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ON ASYMPTOTIC PROPERTY FOR NONLINEAR DIFFERENCE SYSTEMS

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Abstract. We study asymptotic equivalence between nonlinear difference system

$$\Delta x(n) = f(n, x(n))$$

and its variational system

$$\Delta v(n) = f_x(n, 0)v(n).$$

1. Introduction

We consider the nonlinear difference system

$$\Delta x(n) = f(n, x(n)),\ x(n_0) = x_0, \tag{1.1}$$

where Δ is the forward difference operator, i.e., $\Delta x(n) = x(n+1) - x(n)$, $f : \mathbb{N}(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$, $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \cdots, n_0 + k, \cdots\}$, n_0 is a nonnegative integer, and $\mathbb{R}^m$ is the $m-$dimensional real Euclidean space.

System (1.1) has *asymptotic equilibrium* if there exists a single $\xi \in \mathbb{R}^m$ such that any solution $x(n) = x(n, n_0, x_0)$ of (1.1) for all $|x_0|$ sufficiently small satisfies

$$x(n) = \xi + o(1) \text{ as } n \to \infty, \tag{1.2}$$

and, for every $\xi \in \mathbb{R}^m$, there exists a solution of (1.1) such that satisfies (1.2).

Also, for the other nonlinear difference system

$$\Delta y(n) = g(n, y(n)), \tag{1.3}$$

where $g : \mathbb{N}(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$, two systems (1.1) and (1.3) are said to be *asymptotically equivalent* if, for every solution $x(n)$ of (1.1), there exists

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a solution $y(n)$ of (1.3) such that

$$x(n) = y(n) + o(1) \text{ as } n \to \infty, \tag{1.4}$$

and conversely, for every solution $y(n)$ of (1.3), there exists a solution $x(n)$ of (1.1) such that (1.4) holds.

To study asymptotic relationship between two linear systems, the problem of asymptotic equivalence have been initiated by H. Poincaré (1885) and O. Perron (1921). In [8] Medina and Pinto studied asymptotic equivalence using dichotomy conditions. Also, Medina [7] investigated this problem using some discrete inequalities combined with the Schauder's fixed point theorem.

We established asymptotic equivalence between two nonlinear difference systems via their variational systems in [2]. Also, in [4], we characterized asymptotic equilibrium for nonlinear system by means of n_∞-similarity, which is defined in [3] as a different discrete analogue of t_∞-similarity by Conti.

In this paper we study asymptotic equivalence between (1.1) and its variational system

$$\Delta v(n) = f_x(n,0)v(n), \tag{1.5}$$

where $f_x = \frac{\partial f}{\partial x}$, and give an appropriate example.

2. Main results

We are concerned with the nonlinear difference system

$$\Delta x(n) = f(n, x(n)),\ x(n_0) = x_0. \tag{2.1}$$

We assume that

(i) $f_x = \frac{\partial f}{\partial x}$ exists,

(ii) $I + f_x$ is continuous and invertible on $\mathbb{N}(n_0) \times \mathbb{R}^m$, where I is the $m \times m$ identity real matrix,

(iii) $f(n, 0) = 0$.

Let $x(n) = x(n, n_0, x_0)$ be the solution of (2.1) with $x(n_0, n_0, x_0) = x_0$.

To study asymptotic behavior for (2.1) it is useful to consider its variational system

$$\Delta v(n) = f_x(n,0)v(n) = A(n)v(n) \tag{2.2}$$

and rewrite (2.1), by setting $f_x(n,0) = A(n)$ and using the mean value theorem, as

$$\begin{aligned} \Delta x(n) &= A(n)x(n) + f(n,x(n)) - f_x(n,0)x(n) \\ &= A(n)x(n) + G(n,x(n)), \quad x(n_0) = x_0, \end{aligned} \tag{2.3}$$

where

$$G(n,x) = \int_0^1 [f_x(n,\theta x) - f_x(n,0)]d\theta x.$$

Then, by the variational of constants formula [1], the solution of (2.3) is given by

$$x(n) = \Phi(n,n_0)x_0 + \sum_{s=n_0}^{n-1} \Phi(n,s+1)G(s,x(s)), \tag{2.4}$$

where $\Phi(n,n_0)$ is the fundamental matrix solution of the linear difference system $\Delta x(n) = A(n)x(n)$.

For system (2.3), we assume that

$$|G(n,x)| \le \omega(n,|x|),\ n \ge n_0,\ x \in \mathbb{R}^m, \tag{2.5}$$

where $\omega : \mathbb{N}(n_0) \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing in $u \in \mathbb{R}_+$ for $n \ge n_0$.

To use the comparison principle we consider the scalar difference equation

$$\Delta u(n) = M^2\omega(n,u(n)), \quad u(n_0) = u_0 > 0, \tag{2.6}$$

where M is a positive constant, by the function $\omega(n,u)$.

We need the following comparison principle which is a slight modification of [6] :

LEMMA 2.1. [3] *Let $p(n,r)$ be a nondecreasing function in r for any fixed $n \in \mathbb{N}(n_0)$. Suppose that for $n \ge n_0$, nonnegative functions $u(n)$ and $v(n)$ defined on $\mathbb{N}(n_0)$ satisfies the following inequalities :*

$$v(n) - \sum_{l=n_0}^{n-1} p(l,v(l)) < u(n) - \sum_{l=n_0}^{n-1} p(l,u(l)).$$

If $v(n_0) < u(n_0)$, then $v(n) < u(n)$ for all $n \ge n_0$.

Now, we present the result.

THEOREM 2.2. *Assume that (2.5) holds and all solutions of (2.6) are bounded on $\mathbb{N}(n_0)$. If the variational system (2.2) has asymptotic equilibrium, then the original system (2.1) has also asymptotic equilibrium provided $d = M|x_0| < u_0$.*

PROOF. Let $x(n, n_0, x_0)$ be any solution of (2.1). From the variation of constants formula in [1] and conditions (i) and (ii), we obtain

$$\begin{aligned} |x(n)| &= |\Phi(n,n_0)[x_0 + \sum_{s=n_0}^{n-1} \Phi^{-1}(s+1,n_0)G(s,x(s))]| \\ &\le |\Phi(n,n_0)||x_0| + |\Phi(n,n_0)| \sum_{s=n_0}^{n-1} |\Phi^{-1}(s+1,n_0)||G(s,x(s))| \\ &\le M|x_0| + M^2 \sum_{s=n_0}^{n-1} \omega(s,|x(s)|) \\ &= d + M^2 \sum_{s=n_0}^{n-1} \omega(s,|x(s)|), \end{aligned}$$

where $\Phi(n, n_0) \le M$ for $n, m \ge n_0$ and $d = M|x_0|$. It follows that

$$\begin{aligned} |x(n)| - M^2 \sum_{s=n_0}^{n-1} \omega(s,|x(s)|) &= d \\ &< u_0 = u(n) - M^2 \sum_{s=n_0}^{n-1} \omega(s,u(s)). \end{aligned}$$

Applying Lemma 2.1 for $p(n, u) = M^2\omega(n, u)$, we conclude that

$$|x(n)| \le u(n) \quad \text{for each } n \ge n_0,$$

provided $d < u_0$.

We claim that the solution $x(n)$ of (2.1) which can be written as (2.3) is convergent. If we consider the sequence

$$v(n) = \sum_{s=n_0}^{n-1} \Phi^{-1}(s+1,n_0)G(s,x(s)),$$

then we have

$$\begin{aligned} |v(n) - v(m)| &\le \sum_{s=m}^{n-1} |\Phi^{-1}(s+1,n_0)||G(s,x(s))| \\ &\le M \sum_{s=m}^{n-1} \omega(s,|x(s)|) \end{aligned}$$

$$\begin{aligned} &\le M\sum_{s=m}^{n-1}\omega(s,u(s)) \\ &= M(u(n)-u(m)),\ n\ge m\ge n_0, \end{aligned}$$

by the monotonicity of $\omega(n,u)$. It follows that $v(n)$ is Cauchy since $u(n)$ is convergent. Thus there exists a vector $\xi\in\mathbb{R}^m$ such that any solution $x(n,n_0,x_0)$ of (2.3) for all $|x_0|$ sufficiently small satisfies the following asymptotic relationship :

$$x(n)=\xi+o(1)\quad \text{as } n\to\infty.$$

For the converse asymptotic relationship, we let $\xi\in\mathbb{R}^m$ be any vector and set

$$x_0=\Phi_\infty^{-1}\xi-v_\infty,$$

where $\Phi_\infty=\lim_{n\to\infty}\Phi(n)$ and $v_\infty=\lim_{n\to\infty}v(n)$. Then any solution $x(n)$ of (2.3) which is equivalent to (2.1) satisfies

$$\begin{aligned} x(n,n_0,x_0)) &= \Phi(n,n_0)[x_0+\sum_{s=n_0}^{\infty}\Phi^{-1}(s+1,n_0)G(s,x(s))] \\ &- \Phi(n,n_0)\sum_{s=n}^{\infty}\Phi^{-1}(s+1,n_0)G(s,x(s)) \\ &= \xi+o(1)\quad \text{as } n\to\infty, \end{aligned}$$

since

$$\Phi(n,n_0)\sum_{s=n}^{\infty}\Phi^{-1}(s+1,n_0)G(s,x(s))\to 0\quad \text{as } n\to\infty.$$

This completes the proof. □

As an illustration of Theorem 2.2 we give the following example.

EXAMPLE 2.3. We consider the nonlinear difference equation

$$\Delta x(n)=f(n,x(n))=\frac{a^n x(n)}{\sqrt{1+2x^2(n)}} \tag{2.7}$$

and its associated variational difference equation

$$\Delta v(n)=f_x(n,0)v(n)=a^n v(n), \tag{2.8}$$

where a is a constant with $0<|a|<1$. Thus

$$f(n,x)=\frac{a^n x}{\sqrt{1+2x^2}}\ \text{ and } f_x(n,x)=\frac{a^n}{(1+2x^2)^{\frac{3}{2}}}.$$

First, we can rewrite (2.7) as (2.3) :

$$\Delta x(n) = a^n x(n) + a^n[\frac{1}{\sqrt{1+2x^2(n)}} - 1]x(n). \tag{2.9}$$

Since

$$G(n,x) = \int_0^1 [f_x(n,\theta x) - f_x(n,0)]d\theta x,$$

we have

$$\begin{aligned} |G(n,x)| &\le |a^n[\int_0^1 \frac{d\theta}{\sqrt{1+2(\theta x)^2}} - 1]d\theta x| \\ &= |a^n[\frac{1}{\sqrt{1+2x^2}} - 1]x| \\ &\le a^n|x|. \end{aligned}$$

So $\omega(n,|x|) = a^n u$ and it is continuous nondecreasing in $u > 0$.

Now, (2.6) becomes

$$\Delta u(n) = M^2 a^n u(n),\ u(0) = u_0 > 0. \tag{2.10}$$

Since any solution $u(n)$ of (2.10) is given by

$$\begin{aligned} u(n) &= u_0 + M^2 \sum_{s=0}^{n-1} a^s \\ &= u_0 + M^2 \left(\frac{1-a^n}{1-a}\right),\ n \ge n_0 = 0, \end{aligned}$$

$u(n)$ is bounded on $\mathbb{N}(n_0)$. Thus, by Theorem 2.2, (2.7) has asymptotic equilibrium by setting

$$d = \lim_{n\to\infty} \prod_{s=0}^{n-1} (1+a^s)|x_0|.$$

THEOREM 2.4. *Two systems* (2.1) *and* (2.2) *are asymptotically equivalent under the conditions in* Theorem 2.2.

PROOF. Let $v(n)$ be any solution of (2.2). Then we have $\lim_{n\to\infty} v(n) = v_\infty$ by asymptotic equilibrium of (2.2). Setting

$$x_0 = \Phi_\infty^{-1} v_\infty - p_\infty,$$

where $\lim_{n\to\infty}\Phi(n) = \Phi_\infty$ and $p_\infty = \sum_{s=n_0}^{\infty}\Phi^{-1}(s+1,n_0)G(s,x(s))$, then there exists a solution $x(n,n_0,x_0)$ of (2.1) such that

$$\begin{aligned}\lim_{n\to\infty}[x(n)-v(n)] &= \lim_{n\to\infty}[\Phi(n)(\Phi_\infty^{-1}v_\infty - p_\infty \\ &\quad + \sum_{n\to\infty}^{n-1}\Phi^{-1}(s+1,n_0)G(s,x(s))) - \Phi(n,n_0)v_0] \\ &= \Phi_\infty(\Phi_\infty^{-1}v_\infty) - v_\infty \\ &= 0.\end{aligned}$$

Conversely, we easily see that the asymptotic relationship also holds by setting $v_0 = x_0 + p_\infty$. This completes the proof. □

EXAMPLE 2.5. Two difference equations (2.7) and (2.8) in Example 2.3 are asymptotically equivalent by Theorem 2.4.

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