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CHARACTERIZATION OF PHANTOM GROUPS

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Abstract. We give another characteristic feature of the set of phantom maps: After constructing an isomorphism between derived functors, we show that the set of homotopy classes of phantom maps could be restated as the extension product of subinverse towers induced by the given inverse towers.

1. Introduction and result

Let $\Gamma^n, n \geq 0$ be the set of all increasing sequences $\bar{\gamma} = (\gamma_0, \gamma_1, \cdots, \gamma_n)$, $\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_n, \gamma_i \in \Gamma$ and let $\bar{\gamma}_j \in \Gamma^{n-1}$, $0 \leq j \leq n$, be obtained from $\bar{\gamma} \in \Gamma^n$ by deleting the jth factor γ_j, i.e., $\bar{\gamma}_j = (\gamma_0, \cdots, \gamma_{j-1}, \gamma_{j+1}, \cdots, \gamma_n)$. And for each $\bar{\gamma} \in \Gamma^n$, we associate an abelian group $A_{\bar{\gamma}}$ by the abelian group A_{γ_0} of the first index γ_0 in the category of abelian groups, i.e., $A_{\bar{\gamma}} = A_{\gamma_0}$.

Let $\mathfrak{A} = (A_\gamma, a_{\gamma\gamma'}, \Gamma)$ be an inverse system of abelian groups A_γ and group homomorphisms $a_{\gamma\gamma'} : A_{\gamma'} \to A_\gamma$, $\gamma \leq \gamma'$ over the directed set Γ. We define an n-cochain group $C^n(\mathfrak{A}), n \geq 0$ of $\mathfrak{A}$ by

$$C^n(\mathfrak{A}) = \prod_{\bar{\gamma} \in \Gamma^n} A_{\bar{\gamma}}, \ n \geq 0,$$

where $A_{\bar{\gamma}} = A_{\gamma_0}$ as just mentioned above.

Let $pr_{\bar{\gamma}} : C^n(\mathfrak{A}) \to A_{\bar{\gamma}}$ be a projection. If y is an element of $C^n(\mathfrak{A})$, then we denote the element $y_{\bar{\gamma}}$ of $A_{\bar{\gamma}}$ by $y_{\bar{\gamma}} = pr_{\bar{\gamma}}(y)$. The coboundary operator $\delta^n : C^{n-1}(\mathfrak{A}) \to C^n(\mathfrak{A}), n \geq 1$ is defined by

$$(\delta^n y)_{\bar{\gamma}} = a_{\gamma_0\gamma_1}(y_{\bar{\gamma}_0}) + \sum_{j=1}^{n} (-1)^j y_{\bar{\gamma}_j},$$

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where $y \in C^{n-1}(\mathfrak{A})$. For $n = 0$, if we put $\delta^0 = 0 : 0 \to C^0(\mathfrak{A})$, then we have a cochain complex

$$(C^*(\mathfrak{A}), \delta) : \quad 0 \to C^0(\mathfrak{A}) \xrightarrow{\delta^1} C^1(\mathfrak{A}) \to \cdots \to C^{n-1}(\mathfrak{A}) \xrightarrow{\delta^n} C^n(\mathfrak{A}) \to \cdots .$$

We now define the nth *derived limit* (see [6]) denoted by $H^n(\mathfrak{A})$ of the inverse system $\mathfrak{A} = (A_\gamma, a_{\gamma\gamma'}, \Gamma)$ of abelian groups by the cohomology group of the above cochain complex $(C^*(\mathfrak{A}), \delta)$. That is to say $H^n(\mathfrak{A}) = \ker(\delta^{n+1})/\mathrm{im}(\delta^n)$.

What is even more interesting is to see that the derived limit is not contravariant but a covariant functor although it was defined as the cohomology of the cochain complex $(C^*(\mathfrak{A}), \delta)$. The author [2] showed that a ring could be constructed under the derived cup products arising from a certain inverse system.

Given a connected CW-complex X, a pointed map $f : X \to Y$ is called a *phantom map* if the restriction map $f|_{X_n} : X_n \to Y$ on the skeleton X_n is homotopic to the constant map for each n. Let $\mathrm{Ph}(X, Y)$ denote the set of pointed homotopy classes of phantom maps from X to Y.

Let $\lim^n(-), n = 0, 1$ be the derived functor in the sense of Bousfield-Kan [1, p.251]: Considering an inverse tower $\mathfrak{G} = (G_n, g_n^{n+1}, \mathbb{N})$ of (possibly non-abelian) groups G_n and homomorphisms $g_n^{n+1} : G_{n+1} \to G_n$, we can define a left action of $\prod G_n$ on $\prod G_n$ by the formula

$$(\cdots, s_n, s_{n+1}, \cdots) \circ (\cdots, t_n, t_{n+1}, \cdots) = (\cdots, s_n t_n g_n^{n+1}(s_{n+1}^{-1}), \cdots).$$

We now define $\lim^1 \mathfrak{G}$ as the set of orbits of $\prod G_n$ under this action. We can also define the inverse limit $\lim \mathfrak{G}$ of the inverse tower $\mathfrak{G}$ by using this action. Moreover, the set $\lim^1 \mathfrak{G} = \prod G_n / \simeq$ can be viewed as the quotient set of the direct product $\prod G_n$ by an equivalence relation $\simeq$ defined as the following: For $x = (\cdots, x_n, \cdots), y = (\cdots, y_n, \cdots) \in \prod G_n$, one has $x \simeq y$ if and only if there exists an element $s = (\cdots, s_n, \cdots) \in \prod G_n$ such that $y = s \circ x$.

The following [4] has been playing a vital role in the computation of the set of homotopy classes of phantom maps among the various kinds of results in phantom map theory (also see [1]).

PROPOSITION 1. *If X and Y are pointed CW-spaces, then there are set-theoretic bijections of pointed sets*

$$\lim{}^1[X, \Omega Y^{(n)}] \approx Ph(X,Y) \approx \lim{}^1[\Sigma X_n, Y],$$

where $Y^{(n)}$ is the nth Postnikov approximation of Y through dimension n.

If X is a co-H-group or if Y is a homotopy associative H-space, then the above set-theoretic bijections are group isomorphisms and $\mathrm{Ph}(X,Y)$ is a normal subgroup of $[X,Y]$.

We can find the following characteristic phenomenon [3, Theorem 1] regarding the description of the set of phantom maps proved by McGibbon-Roitberg [5, Theorem 1] in 1998 and by Roitberg-Touhey [7, Theorems 1.1 and 1.5] in 2000.

THEOREM 2. *Let X and Y be nilpotent CW-complexes of finite type. If X has a rational homotopy type of a suspension or Y has a rational homotopy type of a loop space, then there exists a bijection as sets*

$$Ph(X,Y) \approx Ext(\Delta(n), [X, \Omega Y^{(n)}]/F_n).$$

Here $\Delta(n)$ is an inverse tower and F_n is a finite commutator subgroup of $[X, \Omega Y^{(n)}]$.

Let $([X, \Omega Y^{(n)}], s_n^{n+1}, \mathbb{N})$ be an inverse tower of groups $[X, \Omega Y^{(n)}]$ and group homomorphism $s_n^{n+1} : [X, \Omega Y^{(n+1)}] \to [X, \Omega Y^{(n)}]$ over $\mathbb{N}$ (the set of positive integers) induced by the Postnikov approximations. And let $[X, \Omega Y^{(n)}]^{[n+1]}$ be the image of the map $s_n^{n+1} : [X, \Omega Y^{(n+1)}] \to [X, \Omega Y^{(n)}]$. Then we are able to induce a subinverse tower

$$([X, \Omega Y^{(n)}]^{[n+1]}, \bar{s}_n^{n+1}, \mathbb{N}) :$$
$$\cdots \to [X, \Omega Y^{(n)}]^{[n+1]} \xrightarrow{\bar{s}_n^{n+1}} [X, \Omega Y^{(n-1)}]^{[n]} \to \cdots$$

of the given inverse tower $([X, \Omega Y^{(n)}], s_n^{n+1}, \mathbb{N})$.

In this paper, we deduce another expression (to which the title of this paper refers) of the set of phantom maps which is more convenient device just a little bit to handle than the previous one in that we can use the subinverse tower instead of the given inverse tower.

THEOREM 3. *Let X and Y be connected CW-complexes. If X has a homotopy type of a suspension or Y has a homotopy type of a loop space, then there exists an isomorphism*

$$Ph(X,Y) \cong Ext(\Delta(n), [X, \Omega Y^{(n)}]^{[n+1]})$$

as abelian groups.

2. Proof of Theorem 3

Note that if X and Y are connected nilpotent CW-complexes of finite type, then the group $[X, \Omega Y^{(n)}]$ is a finitely generated nilpotent group. Moreover, under the hypothesis of Theorem 3, the group $[X, \Omega Y^{(n)}]$ is now abelian. In order to prove this theorem, we need the following two lemmas.

LEMMA 3.1. *For $r = 0, 1$ there exists an isomorphism*

$$\lim^r [X, \Omega Y^{(n)}] \cong \lim^r [X, \Omega Y^{(n)}]^{[n+1]}$$

as abelian groups.

PROOF. We define a map

$$u : \prod_{n=1}^{\infty} [X, \Omega Y^{(n)}] \to \prod_{n=1}^{\infty} [X, \Omega Y^{(n)}]$$

by

$$(ux)_n = s_n^{n+1}(x_{n+1}) - x_n,$$

where $x \in \prod_{n=1}^{\infty}[X, \Omega Y^{(n)}]$ and $x_n \in [X, \Omega Y^{(n)}]$, $n \geq 0$. Considering the following commutative diagram

$$\begin{array}{ccc}
0 \to U^0 = \prod_{n=1}^{\infty}[X, \Omega Y^{(n)}] & \xrightarrow{u} & U^1 = \prod_{n=1}^{\infty}[X, \Omega Y^{(n)}] \to 0 \\
\downarrow \pi & & \downarrow \pi \\
0 \to \bar{U}^0 = \prod_{n=1}^{\infty}[X, \Omega Y^{(n)}]^{[n+1]} & \xrightarrow{\bar{u}} & \bar{U}^1 = \prod_{n=1}^{\infty}[X, \Omega Y^{(n)}]^{[n+1]} \to 0
\end{array}$$

of cochain complexes, where $\pi = \prod_{n=1}^{\infty} s_n^{n+1}$, we now obtain the following:

(1) $\ker(u) \approx \lim([X, \Omega Y^{(n)}], s_n^{n+1}, \mathbb{N})$
(2) $\ker(\bar{u}) \approx \lim([X, \Omega Y^{(n)}]^{[n+1]}, \bar{s}_n^{n+1}, \mathbb{N})$
(3) $\operatorname{coker}(u) \approx \lim^1([X, \Omega Y^{(n)}], s_n^{n+1}, \mathbb{N})$
(4) $\operatorname{coker}(\bar{u}) \approx \lim^1([X, \Omega Y^{(n)}]^{[n+1]}, \bar{s}_n^{n+1}, \mathbb{N})$.

Let $1 : U^1 \to U^0$ and $\bar{1} : \bar{U}^1 \to \bar{U}^0$ be the identity maps on $\prod_{n=1}^{\infty}[X, \Omega Y^{(n)}]$ and $\prod_{n=1}^{\infty}[X, \Omega Y^{(n)}]^{[n+1]}$ respectively. Then for any $x \in U^1$ and $y \in \bar{U}^1$, we have

$$\begin{aligned} u \circ 1(x) &= u(x) \\ &= (\cdots, s_n^{n+1}(x_{n+1}) - x_n, \cdots) \\ &= (\cdots, s_n^{n+1}(x_{n+1}), \cdots) - (\cdots, x_n, \cdots) \\ &= i \circ \pi(x) - 1(x) \end{aligned}$$

and

$$\begin{aligned} \bar{u} \circ \bar{1}(y) &= \bar{u}(y) \\ &= (\cdots, \bar{s}_n^{n+1}(y_{n+1}) - y_n, \cdots) \\ &= (\cdots, \bar{s}_n^{n+1}(y_{n+1}), \cdots) - (\cdots, y_n, \cdots) \\ &= \pi \circ i(y) - 1(y), \end{aligned}$$

where $i : \prod_{n=1}^{\infty}[X, \Omega Y^{(n)}]^{[n+1]} \to \prod_{n=1}^{\infty}[X, \Omega Y^{(n)}]$ is the inclusion. Thus we have cochain homotopies $1 : i \circ \pi \simeq 1$ and $\bar{1} : \pi \circ i \simeq 1$, i.e., π is a cochain homotopy equivalence. □

LEMMA 3.2. *Let X and Y be as in Theorem 3. Then we have the following:*
(a) $\lim^1[X, \Omega Y^{(n)}]^{[n+1]} \cong H^1([X, \Omega Y^{(n)}]^{[n+1]})$
(b) $H^1([X, \Omega Y^{(n)}]^{[n+1]}) \cong Ext(\Delta(n), [X, \Omega Y^{(n)}]^{[n+1]})$.

PROOF. In part (a), if $[p] = [q]$ in $H^1([X, \Omega Y^{(n)}]^{[n+1]})$, then for each $(n, n+1) \in \mathbb{N}^1$, there exists $y \in C^0([X, \Omega Y^{(n)}]^{[n+1]})$ such that

$$\begin{aligned} p_n &= q_n + (\delta^1 y)_{(n,n+1)} \\ &= q_n + a_{n,n+1}(y_{n+1}) - y_n. \end{aligned}$$

Since $[X, \Omega Y^{(n)}]^{[n+1]}$ is abelian, we have

$$\begin{aligned} q_n &= p_n + y_n - a_{n,n+1}(y_{n+1}) \\ &= y_n + p_n - a_{n,n+1}(y_{n+1}). \end{aligned}$$

If we take the multiplicative notation in the above equalities, then we have $p \sim q$ in $\lim^1[X, \Omega Y^{(n)}]^{[n+1]}$ as required.

In part (b), the proof goes to the same way just like the proof of Lemma 3.1 in [3] by applying the groups $G_n = [X, \Omega Y^{(n)}]$ to the subgroups $[X, \Omega Y^{(n)}]^{[n+1]}$. □

Finally we have

$$\begin{aligned}\mathrm{Ph}(X,Y) &\cong \lim{}^1[X,\Omega Y^{(n)}] \text{ (by Proposition 1)}\\ &\cong \lim{}^1[X,\Omega Y^{(n)}]^{[n+1]} \text{ (by Lemma 3.1)}\\ &\cong H^1[X,\Omega Y^{(n)}]^{[n+1]} \text{ (by Lemma 3.2 (a))}\\ &\cong \mathrm{Ext}(\Delta(n),[X,\Omega Y^{(n)}]^{[n+1]}) \text{ (by Lemma 3.2 (b))}\end{aligned}$$

which completes the proof. □

REMARK. In Theorem 3, note that the hypothesis of the homotopy type of a suspension on X (or a loop space on Y) is really necessary because it is not guaranteed to construct abelian groups $[X, \Omega Y^{(n)}]$ if either X has a rational homotopy type of a suspension or Y has a rational homotopy type of a loop space. In this case, if X and Y are nilpotent CW-complexes of finite type, then we just obtain an abelian group only when it is rationalized!

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