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THE STRONG PERRON INTEGRAL IN THE n-DIMENSIONAL SPACE $\mathbb{R}^n$

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Abstract. In this paper, we introduce the SP-integral and the SP_α-integral defined on an interval in the n-dimensional Euclidean space $\mathbb{R}^n$. We also investigate the relationship between these two integrals.

1. Introduction and preliminaries

It is well known [3] that the Perron integral defined on an interval of the real line $\mathbb{R}$ by major and minor functions which are not assumed to be continuous is equivalent to the one defined by continuous major and minor functions and that the strong Perron integral defined on an interval of $\mathbb{R}$ by strong major and minor functions is equivalent to the McShane integral.

In this paper, we introduce Perron-type integrals defined on an interval of the n-dimensional Euclidean space $\mathbb{R}^n$ using the strong major and minor functions, and investigate the relationship between these integrals. We shall call it the strong Perron integral, or briefly SP-integral.

For a subset E of the n-dimensional Euclidean space $\mathbb{R}^n$, the Lebesgue measure of E is denoted by $|E|$. For a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, the norm of x is $\|x\| = \max_{1\leq i\leq n} |x_i|$ and the δ-neighborhood $U(x,\delta)$ of x is an open cube centered at x with sides equal to 2δ.

For an interval $I = [a_1, b_1] \times [a_2, b_2] \times \cdots [a_n, b_n]$ of $\mathbb{R}^n$ with $a_i < b_i$ for $i = 1, 2, \cdots, n$, we call the number $r(I) = \min_i(b_i - a_i)/\max_i(b_i - a_i)$ the *regularity* of I. If $r(I) > \alpha(\alpha \in (0,1))$, then the interval I is said to be *α-regular*.

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Throughout this paper, I_0 denotes a fixed interval in $\mathbb{R}^n$ and $\mathcal{I}$ the family of all subintervals of I_0. A positive function δ defined on a set $E \subset I_0$ is called a *gauge* on E. By $\mathcal{F}$ we denote the free full interval basis $\mathcal{F} = \{(I, x) : I \in \mathcal{I}, x \in I_0\}$. For a given gauge δ and a given $\alpha \in (0, 1)$, we write

$$\mathcal{F}^\alpha = \{(I, x) \in \mathcal{F} : r(I) > \alpha\},$$

$$\mathcal{F}_\delta^\alpha = \{(I, x) \in \mathcal{F} : r(I) > \alpha, I \subset U(x, \delta(x))\}.$$

For a set $E \subset I_0$, we write

$$\mathcal{F}(E) = \{(I, x) \in \mathcal{F} : I \subset E\},$$

$$\mathcal{F}[E] = \{(I, x) \in \mathcal{F} : x \in E\}.$$

A finite subset π of $\mathcal{F}$ is a *$\mathcal{F}$-partition* of I_0 if for distinct pairs (I_1, x_1) and (I_2, x_2) in π, I_1 and I_2 are nonoverlapping and $\cup_{(I,x)\in\pi} I = I_0$.

2. The strong Perron integral

To define the strong Perron integral, we introduce first the definitions of the strong α-regular and strong lower and upper derivatives.

DEFINITION 2.1. Let F be an interval function and let $x \in I_0$. The *strong α-regular lower and upper derivatives* of F at x are defined by

$$\underline{SD}_\alpha F(x) = \sup_\delta \inf \left\{ \frac{F(I)}{|I|} : (I, x) \in \mathcal{F}_\delta^\alpha[\{x\}] \right\},$$

$$\overline{SD}_\alpha F(x) = \inf_\delta \sup \left\{ \frac{F(I)}{|I|} : (I, x) \in \mathcal{F}_\delta^\alpha[\{x\}] \right\}.$$

The function F is *strongly α-regularly differentiable at x* if

$$\underline{SD}_\alpha F(x) = \overline{SD}_\alpha F(x) \neq \pm\infty.$$

This common value is the *strong α-regular derivative* of F at x and is denoted by $SD_\alpha F(x)$.

The *strong lower and upper derivatives* of F at x are defined by

$$\underline{SD}F(x) = \inf_{\alpha\in(0,1)} \underline{SD}_\alpha F(x),$$

$$\overline{SD}F(x) = \sup_{\alpha\in(0,1)} \overline{SD}_\alpha F(x).$$

The function F is *strongly differentiable at* x if $\underline{SD}F(x) = \overline{SD}F(x) \neq \pm\infty$. This common value is the *strong derivative* of F at x and is denoted by $SDF(x)$.

It is easy to see that for any $0 < \alpha < \beta < 1$ and for any $x \in I_0$ we have

$$\underline{SD}F(x) \leq \underline{SD}_\alpha F(x) \leq \underline{SD}_\beta F(x) \leq \overline{SD}_\beta F(x) \leq \overline{SD}_\alpha F(x) \leq \overline{SD}F(x).$$

DEFINITION 2.2. Let f be a point function on I_0.

(a) An interval function M is a *strong α-major function* of f on I_0 if it is superadditive and $\underline{SD}_\alpha M(x) \geq f(x)$ for all $x \in I_0$.

(b) An interval function m is a *strong α-minor function* of f on I_0 if it is subadditive and $\overline{SD}_\alpha m(x) \leq f(x)$ for all $x \in I_0$.

DEFINITION 2.3. A function $f : I_0 \to \mathbb{R}$ is *SP_α-integrable* on I_0 if

$$-\infty < \sup\{m(I_0)\} = \inf\{M(I_0)\} < \infty,$$

where the supremum is taken over all strong α-minor functions of f and the infimum is taken over all strong α-major functions of f. This common value is the *SP_α-integral* of f on I_0 and is denoted by $(SP_\alpha)\int_{I_0} f$.

The following theorem is an immediate consequence of the definition.

THEOREM 2.4. *A function $f : I_0 \to \mathbb{R}$ is SP_α-integrable on I_0 if for each $\epsilon > 0$ there exist a strong α-major function M and a strong α-minor function m on I_0 such that $M(I_0) - m(I_0) < \epsilon$.*

DEFINITION 2.5. Let $\alpha \in (0,1)$. A function f on I_0 is *M_α-integrable* on I_0 with integral A if there exists a gauge δ such that

$$\left| \sum_{(I,x)\in\pi} f(x)|I| - A \right| < \epsilon$$

for every $\mathcal{F}_\delta^\alpha$-partition π of I_0. We write $A = (M_\alpha)\int_{I_0} f$.

THEOREM 2.6. *Let $\alpha \in (0,1)$. If a function f is SP_α-integrable on I_0, then f is M_α-integrable on I_0 and the integrals are equal.*

PROOF. Suppose that f is SP_α-integrable on I_0 and let $\epsilon > 0$. Then there exist a strong α-major function M and a strong α-minor function m of f on I_0 such that

$$-\epsilon < m(I_0) - (SP_\alpha)\int_{I_0} f \le 0 \le M(I_0) - (SP_\alpha)\int_{I_0} f < \epsilon.$$

Since $\overline{SD}_\alpha m \le f \le \underline{SD}_\alpha M$ on I_0, for each $x \in I_0$ there exists $\delta(x) > 0$ such that

$$\frac{M(I)}{|I|} \ge f(x) - \epsilon \quad \text{and} \quad \frac{m(I)}{|I|} \le f(x) + \epsilon$$

whenever $(I,x) \in \mathcal{F}_\delta^\alpha[\{x\}]$.

If $\pi = \{(I_i, x_i) : 1 \le i \le n\}$ is any $\mathcal{F}_\delta^\alpha$-partition of I_0, then we have

$$\begin{aligned}\sum_{i=1}^n f(x_i)|I_i| &- (SP_\alpha)\int_{I_0} f \\ &\le \sum_{i=1}^n [f(x_i)|I_i| - M|I_i)] + M(\Delta) - (SP_\alpha)\int_{I_0} f \\ &< \epsilon \sum_{i=1}^n |I_i| + \epsilon = \epsilon(|I_0| + 1).\end{aligned}$$

Similarly, using the minor function m

$$\sum_{i=1}^n f(x_i)|I_i| - (SP_\alpha)\int_{I_0} f > -\epsilon(|I_0| + 1).$$

It follows that f is M_α-integrable on I_0 and

$$(M_\alpha)\int_{I_0} f = (SP_\alpha)\int_{I_0} f. \qquad \square$$

THEOREM 2.7. *Let $\alpha \in (0,1)$. If f is M_α-integrable on I_0, then f is SP_α-integrable on I_0.*

PROOF. Suppose that f is M_α-integrable on I_0 and let $\epsilon > 0$. Then there exists a gauge δ on I_0 such that

$$\left| \sum_{(I,x)\in\pi} f(x)|I| - (M_\alpha)\int_{I_0} f \right| < \epsilon$$

for every $\mathcal{F}_\delta^\alpha$-partition π of I_0. For each interval I, let

$$M(I) = \sup\left\{ \sum_{(J,x)\in\pi} f(x)|J| : \pi \subset \mathcal{F}_\delta^\alpha(I) \right\},$$

$$m(I) = \inf\left\{ \sum_{(J,x)\in\pi} f(x)|J| : \pi \subset \mathcal{F}_\delta^\alpha(I) \right\}.$$

Then it is easy to show that M is superadditive and m is subadditive. Fix a point $x \in I_0$. For each $(I,x) \in \mathcal{F}_\delta^\alpha[\{x\}]$, $M(I) \geq f(x)|I|$ and $\frac{M(I)}{|I|} \geq f(x)$. It follows that $\underline{SD}_\alpha M(x) \geq f(x)$. Similarly, $\overline{SD}_\alpha m(x) \leq f(x)$. Hence M is a strong α-major function of f on I_0 and m is a strong α-minor function of f on I_0.

Since

$$\left| \sum_{(I,x)\in\pi_1} f(x)|I| - \sum_{(J,y)\in\pi_2} f(y)|J| \right| < 2\epsilon$$

for any two $\mathcal{F}_\delta^\alpha$-partition π_1 and π_2 of I_0, we have $M(I_0) - m(I_0) \leq 2\epsilon$. By Theorem 2.4, f is SP_α-integrable on I_0. □

DEFINITION 2.8. Let f be a point function on I_0.

(a) An interval function M is a *strong major function* of f on I_0 if it is superadditive and $\underline{SD}M(x) \geq f(x)$ for all $x \in I_0$.

(a) An interval function m is a *strong minor function* of f on I_0 if it is subadditive and $\overline{SD}m(x) \leq f(x)$ for all $x \in I_0$.

DEFINITION 2.9. A function f is *SP-integrable* on I_0 if $-\infty < \sup\{m(I_0)\} = \inf\{M(I_0)\} < \infty$, where the supremum is taken over all strong minor functions of f and the infimum is taken over all strong major functions of f. This common value is the *strong Perron integral* (*SP*-integral) of f on I_0 and is denoted by $(SP)\int_{I_0} f$.

DEFINITION 2.10. A function f is M_0*-integrable* on I_0 with integral A if it is M_α-integrable on I_0 with integral A for each $\alpha \in (0,1)$.

THEOREM 2.11. *If a function f is SP-integrable on I_0, then f is SP_α-integrable on I_0 for each $\alpha \in (0,1)$ and the integrals are equal.*

PROOF. Since

$$\underline{SD}F \leq \underline{SD}_\alpha F \leq \overline{SD}_\alpha F \leq \overline{SD}F$$

for each $\alpha \in (0,1)$, every strong major function is a strong α-major function and the same is true for minor functions. Hence it follows that if f is SP-integrable on I_0 with integral A, then f is SP_α-integrable for each $\alpha \in (0,1)$ with integral A. □

From Theorem 2.6, Theorem 2.7 and Theorem 2.11, we get the following chain of inclusions :

$$SP \subset \cap_{\alpha\in(0,1)} SP_\alpha = \cap_{\alpha\in(0,1)} M_\alpha = M_0, \tag{1}$$

where SP denotes the collection of all SP-integrable functions and $\cap_{\alpha\in(0,1)} SP_\alpha$ denotes the collection of all SP_α-integrable functions for each $\alpha \in (0,1)$.

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