epsilon \left|\int_0^t G(s, X^\epsilon(s)) dW(s)\right|^2_{-p},$$

where k_i, $i = 1, 2, 3...$ are some constants from (S2) and (S3). By Gronwall's inequality there exist k_4 and k_5 such that

$$\sup_{0 \le t \le T} |X^\epsilon(t)|^2_{-p} \le \left(k_1 + \sup_{0 \le t \le T} \epsilon \left|\int_0^t G(s, X^\epsilon(s)) dW(s)\right|^2_{-p}\right) k_4$$

$$\le k_5 + k_4 \sup_{0 \le t \le T} \epsilon \left|\int_0^t G(s, X^\epsilon(s)) dW(s)\right|^2_{-p}.$$

Let $M > 0$ be such that $M^2 > k_5$.

$$\mathbf{P}\left\{\sup_{0 \le t \le T} |X^\epsilon(t)|_{-p} \ge M\right\}$$

$$\le \mathbf{P}\left\{\sup_{0 \le t \le T} \epsilon \left|\int_0^t G(s, X^\epsilon(s)) dW(s)\right|^2_{-p} \ge \left(\frac{M^2 - k_5}{k_1}\right)^{\frac{1}{2}}\right\}$$

$$= \exp\left\{-\frac{1}{\epsilon^2}\left(\frac{M^2 - k_5}{4k_1 l_0}\right)\right\},$$

by (3.9) and Theorem 3.1. Hence, for any given r, we can choose M satisfying $r = \frac{M^2 - k_5}{4k_1 l_0}$. $\qquad \square$

PROPOSITION 3.4. *Under the condition* (S), $\forall a > 0$, $\forall \delta > 0$, $\forall \psi \in \mathcal{H}_T$, *there exists* $\epsilon_0 > 0$ *and* $b_0 > 0$ *such that* $\forall \epsilon \in (0, \epsilon_0]$ *we have*
(3.12)
$$
\mathbf{P} \left\{ \sup_{0 \leq t \leq T} \left| X^\epsilon(t) - z^\psi(t, x) \right|_{-p} \geq \delta, \ \sup_{0 \leq t \leq T} \left| \epsilon W(t) - \int_0^t \psi(s) ds \right|_{-p} \leq b \right\}
$$
$$
\leq \exp \left\{ -\frac{a}{\epsilon^2} \right\}.
$$

PROOF. Let
$$
\mathcal{N}(\epsilon, b)
$$
$$
\equiv \left\{ \sup_{0 \leq t \leq T} \left| X^\epsilon(t) - z^\psi(t, x) \right|_{-p} \geq \delta, \ \sup_{0 \leq t \leq T} \left| \epsilon W(t) - \int_0^t \psi(s) ds \right|_{-p} \leq b \right\}
$$

and $\psi \in \mathcal{H}_T$. W^ϵ and P^ϵ be defined as in the proof of Theorem 3.2. For $\lambda > 0$ set
$$
\mathcal{M}(\epsilon, \lambda) = \left\{ \int_0^T \psi(t) dW(t) \geq -\frac{\lambda}{\epsilon} \right\}.
$$

Obviously, for an arbitrary $\lambda > 0$, we have

(3.13) $\qquad \mathbf{P}\{\mathcal{N}(\epsilon, b)\} \leq \mathbf{P}\{\mathcal{N}(\epsilon, b) \cap \mathcal{M}(\epsilon, \lambda)\} + \mathbf{P}\{\mathcal{M}^c(\epsilon, \lambda)\}.$

By Theorem 3.1, we have

(3.14) $\qquad \mathbf{P}\{\mathcal{M}^c(\epsilon, \lambda)\} \leq 3 \exp \left\{ -\frac{\lambda^2}{4\epsilon^2 |\psi|_{\mathcal{H}_T}^2} \right\}.$

Note that
(3.15)
$$
\mathbf{P}\{\mathcal{N}(\epsilon, b) \cap \mathcal{M}(\epsilon, \lambda)\} = E^\epsilon \left[\frac{dP}{dP^\epsilon} : \mathcal{N}(\epsilon, b) \cap \mathcal{M}(\epsilon, \lambda) \right]
$$
$$
\leq \exp \left\{ \frac{\lambda}{\epsilon^2} + \frac{|\psi|_{\mathcal{H}_T}^2}{2\epsilon^2} \right\} \mathbf{P}^\epsilon\{\mathcal{N}(\epsilon, b)\}.
$$

Since under P^ϵ, X^ϵ is the unique solution of the SDE

$$dX^\epsilon(t) = \tilde{B}(t, X^\epsilon(t))dt + \epsilon G(X^\epsilon(t))dW^\epsilon(t), \quad X^\epsilon(0) = x,$$

where $\tilde{B}(t, x) = B(t, x) + G(x)\psi(t)$, we have

$$\mathbf{P}^\epsilon\{\mathcal{N}(\epsilon, b)\} = \mathbf{P}\left\{\sup_{0 \le t \le T}\left|\tilde{X}^\epsilon(t) - z^\psi(t, x)\right| > \delta, \ |\epsilon W(t)| \le b\right\},$$

where $\tilde{X}(t, x)$ is the solution of the SDE

$$d\tilde{X}^\epsilon(t) = \tilde{B}(t, \tilde{X}^\epsilon(t))dt + \epsilon G(\tilde{X}^\epsilon(t))dW^\epsilon(t), \quad \tilde{X}^\epsilon(0) = x.$$

Using the same argument as in the proof of Theorem 3.2 we get

$$\mathbf{P}^\epsilon\{\mathcal{N}(\epsilon, b)\}$$

$$\le \mathbf{P}\left\{\sup_{0 \le t \le T}\left|\epsilon\int_0^t G(\tilde{X}^\epsilon(s, x))dW(s)\right|_{-p} \ge \frac{\delta}{C_1}, \ \sup_{0 \le t \le T}|\epsilon W(t)|_{-p}\right\}$$

$$\le \mathbf{P}\left\{\sup_{0 \le t \le T}\left|\int_0^t G(\tilde{X}^\epsilon(s, x))dW(s)\right|_{-p} \ge \frac{\delta}{\epsilon C_1}\right\}$$

$$\le 3\exp\left\{-\frac{\delta^2}{4\epsilon^2 C_1^2 l_0}\right\},$$

where C_1 is a constant and l_0 is a constant defined as (3.9). Hence for each $a_0 > a$ we can find $b > 0$ such that

$$(3.17) \qquad\qquad \mathbf{P}^\epsilon\{\mathcal{N}(\epsilon, b)\} \le \exp\left\{-\frac{a}{\epsilon^2}\right\}.$$

Combining (3.13) to (3.17), we have

$$\mathbf{P}\{\mathcal{N}(\epsilon, b)\} \le 3\exp\left\{-\frac{\lambda^2}{4\epsilon^2|\psi|_{\mathcal{H}_T}^2}\right\} + \exp\left\{\frac{2\lambda + |\psi|_{\mathcal{H}_T}^2 - 2\tilde{a}}{2\epsilon^2}\right\}.$$

If we first choose λ and then $\tilde{a}$ we get (3.12). $\qquad\qquad\qquad \square$

THEOREM 3.5. (Upper bound) *Under the condition* (S), $\forall T > 0$, $\forall y \in \mathcal{S}_{-p}$, $\forall \delta > 0$, $\forall R > 0$, $\forall r \ge 0$, $\forall \gamma > 0$, *there exists* $\epsilon_0 > 0$ *such that* $\forall x \in \mathcal{S}_{-p} : |x - y|_{-p} \le R$ *and* $\forall \epsilon \in (0, \epsilon_0]$ *we have*

$$\mathbf{P}\left\{dist_{C([0,T];\mathcal{S}_{-p})}\left(X^\epsilon(\cdot, x), \mathcal{K}(r)\right) \ge \delta\right\} \le \exp\left\{-\frac{r - \gamma}{2\epsilon^2}\right\}.$$

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