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INEQUALITIES FOR THE INTEGRAL MEANS OF HOLOMORPHIC FUNCTIONS IN THE STRONGLY PSEUDOCONVEX DOMAIN

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ABSTRACT. We obtain the following two inequalities on a strongly pseudoconvex domain Ω in $\mathbb{C}^n$: for $f \in \mathcal{O}(\Omega)$

$$\int_0^{\delta_0} t^{a|\alpha|+b} M_p^a(t, D^\alpha f)\, dt \lesssim \int_0^{\delta_0} t^b M_p^a(t, f)\, dt$$

$$\int_0^{\delta_0} t^b M_p^a(t, f)\, dt \lesssim \sum_{j=0}^{m} \int_0^{\delta_0} t^{am+b} M_p^a\left(t, \mathcal{N}^j f\right) dt.$$

In [9], Shi proved these results for the unit ball in $\mathbb{C}^n$. These are generalizations of some classical results of Hardy and Littlewood.

1. Introduction and statement of results

Let Ω be a bounded, strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and ρ be a defining function for Ω. We let dV denote the Lebesgue measure on Ω and $d\sigma$ denote the surface measure on the boundary $\partial\Omega$ of Ω. By $\mathcal{O}(\Omega)$ we denote the class of all functions holomorphic in Ω. For $s > -1$, let L_s^p be the L^p-space $L^p(\Omega, dV_s)$, where $dV_s = (-\rho)^s dV$. We denote by A_s^p the space of all holomorphic functions in L_s^p, and we define A_{-1}^p to be the usual Hardy class consisting of holomorphic functions with boundary values in L_{-1}^p. The integral means $M_p(r, f)$ of f, $0 < p < \infty$, are defined by

$$M_p(r, f) = \left\{ \int_{\partial\Omega} |f(r\zeta)|^p \, d\sigma(\zeta) \right\}^{1/p}.$$

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Let N be a real vector field in a neighborhood of $\partial\Omega$ which agrees with the outward unit normal vector field on $\partial\Omega$. For $z \in \partial\Omega$ and $t > 0$ sufficiently small, say $0 < t < \delta_0$, the integral curve of N through z has a unique intersection point with the hypersurface $\{\delta = t\}$. We call this intersection point z_t. For any function f on Ω we define f_t on $\partial\Omega$ by $f_t(z) = f(z_t)$ for $z \in \partial\Omega$.

For $f \in \mathcal{O}(\Omega)$, denote

$$(D^\alpha f)(z) = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}(z).$$

Let $\mathcal{N}$ be the complex normal vector field of type (1,0) defined by

$$\mathcal{N} = \sum_{j=1}^{n} \frac{\partial \rho}{\partial \bar{\zeta}_j} \frac{\partial}{\partial \zeta_j}.$$

The main results of this paper are the following two theorems.

THEOREM 1. *Let $f \in \mathcal{O}(\Omega)$, $1 < p < \infty$ and $-1 < b < \infty$ and $0 < a < \infty$. Then*

$$\int_0^{\delta_0} t^{a|\alpha|+b} M_p^a(t, D^\alpha f)\, dt \lesssim \int_0^{\delta_0} t^b M_p^a(t, f)\, dt.$$

THEOREM 2. *Let $f \in \mathcal{O}(\Omega)$, $1 < a \le p < \infty$ and $-1 < b < \infty$. Then*

$$\int_0^{\delta_0} t^b M_p^a(t, f)\, dt \lesssim \sum_{j=0}^{m} \int_0^{\delta_0} t^{am+b} M_p^a\left(t, \mathcal{N}^j f\right) dt.$$

COROLLARY 3. *Let m be a positive integer and $f \in \mathcal{O}(\Omega)$. Then $f \in L^p, 1 < p < \infty$, if and only if $\delta(z)^m D^\alpha f(z) \in L^p$ for all α with $|\alpha| = m$.*

PROOF. Let $0 \le j \le m$. In [CK1], we proved that there is a compact subset K of Ω such that

$$\int_{\Omega\setminus K} \delta(\zeta)^{mp} |\nabla^j f(\zeta)|^p dV \lesssim \int_{\Omega\setminus K} \delta(\zeta)^{(m+1)p} |\nabla^{j+1} f(\zeta)|^p dV + \sup_K |\nabla^j f|^p.$$

Thus we get the result by Theorems 1 and 2. □

Here we use the notation $A \lesssim B$ for the two expressions A and B which means that there is a constant C, independent of the quantities under consideration, such that $A \le C \cdot B$. When $A \lesssim B$ and $B \lesssim A$, we use the notation $A \sim B$.

These two theorems generalize the classical results of Hardy- Littlewood (see [6]) to the strongly pseudoconvex domain of $\mathbb{C}^n$. In case of the unit ball, see Theorems 1 and 2 in [9]. In [3] and [5], we can see other cases of Hardy-Littlewood type inequalities in bounded domains in $\mathbb{C}^n$.

2. Notations and some technical results

Let Ω be a C^∞-bounded strongly pseudoconvex domain with the defining function ρ. We need some results for our proofs. See [1], [2] and [8] in detail. We let $g(z,\zeta)$ be the associated Levi polynomial

$$g(z,\zeta) = 2\sum_{j=1}^{n} \frac{\partial\rho}{\partial\zeta_j}(\zeta)(\zeta_j - z_j) - \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial\zeta_j\partial\zeta_k}(\zeta)(\zeta_j - z_j)(\zeta_k - z_k).$$

It follows from Taylor's formula and the strict plurisubharmonicity of ρ that there are positive constants C_1 and r and a neighborhood Ω' of $\bar{\Omega}$ such that

$$\text{Re } g(z,\zeta) \geq \rho(\zeta) - \rho(z) + C_1|z-\zeta|^2$$

for $z,\zeta \in \Omega'$ and $|z-\zeta| \leq r$. Setting $\tilde{g}(z,\zeta) = g(z,\zeta) - 2\rho(\zeta)$, it follows that

$$\text{(1)} \qquad \text{Re } \tilde{g}(z,\zeta) = \text{Re } g(z,\zeta) - 2\rho(\zeta) \geq -\rho(\zeta) - \rho(z) + C_1|z-\zeta|^2$$

for $z,\zeta \in \Omega'$ and $|z-\zeta| \leq r$ and $\tilde{g}(z,\zeta) = g(z,\zeta)$ for $\zeta \in \partial\Omega$. Also we have

$$\mathcal{N}\ \tilde{g}(z,\zeta) = \mathcal{O}(|z-\zeta|^2).$$

LEMMA 4. [1] *Let $\tilde{g}$, Ω', r and C_1 be as above. There is a neighborhood $\tilde{\Omega}$ of $\bar{\Omega}$ with $\tilde{\Omega} \supset \Omega'$, a C^∞ function $\tilde{\Phi}$ on $\tilde{\Omega} \times \tilde{\Omega}$, and a positive constant C_2 such that*

(i) *for any $\zeta \in \tilde{\Omega}$ the function $\tilde{\Phi}(\cdot,\zeta)$ is holomorphic on $\tilde{\Omega}$;*
(ii) *$\tilde{\Phi}(\zeta,\zeta) = -2\rho(\zeta)$ for $\zeta \in \tilde{\Omega}$, and $|\tilde{\Phi}(z,\zeta)| \geq C_2$ for $z,\zeta \in \tilde{\Omega}$ with $|z-\zeta| \geq \frac{r}{2}$;*
(iii) *there is a non-vanishing C^∞ function $Q(z,\zeta)$ on*

$$\Delta_{\frac{r}{2}} = \{(z,\zeta) \in \tilde{\Omega} \times \tilde{\Omega} : |z-\zeta| \leq \frac{r}{2}\} \text{ such that}$$

$$\tilde{\Phi}(z,\zeta) = \tilde{g}(z,\zeta)Q(z,\zeta) \text{ on } \Delta_{\frac{r}{2}}.$$

LEMMA 5. [1] *For each $s > -1$, there is a smooth form $\eta_s \in C^\infty(\tilde{\Omega} \times \tilde{\Omega})$ such that*

(i) *$\eta_s(z,\zeta)$ is holomorphic in z on $\tilde{\Omega}$ for any fixed $\zeta \in \tilde{\Omega}$, and*

(ii) *for* $f \in A_s^1(\Omega)$ *and* $z \in \Omega$ *we have*

$$f(z) = \int_\Omega f(\zeta) \frac{\eta_s(z,\zeta)}{\tilde{\bar{\Phi}}(z,\zeta)^{n+s+1}} (-\rho(\zeta))^s \, dV(\zeta).$$

LEMMA 6. *Let* $s > -1$ *and* m *be a non-negative integer. Then for* $f \in A_s^1$ *we have the representation*

$$f(z) = \sum_{j=0}^{m} \int_\Omega \mathcal{N}^j f(\zeta) \frac{\eta_{s+j}(z,\zeta)}{\tilde{\bar{\Phi}}(z,\zeta)^{n+s+1}} (-\rho(\zeta))^{s+m} \, dV(\zeta).$$

PROOF. Let $\omega = |\partial\rho|^{-2} * \bar{\partial}\rho$ where $*$ is the Hodge star operator. Then it follows that

$$\partial\rho \wedge \omega = |\partial\rho|^{-2}\partial\rho \wedge *\bar{\partial}\rho = |\partial\rho|^{-2}\langle\partial\rho, \partial\rho\rangle dV = dV. \tag{2}$$

Also for $g \in C^\infty(\Omega)$ we have

$$\partial g \wedge \omega = \partial g \wedge |\partial\rho|^{-2} * \bar{\partial}\rho = |\partial\rho|^{-2}\langle\partial g, \partial\rho\rangle dV = |\partial\rho|^{-2}\mathcal{N} g \; dV. \tag{3}$$

For the case of $m = 1$ we write

$$(-\rho)^s dV = -1/(s+1)\partial(-\rho)^{s+1} \wedge \omega$$

and apply Stokes' theorem. Then we have

$$\begin{aligned}
f(z) &= \int_\Omega f(\zeta) \frac{\eta_s(z,\zeta)}{\tilde{\bar{\Phi}}(z,\zeta)^{n+s+1}} (-\rho(\zeta))^s \, dV(\zeta) \\
&= -\int_\Omega f(\zeta) \frac{\eta_s(z,\zeta)}{\tilde{\bar{\Phi}}(z,\zeta)^{n+s+1}} \partial(-\rho(\zeta))^s \wedge \omega \\
&= \int_\Omega \partial f(\zeta) \frac{\eta_s(z,\zeta)}{\tilde{\bar{\Phi}}(z,\zeta)^{n+s+1}} (-\rho(\zeta))^s \wedge \omega \\
&\quad + \int_\Omega f(\zeta) \partial\left(\frac{\eta_s(z,\zeta)}{\tilde{\bar{\Phi}}(z,\zeta)^{n+s+1}}\right) (-\rho(\zeta))^s \wedge \omega \\
&\quad + \int_\Omega f(\zeta) \frac{\eta_s(z,\zeta)}{\tilde{\bar{\Phi}}(z,\zeta)^{n+s+1}} (-\rho(\zeta))^s \partial\omega.
\end{aligned} \tag{4}$$

We have

$$\partial_\zeta \left(\frac{\eta_s(z,\zeta)}{\tilde{\bar{\Phi}}(z,\zeta)^{n+s+1}}\right) = \frac{\partial_\zeta \eta_s(z,\zeta)}{\tilde{\bar{\Phi}}(z,\zeta)^{n+s+1}} - (n+s+1)\frac{\eta_s(z,\zeta)\partial_\zeta\tilde{\bar{\Phi}}(z,\zeta)}{\tilde{\bar{\Phi}}(z,\zeta)^{n+s+2}}$$

and

$$\partial_\zeta \tilde{\bar{\Phi}}(z,\zeta) \wedge \omega = |\partial\rho|^{-2}\mathcal{N} \; \tilde{\bar{\Phi}}(z,\zeta) dV.$$

Since $\mathcal{N}\ \tilde{g}(z,\zeta) = \mathcal{O}(|z-\zeta|^2)$, we have

$$\left| \frac{\mathcal{N}\ \tilde{\Phi}(z,\zeta)}{\tilde{\Phi}(z,\zeta)} \right| \lesssim 1.$$

We calculate $\partial\omega$. We have

$$\begin{aligned} \partial\omega &= \partial\left(\frac{1}{|\partial\rho|^2}\right) \wedge *\bar{\partial}\rho + \frac{1}{|\partial\rho|^2}\partial(*\bar{\partial}\rho) \\ &= \mathcal{N}\left(\frac{1}{|\partial\rho|^2}\right) dV + \frac{1}{|\partial\rho|^2}\partial(*\bar{\partial}\rho). \end{aligned} \tag{5}$$

We use the expression

$$*\bar{\partial}\rho = \sum_{j=1}^{n} \frac{1}{\sqrt{-1}} \frac{\partial\rho}{\partial\bar{\zeta}_j} d\bar{\zeta}_j \left(\frac{1}{2\sqrt{-1}}\right)^{n-1} \bigwedge_{\nu\neq j} (d\zeta_\nu \wedge d\bar{\zeta}_\nu).$$

Then we have

$$\begin{aligned} \partial(*\bar{\partial}\rho) &= \sum_{j=1}^{n} \frac{1}{\sqrt{-1}} \frac{\partial^2\rho}{\partial\zeta_j\partial\bar{\zeta}_j} d\zeta_j \wedge d\bar{\zeta}_j \left(\frac{1}{2\sqrt{-1}}\right)^{n-1} \bigwedge_{\nu\neq j} (d\zeta_\nu \wedge d\bar{\zeta}_\nu) \\ &= \sum_{j=1}^{n} 2(-1)^{4(j-1)} \frac{\partial^2\rho}{\partial\zeta_j\partial\bar{\zeta}_j} \left(\frac{1}{2\sqrt{-1}}\right)^{n} \bigwedge_{\nu=1}^{n} (d\zeta_\nu \wedge d\bar{\zeta}_\nu) \\ &= \sum_{j=1}^{n} 2(-1)^{4(j-1)} \frac{\partial^2\rho}{\partial\zeta_j\partial\bar{\zeta}_j} \left(\frac{\sqrt{-1}}{2}\right)^{n} \bigwedge_{\nu=1}^{n} (d\zeta_\nu \wedge d\bar{\zeta}_\nu) \\ &= \sum_{j=1}^{n} 2(-1)^{4(j-1)} \frac{\partial^2\rho}{\partial\zeta_j\partial\bar{\zeta}_j} dV. \end{aligned} \tag{6}$$

Thus we have

$$\partial\omega = \psi\ dV, \tag{7}$$

where

$$\psi = \mathcal{N}\left(\frac{1}{|\partial\rho|^2}\right) + \frac{1}{|\partial\rho|^2} \sum_{j=1}^{n} 2(-1)^{4(j-1)} \frac{\partial^2\rho}{\partial\zeta_j\partial\bar{\zeta}_j} \in C^\infty(\bar{\Omega}).$$

Thus we get the result of the case $m = 1$. By iteration of the above argument, we get the general case. □

LEMMA 7. [7] *Let $a \in \mathbb{R}, s > -1$, and let $\tilde{\Phi}(z,w)$ be the function defined in Lemma 4. Then*

$$\int_\Omega \frac{|\rho(w)|^s}{|\tilde{\Phi}(z,w)|^{n+1+s+a}}\, dV(w) \sim \begin{cases} 1 & \text{if } a < 0 \\ 1/|\rho(z)|^a & \text{if } a > 0 \\ \log(1/|\rho(z)|) & \text{if } a = 0. \end{cases}$$

3. Proofs of main results

LEMMA 8. *Let $f \in \mathcal{O}(\Omega)$ and $1 < p < \infty$, $s > -1$. Then for $z \in \Omega$*

$$|\nabla f(z)|^p \lesssim \frac{1}{|\rho(z)|^{\varepsilon p}} \int_\Omega |f(w)|^p \frac{|\rho(w)|^s}{|\tilde{\Phi}(z,w)|^{n+s+1+(1-\varepsilon)p}}\, dV(w)$$

for some $\varepsilon > 0$.

PROOF. It follows from the reproducing property in Lemma 5 that

$$f(z) = \int_\Omega f(w) \frac{\eta(z,w)}{\tilde{\Phi}(z,w)^{n+s+1}} (-\rho(w))^s\, dV(w).$$

Thus we obtain, for the gradient in z,

$$\begin{aligned} |\nabla f(z)| &= \Big| \int_\Omega f(w) \nabla\Big(\frac{\eta(z,w)}{\tilde{\Phi}(z,w)^{n+s+1}}\Big)(-\rho(w))^s\, dV(w)\Big| \\ &\lesssim \int_\Omega |f(w)| \frac{|\rho(w)|^s}{|\tilde{\Phi}(z,w)|^{n+s+2}}\, dV(w). \end{aligned}$$

Using Hölder's inequality with exponents $1/p$ and $1/p'$ for $p, p' > 1$ and by Lemma 7, we have

$$\begin{aligned} |\nabla f(z)| &\lesssim \Big(\int_\Omega |f(w)|^p \frac{|\rho(w)|^s}{|\tilde{\Phi}(z,w)|^{n+s+1+(1-\varepsilon)p}}\, dV(w)\Big)^{1/p} \\ &\quad \times \Big(\int_\Omega \frac{|\rho(w)|^s}{|\tilde{\Phi}(z,w)|^{n+s+1+\varepsilon p'}}\, dV(w)\Big)^{1/p'} \\ &\lesssim \frac{1}{|\rho(z)|^\varepsilon} \Big(\int_\Omega |f(w)|^p \frac{|\rho(w)|^s}{|\tilde{\Phi}(z,w)|^{n+s+1+(1-\varepsilon)p}}\, dV(w)\Big)^{1/p}. \end{aligned}$$

The lemma is proved. □

LEMMA 9. *Let $f \in \mathcal{O}(\Omega)$ and $1 < a \le p < \infty$, $s > -1$, then*

$$M_p^a\Big(t, \frac{\partial f}{\partial z_k}\Big) \lesssim \frac{1}{t^{\varepsilon a}} \int_o^{\delta_0} \frac{\tau^s}{(t+\tau)^{s+a+1-\varepsilon a}} M_p^a(\tau, f)\, d\tau$$

for some $\varepsilon > 0$.

PROOF. By Lemma 8, we have for $\zeta \in \partial\Omega$

$$\begin{aligned}\Big|\frac{\partial f(\zeta_t)}{\partial z_k}\Big|^p &\le \Big|\nabla f(\zeta_t)\Big|^p \\ &\lesssim \frac{1}{|\rho(\zeta_t)|^{\varepsilon p}} \int_\Omega |f(w)|^p \frac{|\rho(w)|^s}{|\tilde{\Phi}(\zeta_t, w)|^{n+s+1+p-\varepsilon p}}\, dV(w). \qquad (8)\end{aligned}$$

Using (8) and Minkowski's inequality imply

$$\begin{aligned}&M_p^a\Big(t, \frac{\partial f}{\partial z_k}\Big) \\ &= \Big(\int_{\partial\Omega} \Big|\frac{\partial f(\zeta_t)}{\partial z_k}\Big|^p\, d\sigma(\zeta)\Big)^{a/p} \\ &\lesssim \Big(\int_{\partial\Omega}\Big(\frac{1}{|\rho(\zeta_t)|^{\varepsilon a}} \int_\Omega |f(w)|^a \frac{|\rho(w)|^s}{|\tilde{\Phi}(\zeta_t, w)|^{n+s+1+a-\varepsilon a}}\, dV(w)\Big)^{p/a} d\sigma(\zeta)\Big)^{a/p} \\ &\lesssim \frac{1}{t^{\varepsilon a}} \int_\Omega \Big(\int_{\partial\Omega} |f(w)|^p \frac{|\rho(w)|^{s\cdot p/a}}{|\tilde{\Phi}(\zeta_t, w)|^{(n+s+1+a-\varepsilon a)p/a}}\, d\sigma(\zeta)\Big)^{a/p} dV(w) \\ &= \frac{1}{t^{\varepsilon a}} \int_\Omega |f(w)|^a |\rho(w)|^s \Big(\int_{\partial\Omega} \frac{1}{|\tilde{\Phi}(\zeta_t, w)|^{(n+s+1+a-\varepsilon a)p/a}}\, d\sigma(\zeta)\Big)^{a/p} dV(w).\end{aligned}$$

Using the fact $p \ge a$ and the definition of $\tilde{\Phi}$, we have

$$\begin{aligned}&M_p^a\Big(t, \frac{\partial f}{\partial z_k}\Big) \\ &\lesssim \frac{1}{t^{\varepsilon a}} \int_\Omega |f(w)|^a |\rho(w)|^s \Big(\int_{\partial\Omega} \frac{1}{|\tilde{\Phi}(\zeta_t, w)|^{n+s+1+a-\varepsilon a}}\, d\sigma(\zeta)\Big) dV(w) \\ &= \frac{1}{t^{\varepsilon a}} \int_0^{\delta_0} \int_{\partial\Omega} |f(\eta_\tau)|^a |\rho(\eta_\tau)|^s\, d\sigma(\eta)\, d\tau \\ &\quad \times \int_{\partial\Omega} \frac{1}{|\tilde{\Phi}(\zeta_t, \eta_\tau)|^{n+s+1+a-\varepsilon a}}\, d\sigma(\zeta) \\ &\lesssim \frac{1}{t^{\varepsilon a}} \int_0^{\delta_0} \int_{\partial\Omega} |f(\eta_\tau)|^a |\rho(\eta_\tau)|^s\, d\sigma(\eta)\, d\tau \\ &\quad \times \int_{\partial\Omega} \frac{1}{\Big(\rho(\zeta_t) + \rho(\eta_\tau) + |\zeta_t - \eta_\tau|^2\Big)^{n+s+1+a-\varepsilon a}}\, d\sigma(\zeta).\end{aligned}$$

Applying the change of coordinates and Hölder's inequality, we obtain

$$\begin{aligned} M_p^a\Big(t, \frac{\partial f}{\partial z_k}\Big) &\lesssim \frac{1}{t^{\varepsilon a}} \int_0^{\delta_0} \frac{\tau^s}{(t+\tau)^{s+1+a-\varepsilon a}} \int_{\partial\Omega} |f(\eta_\tau)|^a \, d\sigma(\eta)\, d\tau \\ &\lesssim \frac{1}{t^{\varepsilon a}} \int_o^{\delta_0} \frac{\tau^s}{(t+\tau)^{s+a+1-\varepsilon a}} M_p^a(\tau, f)\, d\tau. \end{aligned}$$

This completes Lemma 9. □

Proof of Theorem 1. Suppose $|\alpha| = 1$. We first prove the following

$$\int_0^{\delta_0} t^{a+b} M_p^a\Big(t, \frac{\partial f}{\partial z_k}\Big)\, dt \lesssim \int_0^{\delta_0} t^b M_p^a(t, f)\, dt. \tag{9}$$

If $a \le p$, taking $s > b$ in Lemma 9, we have from Lemma 9

$$\begin{aligned} &\int_0^{\delta_0} t^{a+b} M_p^a\Big(t, \frac{\partial f}{\partial z_k}\Big)\, dt \\ &\lesssim \int_0^{\delta_0} t^{a+b} \Big(\frac{1}{t^{\varepsilon a}} \int_o^{\delta_0} \frac{\tau^s}{(t+\tau)^{s+a+1-\varepsilon a}} M_p^a(\tau, f)\, d\tau\Big)\, dt \\ &= \int_0^{\delta_0} \tau^s M_p^a(\tau, f) \Big(\int_o^{\delta_0} \frac{t^{a+b-\varepsilon a}}{(t+\tau)^{s+a+1-\varepsilon a}}\, dt\Big)\, d\tau \end{aligned}$$

for some $\varepsilon > 0$. Integrating the right side of inequality, we get the result of the case $a \le p$.

Now let $a > p$. By Lemma 9, we have

$$M_p^p\Big(t, \frac{\partial f}{\partial z_k}\Big) \lesssim \frac{1}{t^{\varepsilon p}} \int_o^{\delta_0} \frac{\tau^s}{(t+\tau)^{s+p+1-\varepsilon p}} M_p^p(\tau, f)\, d\tau.$$

Thus we get, by Hölder inequality,

$$\begin{aligned} &M_p^a\Big(t, \frac{\partial f}{\partial z_k}\Big) \\ &\lesssim \frac{1}{t^{\varepsilon a}} \Big(\int_o^{\delta_0} \frac{\tau^s}{(t+\tau)^{s+p+1-\varepsilon p}} M_p^p(\tau, f)\, d\tau\Big)^{a/p} \\ &\lesssim \frac{1}{t^{\varepsilon a}} \Big(\int_o^{\delta_0} \frac{\tau^s}{(t+\tau)^{s+p+1-\varepsilon p}} M_p^a(\tau, f)\, d\tau\Big) \\ &\quad \times \Big(\int_0^{\delta_0} \frac{\tau^s}{(t+\tau)^{s+p+1-\varepsilon p}} d\tau\Big)^{(a-p)/p} \\ &\lesssim \frac{1}{t^{a-(1-\varepsilon)p}} \int_o^{\delta_0} \frac{\tau^s}{(t+\tau)^{s+p+1-\varepsilon p}} M_p^a(\tau, f)\, d\tau. \end{aligned}$$

Taking $s > b$ we have

$$\int_0^{\delta_0} t^{a+b} M_p^a\Big(t, \frac{\partial f}{\partial z_k}\Big) dt \lesssim \int_0^{\delta_0} \tau^b M_p^a(\tau, f) d\tau.$$

Thus we get the rersult of the case $a > p$.

For $|\alpha| = 2$, using (9) twice gives

$$\begin{aligned}\int_0^{\delta_0} t^{2a+b} M_p^a\Big(t, \frac{\partial^2 f}{\partial z_j \partial z_k}\Big)\, dt &\lesssim \int_0^{\delta_0} t^{a+b} M_p^a\Big(t, \frac{\partial f}{\partial z_k}\Big)\, dt \\ &\lesssim \int_0^{\delta_0} t^b M_p^a(t, f)\, dt.\end{aligned}$$

The general case can be proved by induction. □

Proof of Theorem 2. By Lemma 6, we have for a non-negative integer q and $z \in \Omega$

$$f(z) = \sum_{j=0}^m \int_\Omega \mathcal{N}^j f(\zeta) \frac{\eta_{s+j}(z,\zeta)}{\tilde{\Phi}(z,\zeta)^{n+s+1}} (-\rho(\zeta))^{s+m}\, dV(\zeta).$$

Using Hölder's inequality for $1/p + 1/p' = 1,\ p, p' > 1$ and for some $\varepsilon > 0$ we have

$$\begin{aligned}|f(z)| &\lesssim \sum_{j=0}^m \int_\Omega |\mathcal{N}^j f(\zeta)||\rho(\zeta)|^m \frac{|\rho(\zeta)|^s}{|\tilde{\Phi}(z,\zeta)|^{n+s+1+\varepsilon-\varepsilon}}\, dV(\zeta) \\ &= \sum_{j=0}^m \int_\Omega \frac{|\mathcal{N}^j f(\zeta)||\rho(\zeta)|^m}{|\tilde{\Phi}(z,w)|^{-\varepsilon}} \frac{1}{|\tilde{\Phi}(z,\zeta)|^{\epsilon}} \frac{|\rho(\zeta)|^s}{|\tilde{\Phi}(z,\zeta)|^{n+s+1}}\, dV(\zeta) \\ &\lesssim \sum_{j=0}^m \Big(\int_\Omega \frac{|\mathcal{N}^j f(\zeta)|^p|\rho(\zeta)|^{mp}}{|\tilde{\Phi}(z,w)|^{-\varepsilon p}} \frac{|\rho(\zeta)|^s}{|\tilde{\Phi}(z,\zeta)|^{n+s+1}}\, dV(\zeta)\Big)^{1/p} \\ &\quad \times \Big(\int_\Omega \frac{1}{|\tilde{\Phi}(z,\zeta)|^{\varepsilon p'}} \frac{|\rho(\zeta)|^s}{|\tilde{\Phi}(z,\zeta)|^{n+s+1}}\, dV(\zeta)\Big)^{1/p'} \\ &= \sum_{j=0}^m \Big(\int_\Omega \frac{|\mathcal{N}^j f(\zeta)|^p|\rho(\zeta)|^{mp+s}}{|\tilde{\Phi}(z,\zeta)|^{n+s+1-\varepsilon p}}\, dV(\zeta)\Big)^{1/p} \\ &\quad \times \Big(\int_\Omega \frac{|\rho(\zeta)|^s}{|\tilde{\Phi}(z,\zeta)|^{\varepsilon p'+n+s+1}}\, dV(\zeta)\Big)^{1/p'}.\end{aligned}$$

By Lemma 7, we obtain

$$|f(z)|^p \lesssim \int_\Omega \frac{|\mathcal{N}^j f(\zeta)|^p|\rho(\zeta)|^{mp+s}}{|\tilde{\Phi}(z,\zeta)|^{n+s+1-\varepsilon p}}\, dV(\zeta) \times \frac{1}{|\rho(z)|^{\varepsilon p}}. \tag{10}$$

Then the integral mean has the following inequality by (10)

$$\begin{aligned}
&\int_0^{\delta_0} t^b M_p^a(t,f)\,dt \\
&= \int_0^{\delta_0} t^b \Big(\int_{\partial\Omega} |f_t(z)|^p d\sigma(z)\Big)^{a/p} dt \\
&\lesssim \sum_{j=0}^m \int_0^{\delta_0} t^b \\
&\quad\times \bigg(\int_{\partial\Omega}\Big(\int_\Omega \frac{|\mathcal{N}^j f(\zeta)|^a |\rho(\zeta)|^{ma+s}}{|\tilde\Phi(z_t,\zeta)|^{n+s+1-\varepsilon a}} dV(\zeta) \frac{1}{|\rho(z_t)|^{\varepsilon a}}\Big)^{p/a} d\sigma(z)\bigg)^{a/p} dt \\
&= \sum_{j=0}^m \int_0^{\delta_0} \frac{t^b}{t^{\varepsilon a}} \\
&\quad\times \bigg(\int_{\partial\Omega}\Big(\int_0^{\delta_0}\int_{\partial\Omega} \frac{|\mathcal{N}^j f(\zeta_\tau)|^a |\rho(\zeta_\tau)|^{ma+s}}{|\tilde\Phi(z_t,\zeta_\tau)|^{n+s+1-\varepsilon a}}\, d\sigma(\zeta)\, d\tau\Big)^{p/a} d\sigma(z)\bigg)^{a/p} dt \\
&\lesssim \sum_{j=0}^m \int_0^{\delta_0} \frac{t^b}{t^{\varepsilon a}} \int_0^{\delta_0} \\
&\quad\times \bigg(\int_{\partial\Omega}\Big(\int_{\partial\Omega} \frac{|\mathcal{N}^j f(\zeta_\tau)|^a\ \tau^{ma+s}}{|\tilde\Phi(z_t,\zeta_\tau)|^{n+s+1-\varepsilon a}}\, d\sigma(\zeta)\Big)^{p/a} d\sigma(z)\bigg)^{a/p} d\tau\, dt \\
&= \sum_{j=0}^m \int_0^{\delta_0} \frac{t^b}{t^{\varepsilon a}} \int_0^{\delta_0} \tau^{ma+s} \\
&\quad\times \bigg(\int_{\partial\Omega}\Big(\int_{\partial\Omega} \frac{|\mathcal{N}^j f(\zeta_\tau)|^a}{|\tilde\Phi(z_t,\zeta_\tau)|^{n+s+1-\varepsilon a}}\, d\sigma(\zeta)\Big)^{p/a} d\sigma(z)\bigg)^{a/p} d\tau\, dt.
\end{aligned}$$

Using Minkowski's inequality and Lemma 7 again, the integral in the brace of the right side inequality is for $a \le p$

$$\begin{aligned}
&\bigg(\int_{\partial\Omega}\Big(\int_{\partial\Omega} \frac{|\mathcal{N}^j f(\zeta_\tau)|^a}{|\tilde\Phi(z_t,\zeta_\tau)|^{n+s+1-\varepsilon a}}\, d\sigma(\zeta)\Big)^{p/a} d\sigma(z)\bigg)^{a/p} \\
&\lesssim \int_{\partial\Omega}\Big(\int_{\partial\Omega} \frac{|\mathcal{N}^j f(\zeta_\tau)|^p}{|\tilde\Phi(z_t,\zeta_\tau)|^{(n+s+1-\varepsilon a)p/a}}\, d\sigma(z)\Big)^{a/p} d\sigma(\zeta) \\
&\lesssim \int_{\partial\Omega} |\mathcal{N}^j f(\zeta_\tau)|^a \Big(\int_{\partial\Omega} \frac{1}{|\tilde\Phi(z_t,\zeta_\tau)|^{n+s+1-\varepsilon a}}\, d\sigma(z)\Big)\, d\sigma(\zeta) \\
&\lesssim \frac{1}{(t+\tau)^{s+1-\varepsilon a}} \int_{\partial\Omega} |\mathcal{N}^j f(\zeta_\tau)|^a d\sigma(\zeta).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\int_0^{\delta_0} t^b M_p^a(t,f)\,dt \\
&\lesssim \sum_{j=0}^{m} \int_0^{\delta_0} \frac{t^b}{t^{\varepsilon a}} \left(\int_0^{\delta_0} \frac{\tau^{ma+s}}{(t+\tau)^{s+1-\varepsilon a}} \left(\int_{\partial\Omega} |\mathcal{N}^j f(\zeta_\tau)|^a d\sigma(\zeta) \right) d\tau \right) dt \\
&\lesssim \sum_{j=0}^{m} \int_0^{\delta_0} \frac{t^b}{t^{\varepsilon a}} \left(\int_0^{\delta_0} \frac{\tau^{ma+s}}{(t+\tau)^{s+1-\varepsilon a}} M_p^a(\tau, \mathcal{N}^j f)\,d\tau \right) dt.
\end{aligned}$$

Therefore the straight calculation for the right side inequality completes the proof. □

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