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RANDOM FIXED POINT THEOREMS AND LERAY–SCHAUDER ALTERNATIVES FOR $\mathcal{U}_c^\kappa$ MAPS

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Abstract. This paper presents new random fixed point theorems for $\mathcal{U}_c^\kappa$ maps and new random Leray–Schauder alternatives for $\mathcal{U}_c^\kappa$ type maps. Our arguments rely on recent deterministic fixed point theorems and on a result on hemicompact maps in the literature.

1. Introduction

The theory of random fixed points plays a very important role in probabilistic analysis, and a systematic study was initiated in the 1950's by the Prague school of probabilists. In this paper we present in Section 1 new random fixed point theorems for $\mathcal{U}_c^\kappa$ type maps and in Section 2 we present a variety of random Leray–Schauder alternatives for $\mathcal{U}_c^\kappa$ type maps. All the results are new and contain as special cases most of the well known random fixed point theory in the literature (see [5, 8, 11 12, 15, 17] and the references therein).

Next in this section we present some preliminary results which will be needed. Let $(\Omega, \mathcal{A})$ be a measurable space and C a nonempty subset of a metric space $X = (X, d)$. Let 2^C denote the family of nonempty subsets of C and $CD(C)$ the family of all nonempty closed subsets of C. A mapping $G : \Omega \to 2^C$ is said to be measurable if

$$G^{-1}(U) = \{w \in \Omega : \; G(w) \cap U \neq \emptyset\} \in \mathcal{A}$$

for each open subset U of C. A mapping $\xi : \Omega \to C$ is called a measurable selector of the measurable mapping $G : \Omega \to 2^C$ if ξ is measurable and $\xi(w) \in G(w)$ for each $w \in \Omega$. A mapping $F : \Omega \times C \to 2^X$ is called a random operator if, for any fixed $x \in C$, the map $F(\,.\,, x) : \Omega \to 2^X$ is measurable. A measurable mapping $\xi : \Omega \to C$ is

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said to be a random fixed point of a random operator $F : \Omega \times C \to 2^X$ if $\xi(w) \in F(w, \xi(w))$ for each $w \in \Omega$. Let $P_B(X)$ be the bounded subsets of X. The Kuratowskii measure of noncompactness is the map $\alpha : P_B(X) \to [0, \infty)$ defined by

$$\alpha(A) = \inf\{\epsilon > 0 : \ A \subseteq \cup_{i=1}^{n} X_i \ \text{ and } \ diam\,(X_i) \leq \epsilon\};$$

here $A \in P_B(X)$. Let S be a nonempty subset of X, and for each $x \in X$ define $d(x, S) = \inf_{y \in S} d(x, y)$. Let $H : S \to 2^X$. H is called (i). countably k–set contractive ($k \geq 0$) if $H(S)$ is bounded and $\alpha(H(Y)) \leq k\,\alpha(Y)$ for all countably bounded sets Y of S; (ii). countably condensing if $H(S)$ is bounded and $\alpha(H(Y)) < \alpha(Y)$ for all countably bounded sets Y of S with $\alpha(Y) \neq 0$; (iii). hemicompact if each sequence $(x_n)_{n=1}^{\infty}$ in S has a convergent subsequence whenever $d(x_n, H(x_n)) \to 0$ as $n \to \infty$.

A random operator $F : \Omega \times C \to CD(X)$ is said to be continuous (countably k–set contractive etc.) if for each $w \in \Omega$, the map $F(w, .) : C \to CD(X)$ is continuous (countably k–set contractive etc.).

Next we state a well known result of Tan and Yuan [15].

THEOREM 1.1. *Let $(\Omega, \mathcal{A})$ be a measurable space and Z a nonempty separable complete subset of a metric space $X = (X, d)$. Suppose the map $F : \Omega \times Z \to CD(X)$ is a continuous, hemicompact random operator. If F has a deterministic fixed point then F has a random fixed point.*

REMARK. A single valued map $\phi : \Omega \to X$ is said to be a deterministic fixed point of F if $\phi(w) \in F(w, \phi(w))$ for each $w \in \Omega$. In [5], we established the following convergence result.

THEOREM 1.2. *Let (X, d) be a Fréchet space, D a closed subset of X and $F : D \to 2^X$ a countably condensing map. Then F is hemicompact.*

Recently [9] another version of a random operator $F : \Omega \times Z \to 2^X$ was considered by Gorniewicz. Let X be a separable metric space, Ω a complete measurable space, and Z a closed subset of X. A map $F : \Omega \times Z \to 2^X$ with compact values is said to be a random in the sense of Gorniewicz operator if

(i) F is product measurable and

(ii) $F(w, .)$ is upper semicontinuous for every $w \in \Omega$

hold. The following result is taken from [9, p.156].

THEOREM 1.3. *Let X be a separable metric space, $(\Omega, \mathcal{A})$ a complete measurable space and Z a closed subset of X. Suppose the map*

$F:\Omega\times Z\to 2^X$ has compact values and is a random in the sense of Gorniewicz operator. If F has a deterministic fixed point then F has a random fixed point.

In view of Theorem 1.1 and Theorem 1.3 it is easy to use well known fixed point theory to establish random fixed point theory. Before we do this we need to describe these deteministic fixed point theorems. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 respectively. We will look at maps $F:X\to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say $F:X\to K(Y)$ is *Kakutani* if F is upper semicontinuous with convex values. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F:X\to K(Y)$ is *acyclic* if F is upper semicontinuous with acyclic values. $F:X\to K(Y)$ is said to be an *O'Neill* map if F is continuous and if the values of F consist of one or m acyclic components. (here m is fixed)

Given two open neighborhoods U and V of the origins in E_1 and E_2 repectively, a (U,V)-approximate continuous selection [6] of $F:X\to K(Y)$ is a continuous function $s:X\to Y$ satisfying

$$s(x)\in(F[(x+U)\cap X]+V)\cap Y\quad\text{for every } x\in X.$$

We say $F:X\to K(Y)$ is *approximable* if it is a closed map and if its restriction $F|_K$ to any compact subset K of X admits a (U,V)-approximate continuous selection for every open neighborhood U and V of the origins in E_1 and E_2 repectively.

For our next definition let X and Y be metric spaces. A continuous single valued map $p:Y\to X$ is called a Vietoris map if the following two conditions are satisfied:

(i) for each $x\in X$, the set $p^{-1}(x)$ is acyclic
(ii) p is a proper map i.e. for every compact $A\subseteq X$ we have that $p^{-1}(A)$ is compact.

DEFINITION 1.1. A multifunction $\phi:X\to K(Y)$ is *admissible* (strongly) in the sense of Gorniewicz, if $\phi:X\to K(Y)$ is upper semicontinuous, and if there exists a metric space Z and two continuous maps $p:Z\to X$ and $q:Z\to Y$ such that
(i) p is a Vietoris map and
(ii) $\phi(x)=q(p^{-1}(x))$ for any $x\in X$.

REMARK 1.1. It should be noted [10, p.179] that ϕ upper semicontinuous is redundant in Definition 1.1.

Suppose X and Y are Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:

(i) $\mathcal{U}$ contains the class $\mathcal{C}$ of single valued continuous functions;
(ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued;
(iii) for any polytope P, $F \in \mathcal{U}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

DEFINITION 1.2. $F \in \mathcal{U}_c^\kappa(X,Y)$ if for any compact subset K of X, there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Examples of $\mathcal{U}_c^\kappa$ maps are the Kakutani maps, the acyclic maps, the O'Neill maps, and the maps admissible in the sense of Gorniewicz.

For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α–close, if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

By a space we mean a Hausdorff topological space. In what follows Q denotes a class of topological spaces. A space Y is an *extension space* for Q (written $Y \in ES(Q)$) if for any pair (X,K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$.

A space Y is an *approximate extension space* for Q (and we write $Y \in AES(Q)$) if for any $\alpha \in Cov(Y)$ and any pair (X,K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$, there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0.

DEFINITION 1.3. Let V be a subset of a Hausdorff topological space E. Then we say V is *Schauder admissible* if for every compact subset K of V and every covering $\alpha \in Cov_V(K)$, there exists a continuous function (called the Schauder projection) $\pi_\alpha : K \to V$ such that

(i) π_α and $i : K \to V$ are α–close;
(ii) $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES(\text{compact})$.

If $V \in AES(\text{compact})$ then V is trivially Schauder admissible. If V is an open convex subset of a Hausdorff locally convex topological space E, then it is well known that V is Schauder admissible.

The following fixed point result was established in [4].

THEOREM 1.4. *Let V be a uniform space and assume V is Schauder admissible. Suppose $F \in \mathcal{U}_c^{\kappa}(V, V)$ a compact upper semicontinuous map with closed values. Then F has a fixed point.*

A nonempty subset X of a Hausdorff topological vector space E is said to be *admissible* if for every compact subset K of X and every neighborhood V of 0, there exists a continuous map $h : K \to X$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E. X is said to be *q–admissible* if any nonempty compact, convex subset Ω of X is admissible. X is said to be *q–Schauder admissible* if any nonempty compact, convex subset Ω of X is Schauder admissible.

The following fixed point result was established in [2].

THEOREM 1.5. *Let Ω be a q–Schauder admissible, closed, convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose $F \in \mathcal{U}_c^{\kappa}(\Omega, \Omega)$ is a upper semicontinuous map with cloised values and assume the following property holds:*

$$A \subseteq \Omega,\ A = \overline{co}\,(\{x_0\} \cup F(A)) \text{ implies } A \text{ is compact.} \tag{1.1}$$

Then F has a fixed point in Ω.

Let (E, d) be a pseudometric space. For $S \subseteq E$, let $B(S, \epsilon) = \{x \in E : d(x, S) \leq \epsilon\}$ for $\epsilon > 0$. The measure of noncompactness [7] of the set $M \subseteq E$ is defined by $\alpha(M) = \inf Q(M)$, where

$$Q(M) = \{\epsilon > 0 : M \subseteq B(A, \epsilon) \text{ for some finite subset } A \text{ of } E\}.$$

Let E be a locally convex Hausdorff topological vector space, and let P be a defining system of seminorms on E. Suppose $F : S \to 2^E$; here $S \subseteq E$. The map F is said to be a countably P–concentrative mapping [7] if $F(S)$ is bounded, and for $p \in P$ for each countably bounded subset X of S we have $\alpha_p(F(X)) \leq \alpha_p(X)$, and for $p \in P$ for each countably bounded non–p–precompact subset X of S (i.e. X is not precompact in the pseudonormed space (E, p)) we have $\alpha_p(F(X)) < \alpha_p(X)$; here $\alpha_p(\,.\,)$ denotes the measure of noncompactness in the pseudonormed space (E, p). In this paper when we consider countably P–concentrative maps it is worth remarking here that in fact the results hold if the maps are countably condensing in the sense of [16, p.353, 356], so in particular for countably condensing maps defined before Theorem 1.1.

The following fixed point result was established in [13].

THEOREM 1.6. *Let Ω be a nonempty, closed, convex subset of a Fréchet space E $(Pisadefiningsystemofseminorms)$. Suppose $F \in$*

$\mathcal{U}_c^{\kappa}(\Omega, \Omega)$ *is a countably P–concentrative mapping. Then F has a fixed point in* Ω.

The following fixed point results were established in [3].

THEOREM 1.7. *Let* Ω *be a q-Schauder admissible closed, convex subset of a Hausdorff topological vector space E with* $x_0 \in \Omega$. *Suppose* $F \in \mathcal{U}_c^{\kappa}(\Omega, \Omega)$ *is a upper semicontinuous map with closed values with the following conditions holding:*

$$(1.2) \qquad \begin{cases} \text{for any relatively compact, convex subset } A \text{ of } \Omega \\ \text{with } co\,(F(A)) \subseteq A \text{ we have } F(\overline{A}) \subseteq \overline{co\,(F(A))} \end{cases}$$

and

$$(1.3) \qquad A \subseteq \Omega,\ A = co\,(\{x_0\} \cup F(A)) \text{ implies } \overline{A} \text{ is compact.}$$

Then F has a fixed point in Ω.

THEOREM 1.8. *Let* Ω *be a q-Schauder admissible closed, convex subset of a Hausdorff topological vector space E with* $x_0 \in \Omega$. *Suppose* $F \in \mathcal{U}_c^{\kappa}(\Omega, \Omega)$ *is a upper semicontinuous map with closed values which maps compact sets into relatively compact sets and assume* (1.2) *holds. In addition suppose the following conditions are satisfied:*

$$(1.4) \qquad \begin{cases} A \subseteq \Omega,\ A = co\,(\{x_0\} \cup F(A)) \text{ with } \overline{A} = \overline{C} \\ \text{and } C \subseteq A \text{ countable, implies } \overline{A} \text{ is compact} \end{cases}$$

$$(1.5) \qquad \begin{cases} \text{for any relatively compact subset } A \text{ of } \Omega \text{ there} \\ \text{exists a countable set } B \subseteq A \text{ with } \overline{B} = \overline{A} \end{cases}$$

and

$$(1.6) \qquad \text{if } A \text{ is a compact subset of } \Omega \text{ then } \overline{co}\,(A) \text{ is compact.}$$

Then F has a fixed point in Ω.

REMARK 1.2. If F is a Kakutani map then (1.2) is not needed [3] in Theorem 1.7 and Theorem 1.8.

REMARK 1.3. If $F : \Omega \to 2^{\Omega}$ is lower semicontinuous then (1.2) holds (see [3]).

First we obtain a random analogue of Theorem 1.4.

THEOREM 1.9. *Let* $(\Omega, \mathcal{A})$ *be a measurable space, E a metric space, X a Schauder admissible complete separable subset of E,* $F : \Omega \times X \to CD(X)$ *a random continuous, compact operator with* $F(w, .) \in \mathcal{U}_c^{\kappa}(X, X)$ *for each* $w \in \Omega$. *Then F has a random fixed point.*

PROOF. Now [14] implies that $F : \Omega \times X \to CD(X)$ is hemicompact and Theorem 1.4 guarantees that F has a deterministic fixed point. The result now follows from Theorem 1.1. □

THEOREM 1.10. *Let* $(\Omega, \mathcal{A})$ *be a complete measurable space,* E *a separable metric space,* X *a closed Schauder admissible subset of* E, $F : \Omega \times X \to K(X)$ *a random in the sense of Gorniewicz compact operator with* $F(w, .) \in \mathcal{U}_c^\kappa(X, X)$ *for each* $w \in \Omega$. *Then* F *has a random fixed point.*

PROOF. The result follows from Theorem 1.3 and Theorem 1.4. □

We next obtain a random(in the usual sense) analogue of the other fixed point theorems in this section (the random analogue of these theorems for operators random in the sense of Gorniewicz is left to the reader).

THEOREM 1.11. *Let* $(\Omega, \mathcal{A})$ *be a measurable space,* E *a metrizable topological vector space,* X *a q-Schauder admissible closed convex complete separable subset of* E, $x_0 \in X$, $F : \Omega \times X \to CD(X)$ *a random continuous operator with* $F(w, .) \in \mathcal{U}_c^\kappa(X, X)$ *for each* $w \in \Omega$. *Also assume the following properties are satisfied:*

$$(1.7) \qquad \begin{cases} \text{for each } w \in \Omega,\ A \subseteq X \text{ with } A = \overline{co}\,(\{x_0\} \cup F(w, A)) \\ \text{implies } A \text{ is compact} \end{cases}$$

and

$$(1.8) \qquad F : \Omega \times X \to CD(X) \text{ is hemicompact.}$$

Then F *has a random fixed point.*

PROOF. The result follows from Theorem 1.1 and Theorem 1.5. □

THEOREM 1.12. *Let* $(\Omega, \mathcal{A})$ *be a measurable space,* E *a Fréchet space* $(P\, a defining system of seminorms)$, X *a nonempty closed convex separable subset of* E, $F : \Omega \times X \to CD(X)$ *a random continuous countably* P*-concentrative operator with* $F(w, .) \in \mathcal{U}_c^\kappa(X, X)$ *for each* $w \in \Omega$. *In addition assume* (1.8) *holds. Then* F *has a random fixed point.*

PROOF. The result follows from Theorem 1.1 and Theorem 1.6. □

REMARK 1.4. If in Theorem 1.12, $F : \Omega \times X \to CD(X)$ is a random continuous countably condensing operator (in the sense of the definition before Theorem 1.1) then (1.8) is satisfied (see Theorem 1.2).

THEOREM 1.13. *Let $(\Omega, \mathcal{A})$ be a measurable space, E a metrizable topological vector space, X a q-Schauder admissible closed convex complete separable subset of E, $x_0 \in X$, $F : \Omega \times X \to CD(X)$ a random continuous operator with $F(w, .) \in \mathcal{U}_c^\kappa(X, X)$ for each $w \in \Omega$. Also assume (1.8) holds and suppose the following property is satisfied:*

$$(1.9) \qquad \begin{cases} \text{for each } w \in \Omega,\ A \subseteq X \text{ with } A = co\,(\{x_0\} \cup F(w, A)) \\ \text{implies } \overline{A} \text{ is compact.} \end{cases}$$

Then F has a random fixed point.

PROOF. The result follows from Theorem 1.1 and Theorem 1.7 (note for each $w \in \Omega$ that $F(w, .) : X \to CD(X)$ is continuous so lower semicontinuous). □

REMARK 1.5. One could also obtain the random analogue of Theorem 1.8 (we leave the details to the reader).

2. Random Leray–Schauder alternatives

In this section we present random Leray–Schauder alternatives for $\mathcal{U}_c^\kappa$ type maps. First however we recall [1] (where we use the results in [14]) some results for essential $\mathcal{U}_c^\kappa$ maps in the deterministic situation.

Here E is a Hausdorff locally convex topological vector space, C is a closed convex subset of E, $U \subseteq C$ is convex, U is an open subset of E, and $0 \in U$. Notice $int_C\, U = U$ since U is open in C. We will consider maps $F : \overline{U} \to K(C)$ (here $\overline{U}$ denotes the closure of U in C). Throughout our map $F : \overline{U} \to K(C)$ will satisfy <u>one</u> of the following conditions:

(H1) F is compact;
(H2) if $D \subseteq \overline{U}$ and $D \subseteq \overline{co}\,(\{0\} \cup F(D))$ then $\overline{D}$ is compact;
(H3) F is countably P–concentrative and E is Fréchet (here P is a defining system of seminorms);
(H4) F is lower semicontinuous and if $D \subseteq \overline{U}$ and $D \subseteq co\,(\{0\} \cup F(D))$ then $\overline{D}$ is compact; or
(H5) F is lower semicontinuous, F maps compact sets into relatively compact sets, and if $D \subseteq \overline{U}$, $D \subseteq co\,(\{0\} \cup F(D))$ with $K \subseteq D$ countable and $\overline{K} = \overline{D}$ then $\overline{D}$ is compact and in this case we also assume (i). for any relatively compact convex set A of E there exists a countable set $B \subseteq A$ with $\overline{B} = \overline{A}$, and (ii). if Q is a compact subset of E then $\overline{co}\,(Q)$ is compact.

Fix $i \in \{1,2,3,4,5\}$.

DEFINITION 2.1. We say $F \in LS^i(\overline{U},C)$ if $F \in \mathcal{U}_c^\kappa(\overline{U},C)$ satisfies (Hi).

DEFINITION 2.2. We say $F \in LS^i_{\partial U}(\overline{U},C)$ if $F \in LS^i(\overline{U},C)$ with $x \notin F\,x$ for $x \in \partial U$; here ∂U denotes the boundary of U in C.

DEFINITION 2.3. A map $F \in LS^i_{\partial U}(\overline{U},C)$ is essential in $LS^i_{\partial U}(\overline{U},C)$ if for every $G \in LS^i_{\partial U}(\overline{U},C)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

THEOREM 2.1. *Fix $i \in \{1,2,3,4,5\}$ and let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, $0 \in U$, $F \in LS^i(\overline{U},C)$ and assume the following condition holds:*

$$x \notin \lambda F\,x \quad \text{for } x \in \partial U \text{ and } \lambda \in (0,1]. \tag{2.1}$$

Then F is essential in $LS^i_{\partial U}(\overline{U},C)$ (in particular F has a fixed point in U).

REMARK 2.1. In (H4) and (H5), F lower semicontinuous can be replaced by

$$\begin{cases} \text{for any relatively compact, convex subset } A \text{ of } \overline{U} \\ \text{with } co\,(F(A)) \subseteq A \text{ we have } F(\overline{A}) \subseteq \overline{F(A)}. \end{cases} \tag{2.2}$$

Our first result is a random Leray–Schauder alternative for $\mathcal{U}_c^\kappa$ type maps.

THEOREM 2.2. *Fix $i \in \{1,2,3,4,5\}$ and let $(\Omega,\mathcal{A})$ be a measurable space, E a separable Fréchet space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, $0 \in U$, $F : \Omega \times \overline{U} \to CD(C)$ a random continuous operator with $F(w,\,.\,) \in LS^i(\overline{U},C)$ for each $w \in \Omega$. Also assume for each $w \in \Omega$ that the following condition is satisfied:*

$$x \notin \lambda F(w,x) \text{ for all } x \in \partial U \text{ and } \lambda \in (0,1]. \tag{2.3}$$

Finally suppose

$$F : \Omega \times \overline{U} \to CD(C) \text{ is hemicompact if } i \in \{2,3,4,5\}. \tag{2.4}$$

Then F has a random fixed point ξ with $\xi(w) \in U$ for each $w \in \Omega$.

REMARK 2.2. For the cases $i = 4$ and $i = 5$ note for each $w \in \Omega$ that $F(w,\,.\,) : \overline{U} \to K(C)$ is automatically lower semicontinuous since $F(w,\,.\,) : \overline{U} \to K(C)$ is continuous.

REMARK 2.3. For the case $i = 3$ if F countably P–concentrative is replaced by F countably condensing (in the sense of the definition before Theorem 1.1) then (2.4) holds when $i = 3$.

PROOF. We know from (2.4) (or [15] if $i = 1$) that $F : \Omega \times \overline{U} \to CD(C)$ is hemicompact. Fix $w \in \Omega$ and notice Theorem 2.1 guarantees that $F(w, .)$ has a fixed point in U. As a result F has a deterministic fixed point so Theorem 1.1 guarantees that F has a random fixed point ξ with $\xi(w) \in \overline{U}$ for each $w \in \Omega$. This with (2.3) completes the proof. □

From an application point of view it is of interest to allow our set U in Theorem 2.2 to vary with w. Our next result was motivated by the papers [8, 12]. We will consider the cases $i = 1$ and $i = 3$ and our space E will be a Banach space, so in this situation when $i = 3$ countably P–concentrative means countably condensing (in the sense of the definition before Theorem 1.1).

THEOREM 2.3. *Let* $(\Omega, \mathcal{A})$ *be a measurable space,* $E = (E, |\,.\,|)$ *a separable Banach space,* $r : \Omega \to \mathbf{R}$ *measurable with* $r(w) > 0$ *for each* $w \in \Omega$ *and let* $Q_{r(w)} = \{x \in E : |x| \le r(w)\}$. *Suppose* $F : \Omega \times E \to CD(E)$ *is a random continuous countably condensing* $(respectively, compact or condensing)$ *operator with* $F(w, .) \in \mathcal{U}_c^{\kappa}(Q_{r(w)}, E)$ *for each* $w \in \Omega$, *and assume for each* $w \in \Omega$ *that the following condition holds:*

$$x \notin \lambda\, F(w, x) \text{ for all } x \in \partial\, Q_{r(w)} \text{ and } \lambda \in (0, 1). \tag{2.5}$$

Then F *has a random fixed point.*

PROOF. Fix $w \in \Omega$. Let $R_w : E \to Q_{r(w)}$ be the continuous retraction given by

$$R_w(x) = \begin{cases} x, & x \in Q_{r(w)} \\ r(w)\,\frac{x}{|x|}, & |x| > r(w). \end{cases}$$

Let $H_w(\,.\,) = F(w, .)$ and

$$G(w, x) = F(w, R_w(x)) \quad \text{for} \quad x \in E.$$

Notice $G(w, .) = H_w \circ R_w(\,.\,)$ so $G(w, .) \in \mathcal{U}_c^{\kappa}(E, E)$ since $\mathcal{U}_c^{\kappa}$ is closed under compositions. Also it is easy to check that $G(w, .) : E \to E$ is countably condensing (respectively, compact or condensing); to see the countably condensing situation notice if A is a countably bounded subset of E with $\alpha(A) \neq 0$, then since $R_w(A)$ is countable and $R_w(A) \subseteq$

$co\,(A \cup \{0\})$ we have

$$\alpha(G(w,A)) = \alpha(F(w,R_w(A))) < \alpha(R_w(A)) \le \alpha(A) \quad \text{if} \quad \alpha(R_w(A)) \ne 0$$

and

$$\alpha(G(w,A)) = \alpha(F(w,R_w(A))) \le \alpha(R_w(A)) < \alpha(A) \quad \text{if} \quad \alpha(R_w(A)) = 0.$$

Also a standard argument [12, p.1967] guarantees that $G(\,.\,,x)$ is measurable for each $x \in E$. As a result $G : \Omega \times E \to CD(E)$ is a random continuous operator with $G(w,\,.\,) \in \mathcal{U}_c^\kappa(E,E)$ for each $w \in \Omega$. We apply Theorem 1.12 (respectively, Theorem 1.9 or Theorem 1.11) to deduce that G has a random fixed point u i.e. $u(w) \in F(w,R_w(u(w)))$ for each $w \in \Omega$ and $u : \Omega \to E$ is measurable. We now claim that $u(w) \in Q_{r(w)}$ for each $w \in \Omega$. If the claim is true then we are finished, since if $u(w) \in Q_{r(w)}$ then $R_w(u(w)) = u(w)$.

It remains to prove the claim. If the claim is false, then there exists a $w_1 \in \Omega$ with $|u(w_1)| > r(w_1)$. Let $v(w_1) = R_{w_1}(u(w_1))$ and notice $u(w_1) \in F(w_1, R_{w_1}(u(w_1)))$ yields

$$v(w_1) \in \lambda\, F(w_1, v(w_1)) \quad \text{with} \quad v(w_1) = r(w_1)\,\frac{u(w_1)}{|u(w_1)|} \in \partial\, Q_{r(w_1)}$$

and $\lambda = \frac{r(w_1}{|u(w_1)|} \in (0,1)$. This contradicts (2.5). □

References

[1] R. P. Agarwal and D. O'Regan, *An essential map theory for $\mathcal{U}_c^\kappa$ and PK maps*, Topol. Methods Nonlinear Anal. **21** (2003), 375–386.

[2] ———, *Collectively fixed point theorems*, Nonlinear Anal. Forum **7** (2002), 167–179.

[3] ———, *Fixed point theory of Mönch type for general classes of admissible maps*, Nonlinear Funct. Anal. Appl. **8** (2003), 609–622.

[4] R. P. Agarwal, D. O'Regan, and S. Park, *Fixed point theory for multimaps in extension type spaces*, J. Korean Math. Soc. **39** (2002), 579–591.

[5] R. P. Agarwal, D. O'Regan, and M. Sambandham, *Random fixed point theory for multivalued countably condensing random operators*, Stochastic Anal. Appl. **16** (2002), 1157–1168.

[6] H. Ben-El-Mechaiekh and P. Deguire, *Approachability and fixed points for nonconvex set valued maps*, J. Math. Anal. Appl. **170** (1992), 477–500.

[7] J. Daneš, *Generalized contractive mappings and their fixed points*, Comment. Math. Univ. Carolin. **11** (1970), 115–136.

[8] K. Deimling, G. S. Ladde, and V. Lakshmikantham, *Sample solutions of stochastic boundary value problems*, Stochastic Anal. Appl. **3** (1985), 153–162.

[9] L. Gorniewicz, *Topological fixed point theory of multivalued mappings*, Kluwer Acad. Publ. Dordrecht, 1999.

[10] L. Gorniewicz and M. Slosarski, *Topological essentiality and differential inclusions*, Bull. Austral. Math. Soc. **45** (1992), 177–193.

[11] T. C. Lin, *Random approximations and random fixed point theorems for continuous 1-set contractive random maps*, Proc. Amer. Math. Soc. **123** (1995), 1167–1176.

[12] D. O'Regan, *A continuation type result for random operators*, Proc. Amer. Math. Soc. **126** (1998), 1963–1971.

[13] ———, *A unified fixed point theory for countably P-concentrative multimaps*, Appl. Anal. **81** (2002), 565–574.

[14] ———, *Fixed point theory for the $\mathcal{B}^k$-admissible maps of Park*, Appl. Anal. **79** (2001), 173–185.

[15] K. K. Tan and X. Z. Yuan, *Random fixed point theorems and approximations in cones*, J. Math. Anal. Appl. **185** (1994), 378–390.

[16] M. Vath, *Fixed point theorems and fixed point index for countably condensing maps*, Topol. Methods Nonlinear Anal. **13** (1999), 341–363.

[17] H. K. Xu, *Some random fixed point theorems for condensing and nonexpansive operators*, Proc. Amer. Math. Soc. **110** (1990), 495–500.

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